Example 2.1.

Given $u(a, b) = \max(a, b)$ and $c(a) = (1 - a^2)^{1/2}$, determine i such that DeMorgan's laws are satisfied. Employing Eq. (2.15), we obtain

$$i(a, b) = (1 - u^2[(1 - a^2)^{1/2}, (1 - b^2)^{1/2}])^{1/2}$$
$$= (1 - \max^2[(1 - a^2)^{1/2}, (1 - b^2)^{1/2}])^{1/2}.$$

Solving Eq. (2.13) for c is more difficult and may result in more than one solution. For example, if the standard max and min operations are employed for u and i, respectively, then every involutive complement satisfies the equation. Hence, max, min, and any of the Sugeno complements (or Yager complements) defined in Sec. 2.2 satisfy DeMorgan's laws.

For the sake of simplicity, we have omitted an examination of the properties of one operation that is important in fuzzy logic—fuzzy implication, $\Rightarrow$. This operation can be expressed in terms of fuzzy disjunction, $\vee$, fuzzy conjunction, $\wedge$, and negation, $^{-}$, by using the equivalences

$$a \Rightarrow b = \bar{a} \vee b \quad \text{or} \quad a \Rightarrow b = \overline{a \wedge \bar{b}}.$$

By employing the correspondences between logic operations and set operations defined in Table 1.5, the equivalences just given can be fully studied in terms of the functions

$$u(c(a), b) \quad \text{or} \quad c(i(a, c(b))).$$

Different fuzzy implications are obtained when different fuzzy complements c and either different fuzzy unions u or different fuzzy intersections i are used.

2.6 GENERAL AGGREGATION OPERATIONS

Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined to produce a single set. In general, any *aggregation operation* is defined by a function

$$h : [0, 1]^n \rightarrow [0, 1]$$

for some $n \geq 2$. When applied to n fuzzy sets $A_1, A_2, \ldots, A_n$ defined on X, h produces an aggregate fuzzy set A by operating on the membership grades of each $x \in X$ in the aggregated sets. Thus,

$$\mu_A(x) = h(\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x))$$

for each $x \in X$.

In order to qualify as an aggregation function, h must satisfy at least the following two axiomatic requirements, which express the essence of the notion of aggregation:

Axiom h1. $h(0, 0, \ldots, 0) = 0$ and $h(1, 1, \ldots, 1) = 1$ (*boundary conditions*).

Axiom h2. For any pair $(a_i \mid i \in \mathbb{N}_n)$ and $(b_i \mid i \in \mathbb{N}_n)$, where $a_i \in [0, 1]$ and $b_i \in [0, 1]$, if $a_i \geq b_i$ for all $i \in \mathbb{N}_n$, then $h(a_i \mid i \in \mathbb{N}_n) \geq h(b_i \mid i \in \mathbb{N}_n)$, that is, h is *monotonic nondecreasing* in all its arguments.

Two additional axioms are usually employed to characterize aggregation operations despite the fact that they are not essential:

Axiom h3. h is a *continuous* function.

Axiom h4. h is a *symmetric* function in all its arguments, that is,

$$h(a_i \mid i \in \mathbb{N}_n) = h(a_{p(i)} \mid i \in \mathbb{N}_n)$$

for any permutation p on $\mathbb{N}_n$.

Axiom h3 guarantees that an infinitesimal variation in any argument of h does not produce a noticeable change in the aggregate. Axiom h4 reflects the usual assumption that the aggregated sets are equally important. If this assumption is not warranted in some application context, the symmetry axiom must be dropped.

We can easily see that fuzzy unions and intersections qualify as aggregation operations on fuzzy sets. Although they are defined for only two arguments, their property of associativity guaranteed by Axioms u4 and i4 provides a mechanism for extending their definition to any number of arguments. Hence, fuzzy unions and intersections can be viewed as special aggregation operations that are symmetric, usually continuous, and required to satisfy some additional boundary conditions. As a result of these additional requirements, fuzzy unions and intersections can produce only aggregates that are subject to restrictions (2.8) and (2.9). In particular, they do not produce any aggregates of $a_1, a_2, \ldots, a_n$ that produce values between $\min(a_1, a_2, \ldots, a_n)$ and $\max(a_1, a_2, \ldots, a_n)$. Aggregates that are not restricted in this way are, however, allowed by Axioms h1 through h4; operations that produce them are usually called *averaging operations*.

Averaging operations are therefore aggregation operations for which

$$\min(a_1, a_2, \ldots, a_n) \leq h(a_1, a_2, \ldots, a_n) \leq \max(a_1, a_2, \ldots, a_n). \quad (2.16)$$

In other words, the standard max and min operations represent boundaries between the averaging operations and the fuzzy unions and intersections, respectively.

One class of averaging operations that covers the entire interval between the min and max operations consists of *generalized means*. These are defined by the formula

$$h_\alpha(a_1, a_2, \ldots, a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha}, \quad (2.17)$$

where $\alpha \in \mathbb{R}$ ($\alpha \neq 0$) is a parameter by which different means are distinguished.

Function h_α clearly satisfies Axioms h1 through h4 and, consequently, it represents a parameterized class of continuous and symmetric ...

erations. It also satisfies the inequalities (2.16) for all $\alpha \in \mathbb{R}$, with its lower bound

$$h_{-\infty}(a_1, a_2, \ldots, a_n) = \min(a_1, a_2, \ldots, a_n)$$

and its upper bound

$$h_{\infty}(a_1, a_2, \ldots, a_n) = \max(a_1, a_2, \ldots, a_n).$$

For fixed arguments, function h_α is monotonic increasing with α. For $\alpha \to 0$, the function h_α becomes the *geometric mean*

$$h_0(a_1, a_2, \ldots, a_n) = (a_1 \cdot a_2 \cdots a_n)^{1/n};$$

furthermore,

$$h_{-1}(a_1, a_2, \ldots, a_n) = \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}$$

is the *harmonic mean* and

$$h_1(a_1, a_2, \ldots, a_n) = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

is the *arithmetic mean*.

Since it is not obvious that h_α represents the geometric mean for $\alpha \to 0$, we use the following theorem.

Theorem 2.15. Let h_α be given by Eq. (2.17). Then,

$$\lim_{\alpha \to 0} h_\alpha = (a_1 \cdot a_2 \cdots a_n)^{1/n}.$$

Proof: First, we determine

$$\lim_{\alpha \to 0} \ln h_\alpha = \lim_{\alpha \to 0} \frac{\ln(a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha) - \ln n}{\alpha}.$$

Using l'Hospital's rule, we now have

$$\lim_{\alpha \to 0} \ln h_\alpha = \lim_{\alpha \to 0} \frac{a_1^\alpha \ln a_1 + a_2^\alpha \ln a_2 + \cdots + a_n^\alpha \ln a_n}{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}$$

$$= \frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n} = \ln(a_1 \cdot a_2 \cdots a_n)^{1/n}.$$

Hence,

$$\lim_{\alpha \to 0} h_\alpha = (a_1 \cdot a_2 \cdots a_n)^{1/n}. \quad \blacksquare$$

When it is desirable to accommodate variations in the importance of individual aggregated sets, the function h_α can be generalized into *weighted generalized*

means, as defined by the formula

$$h_\alpha(a_1, a_2, \ldots, a_n; w_1, w_2, \ldots, w_n) = \left(\sum_{i=1}^{n} w_i a_i^\alpha\right)^{1/\alpha}, \tag{2.18}$$

where $w_i \geq 0$ ($i \in \mathbb{N}_n$) are weights that express the relative importance of the aggregated sets; it is required that

$$\sum_{i=1}^{n} w_i = 1.$$

The weighted means are obviously not symmetric. For fixed arguments and weights, the function h_α given by Eq. (2.18) is monotonic increasing with α.

The full scope of fuzzy aggregation operations is summarized in Fig. 2.5. Included in this diagram are only the generalized means, which cover the entire range of averaging operators, and those parameterized classes of fuzzy unions and intersections given in Table 2.2 that cover the full ranges specified by the inequalities (2.8) and (2.9). For each class of operators, the range of the respective parameter is indicated. Given one of these families of operations, the identification of a suitable operation for a specific application is equivalent to the estimation of the parameter involved.

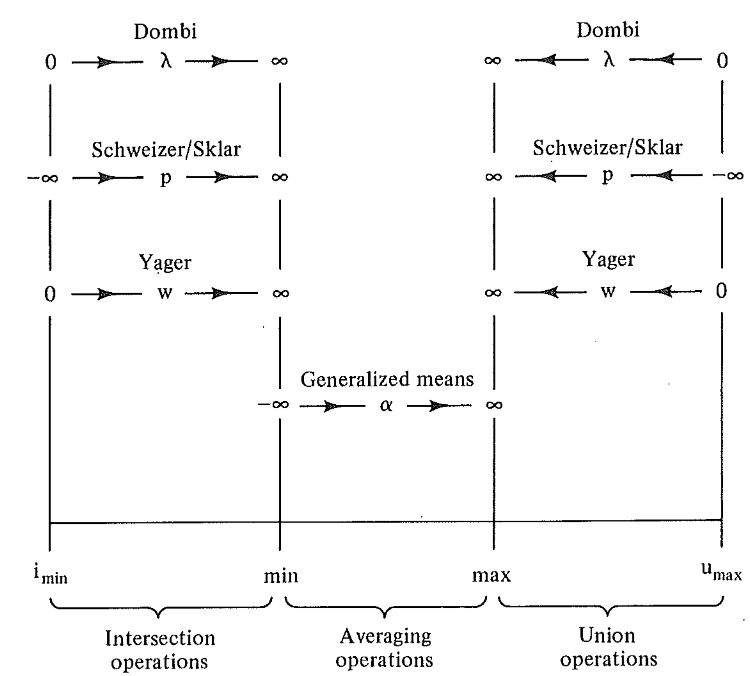

Figure 2.5. The full scope of fuzzy aggregation operations.

FUZZY SETS, UNCERTAINTY, AND INFORMATION

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