An Introduction to Fuzzy Inference

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The *fuzzy inference engine* is the foundation of most fuzzy expert systems and control systems. From a linguistic description of cause and effect of a process, a fuzzy inference engine can be designed to emulate the process.

Fuzzy Sets

Conventional or *crisp* sets are binary. An element either belongs to the set or doesn't. The operations on these sets, such as complement, conjunction (logical OR) and injunction (logical AND) constitute conventional crisp logic. Consider the set of cities that lie more than 3000 miles from Los Angeles. A city lies in the set or it doesn't. *Fuzzy sets*, on the other hand, have grades of memberships. The set of cities `far' from Los Angeles is an example. New York is clearly farther from LA than Chicago. Thus, New

$$\mu_{LA} = 0.0/\mathrm{LA} + 0.5/\mathrm{Chicago} + 0.8/\mathrm{New\ York} + 0.9/\mathrm{London} \qquad (1)$$

York has a greater membership in this set than does Chicago. The grade of membership lies on the interval of zero to one. The assignment of grades of membership is somewhat arbitrary. Different grades will typically be assigned by different people. No matter what the assigned membership function, however, we expect the membership of New York to be greater than that of Chicago. We can write the fuzzy membership function,

Thus, the city of LA has a membership of zero in the set while London has a membership of 0.9.

The term *far* used to define this set is a *fuzzy linguistic variable*. Language is full of such variables. Examples include *close, heavy, light, big, small, smart, fast, slow, hot, cold, tall* and *short*. For a specific element, the membership function for a given fuzzy set, say `*good* Olympic dives', is equivalent to asking the question `On a scale of zero to ten, how *good* was high dive?' When the Olympic judges hold their signs of judgment with numbers from one to ten, their individual numerical assessment, divided by 10, is a membership assessment of the *goodness* of the dive.

Linguistic variables are commonly used in human communication. Consider the case of giving a truck driver oral instructions to back up. A crisp command would be "Back up the truck 6 feet and 7 and one half inches". A more typical command consists of a string of fuzzy linguistic variables. `More ... more ... slow ... slower ... almost there ... stop!'. The procedure works well even though the underlying membership functions for these commands may differ significantly from the instructor to the truck driver. One's specific interpretation of the term `slow', for example, may differ from the other's.

The fuzzy membership function given in Equation (1) is discrete. Membership functions can also be continuous. For example, the set of tall men can have the membership function shown in Figure (2). A continuous membership function for the set, B, of numbers *near* to two is

$$\mu_B(x) = \frac{1}{(x-2)^2}$$

A fuzzy set, A, is said to be a subset of B if

$$\mu_A(x) \le \mu_B(x)$$

The set of *very good* divers is, for example, a subset of *good* divers. The impact of the adjective *very* on the membership function can be a simple squared operation. For example, the set, V, of numbers *very near* to two has a membership function

$$\mu_V(x) = \mu_B^2(x)$$

Note that V is a subset of the fuzzy set B.

Differences Between Fuzzy Membership Functions and Probability Density Functions

On first exposure, fuzzy membership functions are often mistaken as probability density functions. Although both measure uncertainty, they are very different. Here are some illustrations of that difference.

- Billy has ten toes. The probability Billy has nine toes is zero. The membership of Billy in the set of people with nine toes, however, is nonzero. A value of 0.7 might be appropriate.

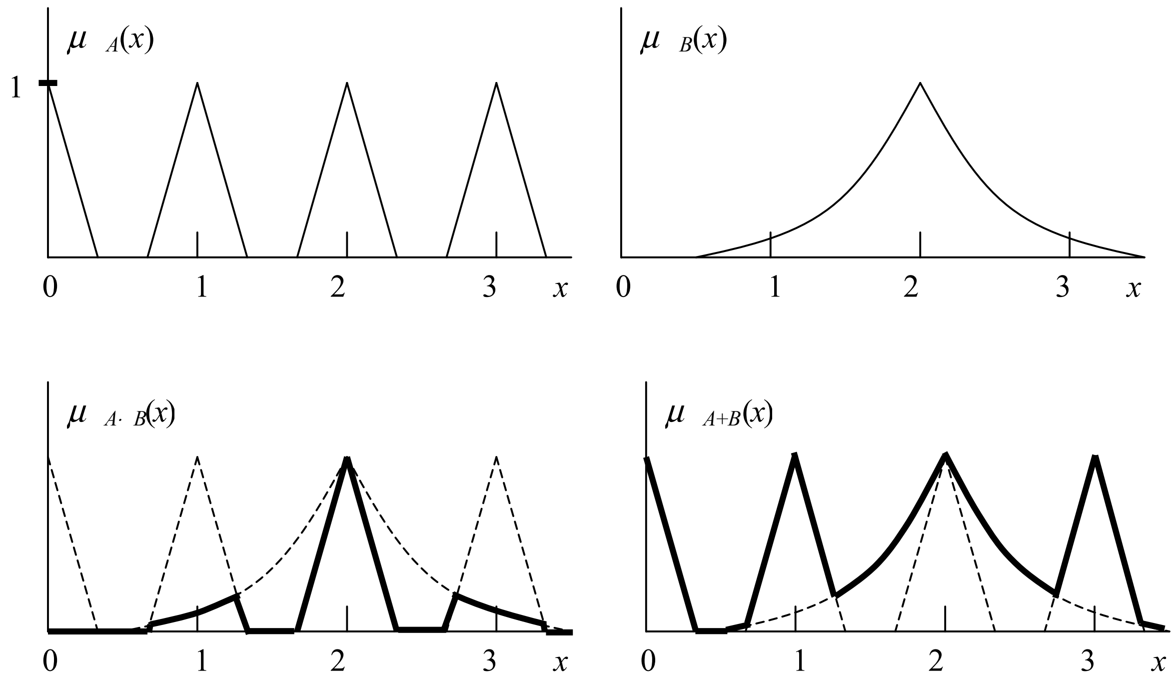

Figure 1. Illustration of the operations of fuzzy intersection and union on two fuzzy membership functions. The fuzzy membership function for the fuzzy set *A*, shown at the upper left, is for numbers that are *close* to integers. The fuzzy membership for the set B, numbers *near* 2, is shown in the upper right. The membership function for the intersection (logical AND) of the two sets, denoted by $A \cdot B$, is the minimum of the two component membership functions. This is illustrated in the bottom left figure with the bold plot. The fuzzy OR of the fuzzy sets, denoted $A+B$, has a membership function equal to the maximum of the two membership functions. The result, shown in the bottom right, corresponds to the membership function of numbers either *close* to an integer OR *near* the number 2.

- A bottle of liquid has a probability of ½ of being rat poison and ½ of being pure water. A second bottle's contents, in the fuzzy set of liquids containing *lots* of rat poison, is ½. The meaning of ½ for the two bottles clearly differs significantly and would impact your choice should you be dieing of thirst.

- Probabilities can be either crisp of fuzzy. The probability that a fair die will show six is one sixth. This is a crisp probability. All credible mathematicians will agree on this exact number. The probability that the result of a die throw will be *near* six, on the other hand, is a fuzzy probability. The weatherman's forecast of a probability of rain tomorrow being 70% is also a fuzzy probability. Using the same meteorological data, another weatherman will typically announce a different probability.

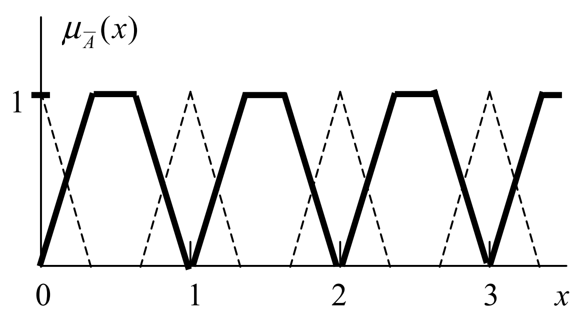

Figure 2. Illustration of the fuzzy complement, or *NOT*, operation. The fuzzy set, *A*, from Figure 1, consists of numbers close to integers. The membership function is shown here with broken lines. The complement is the set of numbers *NOT* close to integers. The membership function for this set is show above by a bold line. The sum of the membership functions of a set and its complement add to one.

Fuzzy Logic

Elementary conventional crisp set operations include complementing, intersection and union. The same operations can be performed on fuzzy sets. For the intersection or *AND* operation, the fuzzy operation commonly used is minimum. For the union or *OR* operation, the maximum is used. Some examples are appropriate.

IF
an undergraduate's GPA is *high* AND their GRE score is *high*,
THEN
an undergraduate student will make an *excellent* graduate student,

OR, IF
their GPA is *high* AND their GRE score is *fair*,
OR
their GPA is *fair* AND their GRE score is *high*.
THEN
an undergraduate student will make a *good* graduate student,

OR, IF
their GPA is *fair* AND their GRE score is *fair*,
THEN
an undergraduate student will make an *average* graduate student,

OR, OTHERWISE,
the undergraduate student will make a *poor* graduate student.

TABLE 1: AN EXAMPLE OF RULES USING FUZZY LINGUISTIC VARIABLES.

In Figure 1, two fuzzy membership functions are shown. The set *A* consists of those numbers *close* to integers. The set is *B* is the set of numbers *near* to two. Using "+" as the fuzzy union operation or, equivalently, a fuzzy *OR*, we form the membership function for *A*+*B* as

$$\mu_{A+B}(x) = \max\,[\mu_{\mathrm{A}}(x), \mu_B(x)]$$

where "max" denotes the maximum operation. Similarly, for the fuzzy *AND*, using "·" as the intersection operation,

$$\mu_{A\cdot B}(x) = \min\,[\mu_{\mathrm{A}}(x), \mu_B(x)]$$

where "min" denotes performing the minimum operation. An example of performing fuzzy intersection and union is illustrated in Figure 1.
The complement of a set has a membership function

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

This is illustrated in Figure 2.

IF	(GPA is *H* AND GRE is *H*),	THEN *E* ,
OR IF	[(GPA is *H* AND GRE is *F*), OR (GPA is *F* AND GRE is *H*)],	THEN *G*
OR, IF	(GPA is *F* AND GRE is *F*),	THEN *A*
OR, IF	(GPA is *P* OR GRE is *P*),	THEN *P*

TABLE 2: A MORE COMPACT STATEMENT OF THE RULES IN TABLE 1.

IF	(½ AND ¾),	THEN *E* ,
OR IF	[(½ AND ¼), OR (½ AND ¾)],	THEN *G*,
OR, IF	(½ AND ¼),	THEN *A*,
OR, IF	(0 OR 0),	THEN *P* .

TABLE 3: NUMERICAL INTERPRETATION OF THE ANTECEDENTS IN TABLE 2.

IF	min(½ ,¾) = ½,	THEN *E* ,
OR	IF [max{min(½,¼), min(½,¾)}] = ½,	THEN *G*,
OR,	IF min(½,¼) = ¼,	THEN *A*,
OR,	IF max(0,0) = 0,	THEN *P* .

TABLE 4: EVALUATION OF TABLE 3.

The intersection and union operations can also be used to assign memberships on the Cartesian product of two sets. Consider the fuzzy membership of a set, G, of liquids that taste *good*.

$$\mu_G = 0.0/\text{Swamp Water} + 0.5/\text{Radish Juice} + 0.9/\text{Grape Juice}$$

Using the set, *LA*, of cities close to Los Angeles in Equation 1, we form the set

$$E = G \cdot LA$$
= liquids that taste *good AND* cities that are *close* to LA

The membership function for *E,* given in the table on the top of the next page, is generated by performing a minimum operation on elements of *G* and *LA*.

The set *G*+*LA* can similarly be defined by taking the maximum in this table.

The operations of fuzzy intersection, union and complement have many properties of their crisp counterparts. Intersection, for example, is distributive over the union operation and the union operation is

	Los Angeles (0.0)	Chicago (0.5)	New York (0.8)	London(0.9)
Swamp Water (0.0)	0.00	0.00	0.00	0.00
Radish Juice (0.5)	0.00	0.25	0.40	0.45
Grape Juice (0.9)	0.00	0.45	0.72	0.81

distributive over intersection. DeMorgan's laws are applicable. The fuzzy logic min and max operations, however, do not obey *the law of excluded middle*. Specifically, since

$$\min(\mu_A, 1-\mu_A) \neq 0,$$

it follows that

$$A \cdot \overline{A} \neq \phi$$

and that fuzzy intersection using min does not obey *the law of contradiction*. Similarly, since

$$\max(\mu_A, 1-\mu_A) \neq 1$$

it follows that

$$A + \overline{A} \neq \mathrm{U} .$$

Therefore, fuzzy intersection using the maximum operator does not obey the *law of excluded middle.*[1]

Fuzzy If-Then Rules

Cause and effect statements of a process are stated through if-then rules. Consider the pedagogical example in Table 1 wherein the success of an undergraduate as a graduate student is inferred through consideration of their undergraduate grade point averages (GPA's) and their performance on the GRE analytic test.

Note, first, the operations of IF, THEN, AND and OR. Each can be interpreted in a fuzzy sense. The fuzzy linguistic variables are written in *italics.* These rules can be simplified using the following linguistic variable abbreviations.

A for *average,*
E for *excellent,*
F for *fair,*
G for *good,*
H for *high,*
L for *low,*
P for *poor.*

The *If-Then* rules can then be written as shown in Table 2. The IF portions of these statements are referred to as *antecedents*. The THEN portions are the *consequents*.

Numerical Interpretation of the Fuzzy Antecedent

The first step in building the fuzzy inference engine is quantification of the linguistic variables. For the GPA and GRE's, the fuzzy membership functions will be as is shown in Figure 3.

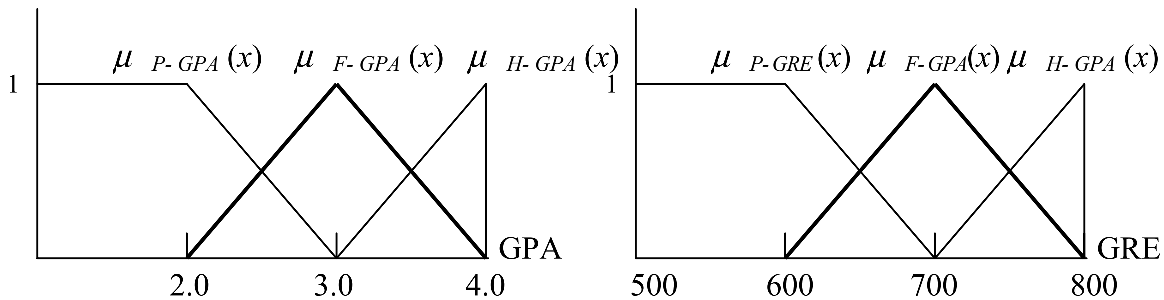

Figure 3. Fuzzy membership functions for *poor, fair* and *high* scores in the GPA and GRE.

To illustrate, suppose Student A has a GPA of 3.5 and a GRE of 725. This crisp numbers are now fuzzified using the membership functions in Figure 3. For the GPA of 3.5, the values from the membership functions on the left plot in Figure 3 are

$$\mu_{P\text{-}GPA}(3.5) = 0, \quad \mu_{F\text{-}GPA}(3.5) = \tfrac{1}{2}, \quad \mu_{H\text{-}GPA}(3.5) = \tfrac{1}{2} \qquad (2)$$

Similarly, for the GRE of 733, from the right hand plot in Figure 3,

$$\mu_{P\text{-}GRE}(725) = 0, \quad \mu_{F\text{-}GRE}(725) = \tfrac{1}{4}, \quad \mu_{H\text{-}GRE}(725) = \tfrac{3}{4} \qquad (3)$$

As illustrated in Table 3, these fuzzy values can then be used in lieu of the fuzzy antecedent statements in the logic statement. Recall the logical AND as a minimum operation and an OR as a maximum. Applying gives the results shown in Table 4. Each of the consequent classes is now assigned a value or weight in accordance to the fuzzy logic operations. *Excellent* is assigned a value of ½, *good* a value of ½, *average* a value of ¼ and *poor* a value of zero. Each consequent is now assigned a value. It remains to

[1] There are other operations used for fuzzy intersection and union other than min and max. Although some of these operations obey the laws of contradiction and excluded middle, the result is invariably a sacrifice of other properties.

combine, or defuzzify, these values into a single crisp number representing the consequent. To do so, fuzzy membership functions must be assigned to each fuzzy linguistic variable defining the consequent. One possibility is shown in Figure 4. The type of graduate student will be measured on a scale of zero to ten with zero being the worst and 10 the most excellent. This range can be assigned fuzzy membership functions as shown in Figure 4. From the analysis, the poor membership weight is assigned a value of 0, the average a value of ¼, and the good and excellent weights are both ½. One popularly used defuzzification method simply multiplies each membership function by its weight and takes the center of mass of the resulting functional sum as the final crisp value. The weighting of the membership functions is shown in Figure 5. The sum of the functions is shown in Figure 6. The center of mass of the weighted membership functions, in general, is given by

$$d = \frac{\int_{-\infty}^{\infty} \sum_n \alpha_n x \mu_n(x) dx}{\int_{-\infty}^{\infty} \sum_n \alpha_n \mu_n(x) dx}$$

where the summation is over all of the consequent membership functions, $\{\mu_n(x)\}$, and their corresponding weights, $\{\alpha_n\}$. If the area of the n^{th} membership function is

$$A_n = \int_{-\infty}^{\infty} \mu_n(x) dx$$

and center of mass

$$c_n = \frac{\int_{-\infty}^{\infty} x \mu_n(x) dx}{A_n}$$

If	GPA is H	And	GRE is H,	Then E,
Or, If	GPA is H	And	GRE is F,	Then G,
Or, If	GPA is H	And	GRE is P,	Then P,
Or, If	GPA is F	And	GRE is H,	Then G,
Or, If	GPA is F	And	GRE is F,	Then A,
Or, If	GPA is F	And	GRE is P,	Then P,
Or, If	GPA is P	And	GRE is H,	Then P,
Or, If	GPA is P	And	GRE is F,	Then P,
Or, If	GPA is P	And	GRE is P,	Then P.

TABLE 5 EXHAUSTIVE LISTING OF THE FUZZY RULES IN TABLE 1.

Then the defuzzification can be written as

$$d = \frac{\sum_n \alpha_n c_n A_n}{\sum_n \alpha_n A_n}$$

For the graduate student example, as illustrated in Figure 6, the defuzzification is at $d = 7.3$.

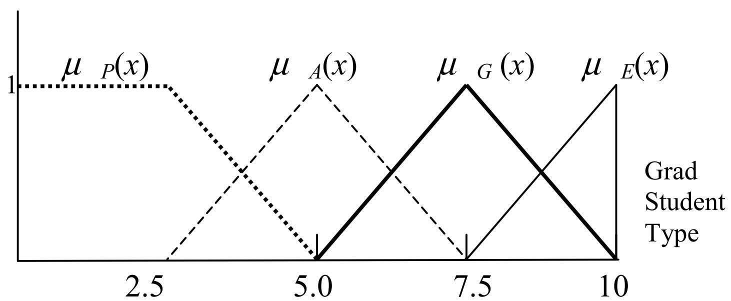

Figure 4. Fuzzy membership functions for *poor, average, good* and *excellent* rated graduate students used as the consequent in the running example.

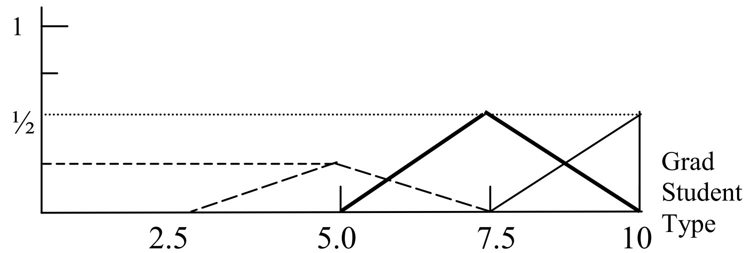

Figure 5. The results of multiplying each of the fuzzy membership functions in Figure 4 by their respective weights.

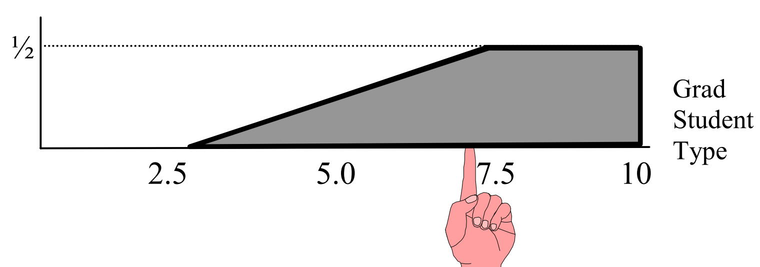

Figure 6. The gray area is that bounded by the sum of the weighted membership functions in Figure 5. The center of mass of this object, marked by the balance point at $d = 7.3$, is the value for the defuzzification.

Matrix Descriptions of Fuzzy Rules

Fuzzy *If-Then* rules can often be conveniently expressed in matrix form. The fuzzy *If-Then* rules of the running example can be concisely written as is shown in Table 5. When thus tabulated, the rules can be expressed nicely in a rule matrix as is shown in Table 6. From this table, we read, for example, "If the GPA is *fair* AND the GRE is *fair*, then the graduate student will be *average*." The matrix structure is convenient for visualization of the numerical results of fuzzification. The numerical results from Equations (1) and (2) for the GPA and GRE of Student A are imposed on the rule matrix in Table 6. Since each rule is linked with a fuzzy AND, the matrix entries are equal to the minimum of the column and row values. This is also illustrated in Table 7.

Grad Student Type		GPA		
		P	F	H
GRE	P	P	P	P
	F	P	A	G
	H	P	G	E

TABLE 6 MATRIX OF FUZZY RULES.

To defuzzify from these table entries, we are reminded from Table 5 that each of the elements of the matrix are linked by fuzzy OR's. Thus, the student is *good* if [(GPA is *high* AND GRE is *fair*) ***OR*** (GPA is *fair* and GRE is *high*)]. The two entries in Table 7 can thus be viewed as being linked by a fuzzy OR or maximum operation. Therefore, the consequent of good is assigned a value of

Grad Student Type		GPA		
		P→0	F→½	H→½
GRE	P→0	P→0	P→0	P→0
	F→¼	P→0	A→¼	G→¼
	H→¾	P→0	G→½	E→½

TABLE 7 MATRIX OF FUZZY RULES WITH EXAMPLE ENTRIES.

good ← maximum(½,¼)= ½

Similarly,

average ← ¼
excellent ← ½
poor ← maximum(0,0,0,0,0) = 0

These consequent weights will defuzzify using the memberships functions in Figure 4, as before, as 7.3.

Application to Control

Fuzzy inference engines can be used effectively in many control problems. For feedback tracking control, a commonly used antecedent the error between the current and desired state. The change in error is also used. To illustrate, consider the simple cruise control illustrated in Figure 7. A desired speed is set and the car's accelerator is to be set to achieve this speed. The car's speed is measured. The difference between the true and desired speed is the error. The error is delayed a short period of time and subtracted from the current error to assess the change in error. Using the speed error, E, and the change in error, ΔE, the rules for the fuzzy cruise control might be as shown in Table 8. The following abbreviations are used.

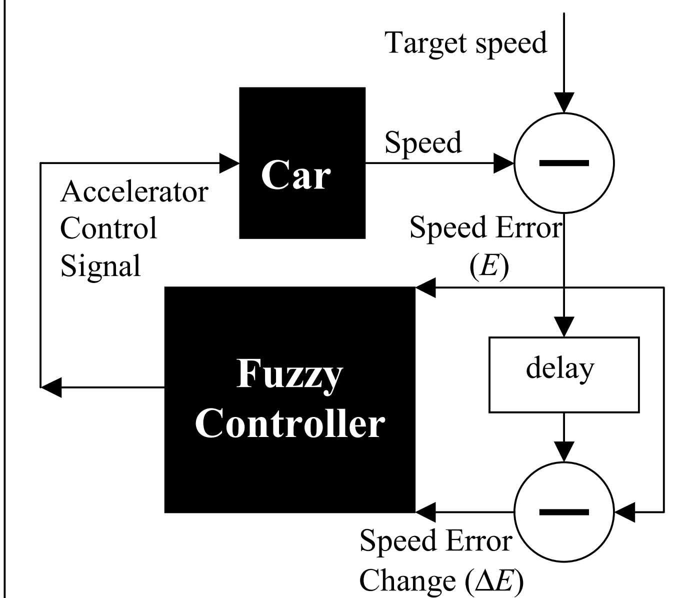

Figure 7. Illustration of a simple fuzzy cruise control.

From Table 8, we read, for example, "If the error is *large positive* and the change in error is *negative*, then the acceleration should be made *small negative*." To complete the fuzzy controller, fuzzy membership functions must be defined – five each for the two antecedents and nine for the consequent. Implementation of the fuzzy controller is then carried out in the same way as in the graduate student quality assessment example. The error and change of error are fuzzified, the weights of the consequent membership functions are computed using fuzzy logic, and the final control action is determined through the process of defuzzification.

Acceleration Change		E				
		LN	N	Z	P	LP
ΔE	LP	Z	SN	N	LN	VLN
	P	SP	Z	SN	N	LN
	Z	P	SP	Z	SN	N
	N	LP	P	SP	Z	SN
	LN	VLP	LP	P	SP	Z

TABLE 8. A RULE MATRIX FOR THE FUZZY CRUISE CONTROL.

As is the case with any design, changes in the controller are typically made as its performance is monitored. Tuning of fuzzy membership functions and/of rule table entries are typical. This can be achieved with a "man in the loop" or by placing the fuzzy control process in that automatically tunes the system parameters for optimal performance. There exists numerous variations of fuzzy logic. Remarkably, a fine tuned fuzzy inference engine is relatively robust to the choice of its underlying fuzzy logic.

Fuzzy control has a great advantage in its simplicity. No knowledge of Laplace transforms or migrating poles in the s plane is required. Classical control is a richly developed field. For linear plants, properly tuned conventional control will typically perform superiorly. Fuzzy control becomes more valuable when applied to nonlinear time-variant plants.

Variations

There exist numerous variations on the fundamental fuzzy inference engine thus far described. Some of them are in need of address.

Alternate Fuzzy Logic

The minimum and maximum operations are one of many different operations that can be used to perform logical ANDS and ORS. There also exist alternate methods to generate the fuzzy complement. One popular alternate is *sum-product inferencing*, wherein the AND operation is performed using a multiplication of membership functions (rather than a minimum) and the OR operation by an add (rather than a maximum). It the sum exceeds one, the value of the union is typically set to one.

Defuzzification

Defuzzification was performed in Figure 6 as the center of mass of the weighted membership functions. More generally, any measure of central tendency of the function can suffice. The median, for example, could have been used. (The defuzzification, in this case, would be at 7.5). Also, the manner in which the consequent membership functions are weighted can differ. The membership functions in Figure 4, for example, were multiplicatively weighted to give the curves shown in Figure 5. The curves in Figure 5 are added and the center of mass used to find the defuzzified output. A commonly used alternate method *clips* the consequence membership functions at their weight value. Recall that the weights for the membership functions for Figure 4 were (P,A,G,E) = (0, ¼, ½, ½). Using these as clip values, the result, when applied to the fuzzy membership functions in

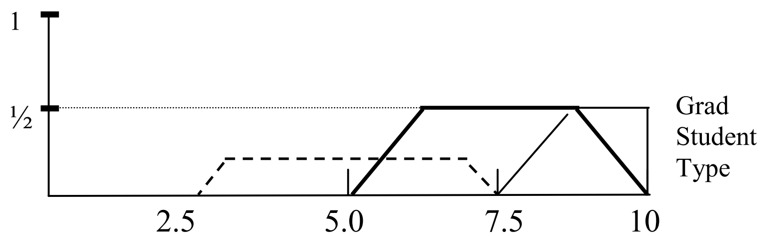

Figure 8. An alternate method of applying the weights of the fuzzy consequents is by clipping the consequent membership functions. Here, the membership functions in Figure 4 are clipped in accordance to the running example.

Figure 4, is shown in Figure 8. The sum of these curves is shown in Figure 9. The center of mass of this curve, or some other measure of the curve's central tendency, is the resultant value for defuzzification.

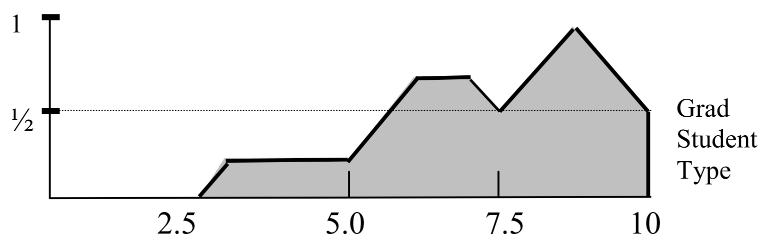

Figure 9. Addition of the clipped membership functions in Figure 8.

Weighting Consequents

In Table 7, a given consequent component's weight was generated be taking the maximum of all the entries for the fuzzy set. The final weight for G = good, for example, was max(½,¼) =½. In practice, a number of alternate combinations or *aggregations* of these weights can be used to determine the composite weight other than the maximum operation.

Alternate Inferencing Methods

There exist alternate methods of fuzzy inferencing. Data base inferencing establishes the fuzzy rule table from a data base rather than based on fuzzy linguistic variables. A powerful generalization proposed by Sugeno replaces the consequents with algebraic expressions.

Further Reading

There are numerous excellent texts of fuzzy systems and fuzzy control. The offerings by George Klir are especially clear and precise. Two books of collections of papers are those edited by Bezdek & Pal and Marks. The papers in Bezdek's anthology focus on pattern recognition. Marks' collection contains papers on applications of fuzzy logic. Papers are included in the application of fuzzy systems to numerous fields. The original 1965 paper by Lotfi Zadeh, included in Bezdek's book, is both remarkably readable and currently relevant. It still serves as a superb introduction to fuzzy sets. The paper by Arabshahi *et al.* presents one of many ways which a fuzzy inference engine can be tuned. Matlab's fuzzy system toolbox is an excellent software tool for fuzzy system simulation.

- P. Arabshahi, R.J. Marks II, S. Oh, T.P. Caudell , J.J. Choi, and B.G. Song, "Pointer Adaptation and Pruning of Min-Max Fuzzy Estimation", **IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing**, vol.44, no.9, September 1997, pp.696-709
- J.C. Bezdek and S.K. Pal, ***Fuzzy Models for Pattern Recognitio***n, (IEEE Press, New York,1992).
- J.C. Bezdek, "Editorial: Fuzzy models - what are they, and why?", **IEEE Transactions on Fuzzy System**s, vol.1, no.1, pp. 1-6 (1993).
- M.E. El-Hawary, editor, **Fuzzy System Theory in Electrical Power Engineerin**g, (IEEE Press,1998), pp.85-111.
- George J. Klir, Bo Yuan, **Fuzzy Sets and Fuzzy Logic: Theory and Applications** (Prentice Hall, 1995).
- R.J. Marks II, Editor, **Fuzzy Logic Technology and Application**s, (IEEE Technical Activities Board, Piscataway, 1994).