R.J. Marks II, G.L. Wise and D.G. Haldeman, "Further results on detection in Laplace noise", Proceedings of the 1977 Midwest Symposium on Circuits and Systems, Texas Tech University, Lubbock, August 1977.

FURTHER RESULTS ON DETECTION IN LAPLACE NOISE

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Abstract

The discrete time detection of a known constant signal in additive white stationary Laplace noise is considered. The receiver operating characteristics of the Neyman-Pearson optimal detector are presented and compared with those of the linear detector. Also, some results obtained using a Gaussian approximation to the distribution of the test statistic are presented.

1. INTRODUCTION

The Gaussian distribution is a popular model for noise statistics, and it is valid in a wide variety of situations. However, in some cases, such as impulsive noise, this model may not be satisfactory. Impulsive noise is typically characterized as noise whose distribution has an associated "heavy tail" behavior. That is, the probability density function (pdf) approaches zero more slowly than a Gaussian pdf. The references in [1] and [2] give a summary of some forms of impulsive noise and situations where it arises.

Certain forms of impulsive noise may be characterized by a Laplace distribution. That is, the pdf of the noise is given by

$$f(n) = \frac{\gamma}{2} e^{-\gamma|n|} \quad . \qquad (1)$$

Notice that this model has the "heavy tail" behavior associated with impulsive noise. The Laplace distribution is popular in statistics and many of its properties have been studied [3]. Furthermore, it is used as a noise model in engineering studies. For example, Miller and Thomas [1] used Laplace noise in a numerical study of relative efficiency. Bernstein, et.al. [4] comment on the non-Gaussian nature of ELF atmospheric noise, and they give a plot of a typical experimentally determined pdf associated with such noise [4, Figure 10]. This experimentally determined pdf is similar to a Laplace pdf, and on a linear graph the difference is barely distinguishable. Also, the limiting case of the Mertz model [5] for the amplitude distribution of impulsive noise is identical to the distribution of the amplitude of Laplace noise [6]. Kanefsky and Thomas [7] considered a class of generalized Gaussian noises, obtained by generalizing the Gaussian density to obtain a variable rate of exponential decay. The Laplace distribution is within this class of generalized Gaussian distributions. Also, Duttweiler and Messerschmitt [8] refer to the Laplace distribution as a model for the distribution of speech.

In this paper we are concerned with the detection of a signal in Laplace noise. Recently, we investigated the distribution function of the test statistic which arises in this problem, and we obtained a convenient expression for it [6]. Here we use these results to investigate various aspects of the detection problem. In the following sections

Presented at the Twentieth Midwest Symposium on Circuits and Systems, August 15-17, 1977; to be published in the Proceedings of the Symposium.

we briefly state the problem and summarize some previous results. For large sample sizes, the Central Limit Theorem is often used to obtain an approximation to the distribution of the test statistic. We compare the results obtained using the actual distribution and using the approximating Gaussian distribution. Then we consider the receiver operating characteristics of the Neyman-Pearson optimal detector. The popular but suboptimal linear detector is also considered and is compared to the optimal detector.

2. PRELIMINARIES

We consider testing for the presence or absence of a positive, constant signal s in additive, stationary, white Laplace noise. The problem is modeled as the following hypothesis testing problem:

$$\begin{aligned} H_0: &\quad x_i = n_i & i = 1,2,\ldots,N \\ H_1: &\quad x_i = s + n_i & s > 0 \end{aligned}$$

Based on N independent identically distributed observations, we are to decide to announce that the signal is absent or present.

For this problem both the Neyman-Pearson optimal detector and the linear detector have the following structure. The N observations are fed into the zero memory nonlinearity, $g(\cdot)$, whereupon they are summed to give the test statistic

$$t = \sum_{i=1}^{N} g(x_i).$$

The test statistic is then compared to a threshold T from which a decision concerning the presence or absence of the signal is made. The choice of the threshold T determines the false alarm probability α and the detection probability β. However, to determine this interrelationship, it is necessary to know the probability distribution of the test statistic t. This, in turn, depends on the nonlinearity $g(\cdot)$. We will now consider two choices for the nonlinearity and will investigate the corresponding detector performance under various conditions.

3. THE NEYMAN-PEARSON OPTIMAL DETECTOR

By a Neyman-Pearson optimal detector we mean a detector which, for a fixed α, maximizes β. The nonlinearity for the optimal detector in our problem is well known [9] and is the amplifier-limiter given by the following expression:

$$g(x) = \begin{cases} \gamma s, & x > s \\ 2\gamma x - \gamma s, & 0 \leq x \leq s \\ -\gamma s, & x < 0 \end{cases} \qquad (2)$$

Consider first the case where H_0 is true; that is, no signal is present. If X_i has the Laplace distribution given in (1), then $g(X_i)$ can easily be shown to have the following distribution function:

$$F_1^{(0)}(x) = \frac{1}{2}u(x+\gamma s) + \frac{1}{4}\int_{\infty}^{x} \exp[-\frac{1}{2}(\gamma x+v)]G(\frac{v}{2\gamma s})dv + \frac{1}{2}e^{-\gamma s}u(x-\gamma x),$$

where $u(x)$ denotes the unit step function which is one for nonnegative x and zero otherwise, and $G(x)$ denotes the gate function which is one for $|x| \leq 1/2$ and zero otherwise. Let $F_N^{(0)}(\cdot)$ denote the distribution function of t for the optimal detector under H_o. Then

$$F_N^{(0)}(x) = \int_{-\infty}^{\infty} F_{N-1}^{(0)}(x-v)\, dF_1^{(0)}(v).$$

By a rather lengthy but straightforward procedure [6], it can be shown that*

$$\begin{aligned} F_N^{(0)}(x) = {} & 2^{-N} \sum_{k=1}^{N} \binom{N}{k} \sum_{p=0}^{k} (-1)^p \binom{k}{p} \sum_{q=0}^{N-k} \binom{N-k}{q} \\ & \cdot \Big[\exp[-(p+q)\gamma s] - \exp[-\frac{1}{2}(x+N\gamma s)] \\ & \cdot e_{k-1}\Big\{\frac{1}{2}[x+(N-2p-2q)\gamma s]\Big\}\Big] \cdot u[x+(N-2p-2q)\gamma s] \\ & + 2^{-N} \sum_{m=0}^{N} \binom{N}{m} \exp(-m\gamma s) \cdot u[x+(N-2m)\gamma s], \end{aligned} \qquad (3)$$

where $e_k(\cdot)$, the incomplete exponential, is defined as

$$e_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!}.$$

We now consider the signal present case, i.e. H_1. We let $F_N^{(1)}(x)$ denote the distribution function of the test statistic under H_1. Since the Laplace pdf is symmetric, it can be shown [1] that

$$F_N^{(1)}(x) = 1 - F_N^{(0)}(-x). \qquad (4)$$

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*A typographical error (omission of a pair of brackets) in [6] has been corrected here.

Eqs. (3) and (4) thus completely determine the performance of the Neyman-Pearson optimal detector.

4. THE LINEAR DETECTOR

By a linear detector, we mean a scheme such as that described previously, but where the function $g(\cdot)$ is $g(x)=x$. That is, the test statistic is simply the sum of the observations. The linear detector is Neyman-Pearson optimal for Gaussian noise and is a commonly used detector.

Consider the signal absent case, i.e. H_0. In this situation, the test statistic is given by

$$t = \sum_{i=1}^{N} X_i ,$$

where, again, the X_i are independent identically distributed random variables with the pdf of Equation (1). Thus, the pdf of t, $p_N(\cdot)$, is obtained from N-1 convolutions of the Laplace pdf with itself. This is given by [3, p. 24] as

$$p_N(x)=\frac{\gamma e^{-\gamma|x|}}{(N-1)!}\sum_{k=0}^{N-1} 2^{-(N+k)} \frac{(N+k-1)!}{k!(N-k-1)!}(\gamma|x|)^{N-k-1}.$$

After a straightforward integration [10], we obtain $G_N^{(0)}(x)$, the distribution function of the test statistic of the linear detector under H_0,

$$G_N^{(0)}(x) = \begin{cases} \frac{1}{2}+\sum_{k=0}^{N-1} 2^{-(N+k)} \binom{N+k-1}{k} \left[1-e^{-\gamma x} \cdot e_{N-k-1}(\gamma x)\right], & x \geq 0 \\ 1 - G_N^{(0)}(-x), & x < 0 \end{cases} \quad (5)$$

In the signal present case, the test statistic is given by

$$t = \sum_{i=1}^{N} X_i + Ns ,$$

where, once again, the X_i are independent identically distributed random variables with the density function of Eq.(1). Let $G_N^{(1)}(x)$ denote the distribution function of the test statistic of the linear detector under H_1. Then we have

$$G_N^{(1)}(x) = G_N^{(0)}(x - Ns) . \quad (6)$$

Equations (5) and (6) completely determine the performance of the linear detector.

5. THE GAUSSIAN APPROXIMATION

In non-Gaussian detection problems of the type considered in this paper, the derivation of an expression for the distribution function of the test statistic for the Neyman-Pearson optimal detector is frequently a mathematically intractable problem. In many such cases, for sufficiently large N, an appeal is made to the Central Limit Theorem to arrive at an approximation for the distribution function of the test statistic. Thus it is instructive in the present case to compare the exact results with those resulting from the Gaussian approximation.

Let X be a random variable with the density function of Eq.(1). Let $g(\cdot)$ be the optimal nonlinearity given by Eq.(2). Then

$$E\{g(X)\} = \int g(x) \frac{\gamma}{2} e^{-\gamma|x|} dx .$$

A straightforward integration yields

$$E\{g(X)\} = 1 - \gamma s - e^{-\gamma s} .$$

Similarly, we get

$$VAR[g(X)] = \int [g(x) - 1 + \gamma s + e^{-\gamma s}]^2 \frac{\gamma}{2} e^{-\gamma|x|} dx$$

$$= 3 - 2e^{-\gamma s} - 4\gamma s\, e^{-\gamma s} - e^{-2\gamma s} .$$

Thus the mean and variance of t under H_0 are, respectively,

$$E_0\{t\} = N[1 - \gamma s - e^{-\gamma s}] = m$$

$$VAR_0[t] = N^2[3 - 2e^{-\gamma s} - 4\gamma s e^{-\gamma s} - e^{-2\gamma s}] = \sigma^2 .$$

Using the relation in Eq.(4), it follows that the corresponding values under H_1 are given by

$$E_1\{t\} = -E_0\{t\} = -m$$

$$VAR_1[t] = VAR_0[t] = \sigma^2 .$$

Let $I_N^{(0)}(x)$ and $I_N^{(1)}(x)$ denote, respectively, the Gaussian approximations to the distribution functions of the test statistic under H_0 and H_1. Then

$$I_N^{(0)}(x) = \Phi(\frac{x-m}{\sigma})$$

and

$$I_N^{(1)}(x) = \Phi(\frac{x+m}{\sigma}) ,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-v^2/2} dv .$$

Let α_G and β_G, respectively, denote the false alarm probability and the detection probability resulting from the Gaussian assumption. Then we have

$$\alpha_G = 1 - \Phi(\frac{T-m}{\sigma}) \qquad (7)$$

and

$$\beta_G = 1 - \Phi(\frac{T+m}{\sigma}) \ .$$

In practice, one may use Eq.(7) to set the value of the threshold T. For example, if $\gamma s=1$, $N=15$, and the desired false alarm probability is 0.3, the Gaussian approximation yields a threshold of approximately 1.208 and a detection probability of approximately 0.628. However, using Eqs.(3) and (4) we find that for this threshold, $\alpha \simeq 0.02$ and $\beta \simeq 0.91$. In fact, for $\alpha = 0.3$, we find that the actual detection probability is greater than 0.99. Thus, in this case, the Gaussian approximation is extremely conservative. In Table I we compare the actual values of α and β for the optimal detector against α_G and β_G for several values of T when N=25 and $\gamma s=0.5$. It is seen from the table that, in this case, the Gaussian approximation is not very good (even though N=25).

T	α	α_G	β	β_G
4.032	0.002	0.278	0.278	0.452
3.024	0.007	0.308	0.443	0.487
1.512	0.035	0.356	0.696	0.540
0.000	0.123	0.407	0.877	0.593
-1.512	0.304	0.460	0.965	0.644
-2.016	0.384	0.477	0.978	0.660

Table I. Exact values of α and β and those resulting from the Gaussian approximation, for several values of the threshold; $\gamma s=0.5$, N=25.

6. DETECTOR PERFORMANCE

One popular way to describe the performance of a detector is by the presentation of the receiver operating characteristics. In this section we give some examples of the receiver operating characteristics of the Neyman-Pearson optimal detector.

Figure 1 illustrates how the performance of the optimal detector varies with the number of samples. Figure 2 illustrates how this performance varies with the signal strength. Figures 3 and 4 compare the performance of the optimal detector to that of the linear detector for different values of the parameters. More detailed studies may be made using the results presented in the earlier sections.

ACKNOWLEDGEMENT

This research was partially supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant AFOSR-76-3062. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.

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BIOGRAPHIES

Robert J. Marks II was born in Sutton, W. Virginia, on August 25, 1950. He received the B.S. and M.S. degrees in electrical engineering from Rose-Hulman Institute of Technology in 1972 and 1973, respectively. From 1973 to 1975 he was a Reliability Engineer for the Department of the Navy. He is presently a Ph.D. candidate in electrical engineering at Texas Tech University. His research

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Douglas G. Haldeman was born in Lakewood, Ohio, on July 19, 1951. He received his B.S. degree in electrical engineering from Rose-Hulman Institute of Technology, Terre Haute, Indiana, in 1973 and his M.S.E.E. degree in communications from Purdue University in 1975.

Since August 1975 he has been employed in the Communications Analysis Department of TRW Defense and Space Systems, Redondo Beach, California. He has been involved in the analysis and design of several communication satellite systems and is currently involved in the area of adaptive antennas.

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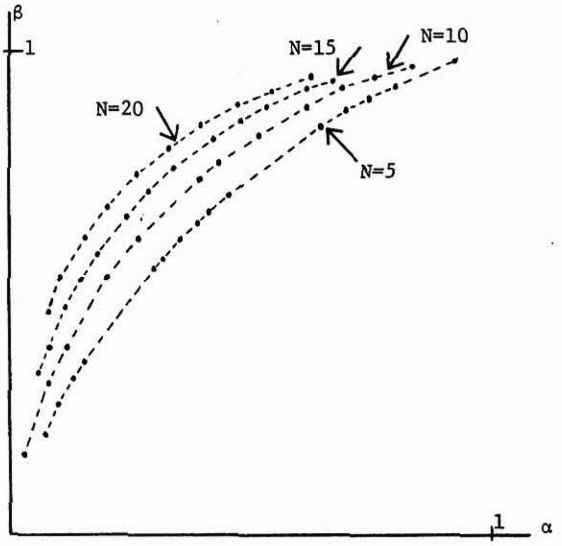

Figure 1. Performance of Optimal Detector for γs=0.3 and Different Values of N.

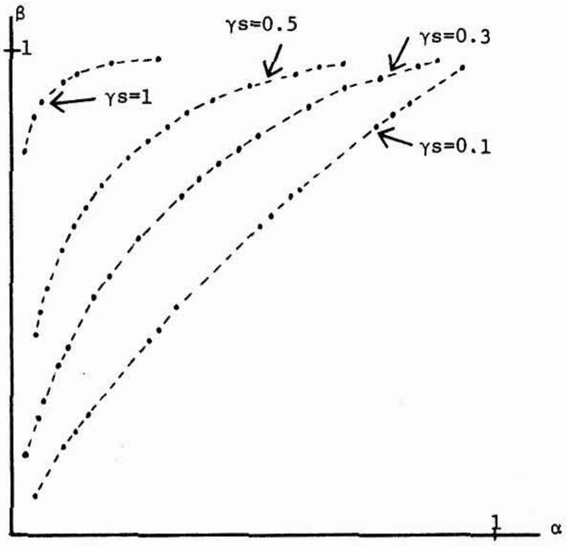

Figure 2. Performance of Optimal Detector for N=10 and Different Values of γs.

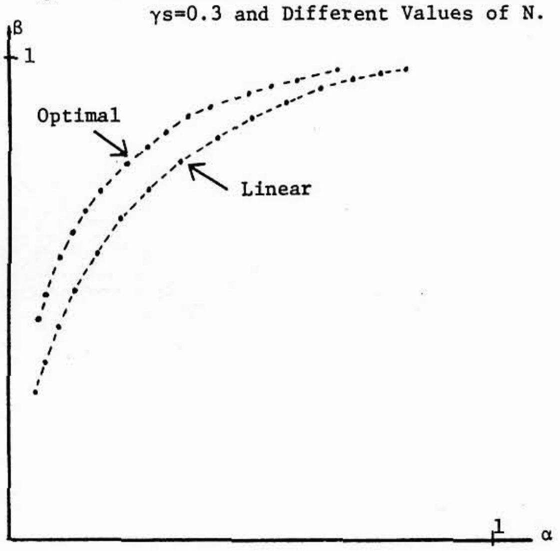

Figure 3. Performance of Optimal Detector Compared to Performance of Linear Detector, N=10, s=1, γ=0.5.

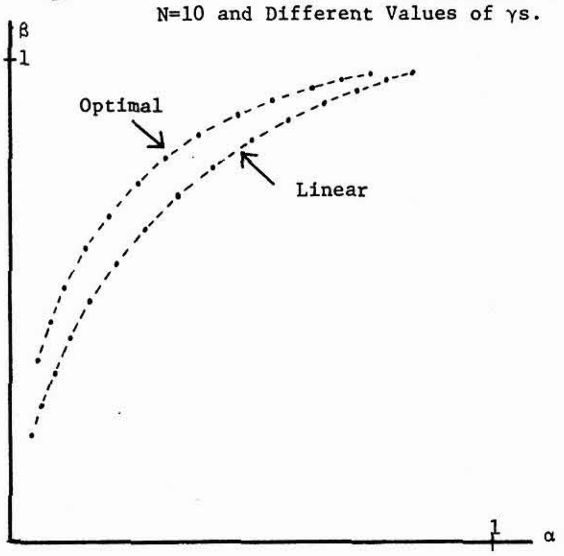

Figure 4. Performance of Optimal Detector Compared to Performance of Linear Detector, N=20, s=0.3, γ=1.