

Signal Synthesis in the Presence of an Inconsistent Set of Constraints

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Abstract—In this paper, we present a novel technique for signal synthesis in the presence of an inconsistent set of constraints. This technique represents a general, minimum norm, solution to the class of synthesis problems in which: the desired signal may be characterized as being an element of some Hilbert Space; each of the N design constraints generates a closed convex set in that space; and those N convex sets generate, or may be resolved into, two disjoint closed convex sets, such that at least one of the two sets is bounded. The synthesis technique employs alternating nearest point maps onto closed convex subsets of a Hilbert Space, and may be viewed as an extension of D. Youla's "Method of Convex Projections"—which addresses the case in which the N closed convex sets, corresponding to the design constraints, possess a nonempty intersection. Section I provides a general introduction to the synthesis problem and to its solution. Section II contains the mathematical justification for the solution technique, while Section III presents an example of the synthesis of a data window for spectral estimation. In Section IV, we discuss potential extensions of this technique within the area of signal synthesis, as well as to the more general class of constrained optimization problems.

I. INTRODUCTION

WITH THE increasing complexity of modern information transmission and processing systems, there have been concomitant increases in the variety, and in the stringency, of design constraints imposed upon the signals which these systems will utilize. It may well be the case that the set of constraints corresponding to the ideal signal for a particular application is inconsistent, in the sense that the satisfaction of any one of the constraints will preclude the satisfaction of other constraints in the set. Examples of such inconsistent constraint sets are abundant, and may be found in even the most fundamental of information systems.

Consider a simple baseband digital communications system, in which a single binary value is to be transmitted every T seconds. In the $(k+1)$ th bit interval, the presence of a "1" at the source is to be indicated by the transmission of a pulse $p(t-kT)$ —where $p(t)$ is identically zero for $t \notin [0, T]$ —while the presence of a "0" is to be indicated by the transmission of $-p(t-kT)$. Hence, $p(t)$ will be of infinite bandwidth, since any band-limited signal is analytic and thus cannot be zero over any contiguous segment of the real line—unless it is identically zero for all values of its argument [1]. The channel over which the

pulse train travels will be band limited, due to physical laws, to some frequency Ω_c . The filtering effect of the channel will thus cause a distorted version of the pulse train to be received, in which each transmitted pulse has spread into the following bit intervals; and the task of detection of the original binary sequence at the receiver is greatly complicated. The design constraints for the ideal $p(t)$ —nonzero only for $t \in [0, T]$ and band limited to Ω_c —form an inconsistent set.

The same dichotomy of time and frequency constraints arises in the field of spectral estimation, where the ideal data window would be of finite duration in time and of vanishingly small bandwidth. A similar problem arises in two dimensions; as in the case of spectroscopy, where signals which are of finite extent in both the spatial and frequency planes would be desirable. In radar and other remote sensing systems, there may well be design constraints imposed upon the magnitude, phase, and spectral content of sensing signals, in such a manner as to form an inconsistent constraint set.

The objective of this paper is the presentation of an iterative signal synthesis technique, of general applicability in the presence of an inconsistent constraint set, which yields a signal most closely satisfying the various constraints. In order to make the meaning of the preceding statement more precise, the mathematical setting of the synthesis technique must be described. It is assumed that any desirable signal may be characterized as being an element of a Hilbert Space— $L_2(-\infty, \infty)$ and $l_2(-\infty, \infty)$, the spaces of all finite-energy continuous-time signals and discrete-time signals, respectively, are representative examples when equipped with the usual, respective, norms and inner products. It is further assumed that each constraint may be phrased in such a manner as to generate a closed convex set in that space. A subset of a linear space is said to be convex if, for any two elements x_1 and x_2 of that subset, the element $\lambda x_1 + (1-\lambda)x_2$ is also in the subset for each value of λ in the interval $(0, 1)$. Subspaces, linear varieties (translates of subspaces), and the linear space itself are all subsumed under the category of convex sets, as well as many other subsets of the space which are meaningful in a signal synthesis context. In $L_2(-\infty, \infty)$: the set of all signals bandlimited to some frequency Ω is closed and convex—a subspace; the set of all signals taking on prescribed values in some interval $[t_1, t_2]$ is closed and convex—a linear variety; the set of all signals whose energy does not exceed some fixed bound is closed and convex—a

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sphere centered at the origin; and the set of all signals satisfying a given dynamic range constraint is closed and convex. This listing is by no means exhaustive.

The final general criterion which the inconsistent constraint set must satisfy, in order that the synthesis technique of this paper be applicable, is that the constraint set must be resolvable into two disjoint closed convex sets. There are two types of constraint sets which will meet this criterion. The first type is that in which the constraint set consists of only two constraints, each generating a closed convex set in the space, such that the two sets are disjoint. The example of the ideal design constraints for a baseband digital communications system, presented earlier in this section, is representative of this first type of constraint set.

The second type of constraint set which will satisfy this criterion consists of $N > 2$ constraints each generating a closed convex set C_i , $i=1, \dots, N$, such that the set $\{C_1, C_2, \dots, C_N\}$ may be represented by the two sets $\{C_{k_1}, C_{k_2}, \dots, C_{k_{N_1}}\}$ and $\{C_{l_1}, C_{l_2}, \dots, C_{l_{N_2}}\}$, where

$$(a) \quad C_{k_i} \in \{C_1, \dots, C_N\}, \quad \text{for } i=1, \dots, N_1$$

$$C_{l_i} \in \{C_1, \dots, C_N\}, \quad \text{for } i=1, \dots, N_2$$

$$(b) \quad \left\{ \bigcap_{i=1}^{N_1} C_{k_i} \right\} \cap C_j \neq \{\emptyset\} \quad \text{or} \quad \left\{ \bigcap_{i=1}^{N_2} C_{l_i} \right\} \cap C_j \neq \{\emptyset\},$$

for each $j=1, \dots, N$

$$(c) \quad \left\{ \bigcap_{i=1}^{N_1} C_{k_i} \right\} \cap \left\{ \bigcap_{i=1}^{N_2} C_{l_i} \right\} = \{\emptyset\}.$$

Since the intersection of closed convex sets is itself closed and convex, the two disjoint sets $\{\bigcap_{i=1}^{N_1} C_{k_i}\}$ and $\{\bigcap_{i=1}^{N_2} C_{l_i}\}$ are each closed and convex. Furthermore, some portion of each constraint set in $\{C_1, \dots, C_N\}$ lies in at least one of the intersections. If two sets $\{C_{k_1}, \dots, C_{k_{N_1}}\}$ and $\{C_{l_1}, \dots, C_{l_{N_2}}\}$ such that conditions (a), (b) and (c) above are satisfied, then $\{\bigcap_{i=1}^{N_1} C_{k_i}\}$ and $\{\bigcap_{i=1}^{N_2} C_{l_i}\}$ will be a resolution of $\{C_1, \dots, C_N\}$ into two disjoint closed convex sets in the sense of the criterion. All following discussion of the synthesis technique of this paper will be phrased in terms of two disjoint closed convex sets, with the understanding that these two sets may have been generated by the resolution of a constraint set with more than two elements, in the manner just described.

A rigorous mathematical discussion of the iterative synthesis technique of this paper will be presented in Section II. A qualitative discussion is, however, in order at this point. It may be shown that to each closed convex set C in a Hilbert Space H , there corresponds an operator, whose domain is H and whose range is C , which is the nearest point map from H into C . Applied to any $x \in H$, this operator will yield the unique element of C closest, in the sense of the norm, to x . The more familiar class of orthogonal projection operators corresponding to subspaces of H is, in fact, subsumed by the category of nearest point maps into closed convex subsets of H .

Consider the case of two disjoint closed convex sets C_1 and C_2 with respective nearest point maps P_1 and P_2 ; and form the composite operator $P = P_1 \circ P_2$ (i.e., $Px =$

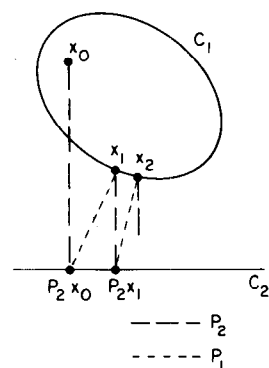


Fig. 1. Alternating nearest point maps between disjoint convex sets.

$P_1(P_2x)$ for $x \in H$). Let x_0 be some element of C_1 , and let $x_1 = Px_0$. P_1 and P_2 are nearest point maps, so it must be the case that $\|x_1 - P_2x_0\| \leq \|x_0 - P_2x_0\| - \|x_1 - P_2x_0\|$ can be no farther from C_2 than x_0 . Similarly, if $x_2 = Px_1 = P^2x_0$, it will be true that

$$\|x_2 - P_2x_1\| \leq \|x_1 - P_2x_1\| \leq \|x_1 - P_2x_0\| \leq \|x_0 - P_2x_0\|.$$

The relationship between x_0 , x_1 , and x_2 is displayed in Fig. 1.

Proceeding on a purely intuitive basis, one might suppose that the sequence of iterates $\{P^n x_0\}_{n=1}^{\infty}$ will converge, or, since $Px \in C_1$ for all $x \in H$, that the sequence $\{P^n(Px)\}_{n=1}^{\infty}$ will converge for an arbitrary element of x of H . If the sequence indeed converges, it will converge, by definition, to a fixed point of P . In order for some $u \in C_1$ to be fixed point of P , u must be $u = Pu = P_1(P_2u) - P_2u$ is the closest element of C_2 to $u \in C_1$, and u is the element of C_1 closest to $P_2u \in C_2$. If u is a fixed point of P , then $\|u - P_2u\|$ will clearly be a local minimum for the distance between elements of C_1 and elements of C_2 .

Section II is primarily concerned with the proof of two statements. Firstly, that if $u \in C_1$ is a fixed point of P , then $\|u - P_2u\|$ is a global minimum for the distance between elements of C_1 and elements of C_2 — thus u and P_2u are signals most closely satisfying the inconsistent constraints represented by C_1 and C_2 . Secondly, that with a simple modification of the operator P , denoted by P_λ , a form of convergence to a fixed point of P can be guaranteed for the sequence of iterates $\{P_\lambda^n(Px)\}_{n=1}^{\infty}$ for any $x \in H$.

Historically, distinctions have not been made between iterative signal synthesis algorithms and iterative signal restoration algorithms in a Hilbert Space setting [2]. In the presence of a consistent constraint set, the difference between a synthesis problem and a restoration problem is merely one of the particular constraints which must be satisfied. In the former case the constraints represent the desired properties of the resultant signal, while in the latter they represent *a priori* knowledge of the characteristics of the signal to be restored. In light of the inconsistency of the constraint set which the technique of this paper employs, this technique will be primarily applicable to problems of synthesis. However, it does have two important antecedents in the class of iterative synthesis and restoration techniques.

The first of these is the so-called Superresolution or Error Energy Reduction algorithm—initially presented by Papoulis [3]–[4] and Gerchberg [5], and later generalized by a number of researchers [6]–[9]. The value of the Gerchberg–Papoulis technique is that it provides a practical algorithm for the reconstruction of a bandlimited signal solely from knowledge of its values over some interval, whereas a formal mathematical technique such as analytic continuation is primarily of pedagogical worth.

The second antecedent of the synthesis technique of this paper is D. Youla’s “Method of Convex Projections” [10]. Youla’s description of this technique was the first systematic presentation of signal synthesis and restoration phrased in terms of closed convex sets in a Hilbert Space. The Method of Convex Projections is applicable to situations in which a consistent set of N constraints generates N closed convex sets, whose intersection is nonempty. Denoting the nearest point maps corresponding to the N convex sets C_1, \dots, C_N as P_1, \dots, P_N , respectively, Youla guarantees a form of convergence for the sequence $\{(P_1 \circ P_2 \circ \dots \circ P_N)^n x\}_{n=1}^\infty$ to an element of $\{\bigcap_{k=1}^N C_k\}$, for any element of x of the respective Hilbert space [10].

One of the most valuable aspects of the paper describing the Method of Convex Projections is the inclusion of a compilation of nearest point maps, for a number of closed convex sets, which are of frequent interest within the context of signal synthesis and restoration [10]. While the orthogonal projection operator corresponding to a particular subspace may often be ascertained by inspection, the nearest point map corresponding to some more general type of convex set may not, in general, be so simply determined. The generation of the nearest point map for a particular convex set will be mentioned again in Sections II and IV.

Section I has, hopefully, provided the reader with some intuition for signal synthesis in a Hilbert Space setting, and with an awareness for the synthesis techniques which have been available to date. Although there are some important mathematical differences between the setting for Youla’s “Method of Convex Projections” and that of signal synthesis in the presence of an inconsistent constraint set, the latter should be thought of as the natural conceptual extension of the former. In Section II, we will provide a rigorous mathematical discussion of signal synthesis in the presence of an inconsistent set of constraints.

II. THEORY

2.1. Introduction

As in Section I, P will be defined to be the composition $P_1 \circ P_2$, where P_1 and P_2 are the nearest point maps corresponding to the two disjoint closed convex sets C_1 and C_2 in a Hilbert Space H . A detailed study will be undertaken of the fixed points of the operator P and of the convergence properties of the sequence

$$\{(\lambda I + (1 - \lambda)P)^n(Px)\}_{n=1}^\infty = \{P_\lambda^n(Px)\}_{n=1}^\infty, \quad \lambda \in (0, 1)$$

for arbitrary $x \in H$.

Most of the important results presented here were culled from articles which appeared in the mathematics literature, and the accompanying proofs were often terse. In the interest of an intuitively satisfying, complete presentation, an effort has been made to minimize the number of assertions stated without proof.

Section II is divided into eight parts. Section 2.2 discusses the idea of uniformly convex Banach Spaces, of which Hilbert Spaces are a subset. Section 2.3 deals with the properties of the nearest point maps P_1 and P_2 , and with the composition operator P ; and Section 2.4 describes the properties of the set of fixed points of P . Sections 2.5 and 2.6 treat the notions of asymptotic regularity and weak convergence, respectively—both of which will be necessary for the convergence proof of Section 2.7. Section 2.8 is a synopsis.

2.2. Uniformly Convex Spaces

The concept of a uniformly convex Banach Space is central to the following development. The notion was first introduced by Clarkson [11] in 1936 and it has been advantageously employed in a variety of contexts. The usual definition is as follows [12]: A Banach Space X is said to be uniformly convex if, for all $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| > \epsilon$, taken together, imply that $\frac{1}{2}\|x + y\| \leq (1 - \delta)$, where x and y are elements of X . An immediate consequence is that the boundary of the unit ball in such a space contains no line segments. All Hilbert Spaces are uniformly convex—this may be observed in the “parallelogram law” which holds in such spaces,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The following Lemma describes some implications of uniform convexity.

Lemma 1: Let X be a uniformly convex Banach Space. Let x and y be elements of X such that $\|x\| \leq M$, $\|y\| \leq M$, — for some fixed $M > 0$; and suppose that $\|x - y\| \geq \epsilon > 0$. Then the following statements are true.

A) There exists a $\delta = \delta(\epsilon) > 0$ such that

$$\frac{1}{2}\|x + y\| \leq (1 - \delta) \max\{\|x\|, \|y\|\}.$$

B) For all $\alpha \in (0, 1)$, there exists a $\delta = \delta(\epsilon, \alpha) > 0$ such that

$$\|\alpha x + (1 - \alpha)y\| \leq (1 - \delta) \max\{\|x\|, \|y\|\}.$$

Proof of A:

Let $\rho = \max\{\|x\|, \|y\|\}$, and consider $\|x/\rho - y/\rho\| \geq \epsilon/\rho > 0$. Now, $\|x/\rho\| \leq 1$ and $\|y/\rho\| \leq 1$, and our previous definition of uniform convexity yields

$$\frac{1}{2}\left\|\frac{x}{\rho} + \frac{y}{\rho}\right\| \leq (1 - \delta), \quad \delta > 0$$

or

$$\frac{1}{2}\|x + y\| \leq (1 - \delta)\rho = (1 - \delta) \max\{\|x\|, \|y\|\}.$$

Thus statement A is true.

Proof of B:

Case 1, $\alpha = 1/2$: In this event, statement B reduces to statement A and is, therefore, valid.

Case 2, $\alpha < 1/2$: We note that $\alpha x + (1 - \alpha)y = y + \alpha(x - y) = (1/2)(y) + (1/2)(y + 2\alpha(x - y))$, i.e., the midpoint of the line segment joining the point y to the point $y + 2\alpha(x - y)$. Since $\|x - y\| \geq \epsilon$, then $\|y - (y + 2\alpha(x - y))\| = \|2\alpha(x - y)\| \geq 2\alpha\epsilon > 0$. Applying statement A, there exists a $\delta > 0$ such that

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\| &= (1/2)\|y + (y + 2\alpha(x - y))\| \\ &\leq (1 - \delta) \max \{ \|y\|, \|y + 2\alpha(x - y)\| \} \end{aligned}$$

Suppose $\|y\| \geq \|x\|$, then

$$\begin{aligned} \|y + 2\alpha(x - y)\| &\leq \|(1 - 2\alpha)y\| + \|2\alpha x\| \\ &\leq (1 - 2\alpha)\|y\| + 2\alpha\|x\| \\ &= \|y\|. \end{aligned}$$

Similarly, if $\|x\| \geq \|y\|$, $\|y + 2\alpha(x - y)\| \leq \|x\|$. So, there exists a $\delta > 0$ such that

$$\|\alpha x + (1 - \alpha)y\| \leq (1 - \delta) \max \{ \|y\|, \|x\| \}, \quad \alpha < 1/2.$$

Case 3, $\alpha > 1/2$: Making the identification $\alpha x + (1 - \alpha)y = (1/2)(x - 2(1 - \alpha)(x - y)) + (1/2)(x)$, and proceeding as in Case 2, we find that

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\| &\leq (1 - \delta) \max \{ \|y\|, \|x\| \}, \\ &\alpha > 1/2, \delta > 0. \end{aligned}$$

So statement B is true, and the proof of the lemma is complete.

In fact, statements A and B are each necessary and sufficient conditions for the uniform convexity of X . That this should be so is straightforward in the case of A, while the verification in the case of B is more involved—the proof, in terms of sequences in X , appears in a paper by Helmut Schaefer [13]. Lemma 1 will be adequate in the present context.

One implication of uniform convexity, which will be invoked in the next lemma, is a theorem due to D. Milman which asserts that every uniformly convex Banach Space is reflexive [12]. The reflexive character of Hilbert Spaces may be ascertained without specific reference to uniform convexity, but in the case of more general Banach Spaces, this result, known as Milman's Theorem, provides a definitive test for the reflexive property.

2.3. Nearest Point Maps in Hilbert Spaces

With the concept of a uniformly convex Space at our disposal, an examination of nearest point maps and compositions of nearest point maps may be undertaken. Let C_1 be a closed convex set in a uniformly convex Banach Space, X . The nearest point map, P_1 , corresponding to C_1 is formally defined for any $x \in X$ as follows [14]:

$$\|x - P_1x\| = \inf_{y \in C_1} \|x - y\|.$$

The following lemma addresses the questions of existence and uniqueness for nearest point maps.

Lemma 2: Let C_1 be a closed convex set in a normed linear space, E .

A) A corresponding nearest point map, P_1 , exists if E is reflexive.

B) If P_1 exists and E is strictly convex, then P_1 is unique in the sense that $P_1x \in C_1$ is unique for any $x \in E$. A normed linear space is said to be strictly convex if, for any two elements x and y such that $\|x - y\| \neq 0$ and $\|x\| = \|y\| = 1$, $\|\lambda x + (1 - \lambda)y\| < 1$ for $\lambda \in (0, 1)$ [15]. Strict convexity is clearly subsumed by uniform convexity, as is the reflexive property by virtue of Milman's Theorem. The result is that if C_1 is a closed convex set in a uniformly convex Banach Space, then the associated nearest point map, P_1 , exists and is unique (in the sense of B). The proof of statement A is beyond the scope of this discussion and the reader is referred to two papers by Phelps [15], [16]. The proof of statement B relies upon the strict convexity of E , and will not be presented at this point, as a similar argument is made in the proof of forthcoming Theorem 3.

The observation should be made that Lemma 2 provides no insight into the functional form of the nearest point map and that the construction of such a mapping operator for a particular convex set is usually a nontrivial task. In practical situations, nearest point maps tend to be nonlinear—some examples in the Hilbert Spaces $L_2(-\infty, \infty)$ and $L_2(-\infty, \infty) \times L_2(-\infty, \infty)$ are given in a paper by Youla [10].

It is clear that in a uniformly convex Banach Space, X , the nearest point map P_1 corresponding to a closed convex set C_1 is a projection in the sense that $P_1x = P_1(P_1x) = P_1^2x$ for any $x \in X$. The terms projection and nearest point map will be used interchangeably from this point on.

The next lemma asserts that the projection operator corresponding to a closed convex set in a Hilbert space is nonexpansive. An operator T in a normed linear space is said to be nonexpansive if, for any two elements of the space in the domain of T , say w_1 and w_2 , the following statement is true:

$$\|Tw_1 - Tw_2\| \leq \|w_1 - w_2\|.$$

The nonexpansivity of such projection operators in a Hilbert Space setting is an important result, but the proof will require the invocation of the fact that if P_1 is the projection operator corresponding to a closed convex set C_1 in a Hilbert Space H , and if x is any element of H , then $\text{Re}(x - P_1x|y - P_1x) \leq 0$ for all y in C_1 —where $\text{Re}\beta$ represents the real part of the, possibly complex, scalar β [14].

The truth of this statement may be verified by noting that the line segment $\lambda y + (1 - \lambda)P_1x$, $\lambda \in [0, 1]$, lies in C_1 for all $y \in C_1$ and all $x \in H$. By the definition of P_1 ,

$$\begin{aligned} \|x - P_1x\|^2 &\leq \|x - (\lambda y + (1 - \lambda)P_1x)\|^2 \\ &= \|(x - P_1x) + \lambda(P_1x - y)\|^2 \\ &= \|x - P_1x\|^2 + \lambda \{ (x - P_1x|P_1x - y)^* \\ &\quad + (P_1x - y|x - P_1x)^* \} + \lambda^2\|P_1x - y\|^2 \\ &= \|x - P_1x\|^2 + 2\lambda \text{Re}(x - P_1x|P_1x - y) \\ &\quad + \lambda^2\|P_1x - y\|^2. \end{aligned}$$

So

$$2 \text{Re}(x - P_1x|y - P_1x) \leq \lambda\|P_1x - y\|^2.$$

Since this holds for all $\lambda \in [0, 1]$, it must hold for $\lambda = 0$, yielding the desired result:

$$\operatorname{Re}(x - P_1x|y - P_1x) \leq 0, \quad x \in H, y \in C_1. \quad (1)$$

Lemma 3: Let C_1 be a closed convex set in a Hilbert Space H with associated projection operator P_1 . Then P_1 is a nonexpansive operator with domain H [10].

Proof: Let x_1 and x_2 be any two elements of H . By (1),

$$\operatorname{Re}(x_1 - P_1x_1|P_1x_2 - P_1x_1) \leq 0$$

and

$$\operatorname{Re}(x_2 - P_1x_2|P_1x_1 - P_1x_2) \leq 0$$

So

$$\begin{aligned} 0 &\geq \operatorname{Re} \{ (x_1 - P_1x_1|P_1x_2 - P_1x_1) \\ &\quad + (x_2 - P_1x_2|P_1x_1 - P_1x_2) \} \\ &= \operatorname{Re} \{ (x_1|P_1x_2 - P_1x_1) - (P_1x_1|P_1x_2 - P_1x_1) \\ &\quad + (x_2|P_1x_1 - P_1x_2) - (P_1x_2|P_1x_1 - P_1x_2) \} \\ &= \operatorname{Re} \{ -(x_1|P_1x_1 - P_1x_2) + (P_1x_1|P_1x_1 - P_1x_2) \\ &\quad + (x_2|P_1x_1 - P_1x_2) - (P_1x_2|P_1x_1 - P_1x_2) \} \\ &= \operatorname{Re} \{ (P_1x_1 - P_1x_2|P_1x_1 - P_1x_2) \\ &\quad - (x_1 - x_2|P_1x_1 - P_1x_2) \}. \end{aligned}$$

Thus

$$\|P_1x_1 - P_1x_2\|^2 - \operatorname{Re}(x_1 - x_2|P_1x_1 - P_1x_2) \leq 0$$

and

$$|(x_1 - x_2|P_1x_1 - P_1x_2)| \geq \operatorname{Re}(x_1 - x_2|P_1x_1 - P_1x_2).$$

Therefore,

$$\|P_1x_1 - P_1x_2\|^2 \leq |(x_1 - x_2|P_1x_1 - P_1x_2)|.$$

Using the Cauchy-Schwarz inequality,

$$\|P_1x_1 - P_1x_2\|^2 \leq \|x_1 - x_2\| \cdot \|P_1x_1 - P_1x_2\|$$

whence $\|P_1x_1 - P_1x_2\| \leq \|x_1 - x_2\|$, completing the proof. The reader should note that no assumptions of linearity on the part of P_1 were required.

The next Theorem establishes the nonexpansive nature of the composite operator P employed in the iterative synthesis procedure.

Theorem 1: Let C_1 and C_2 be two disjoint closed convex sets in a Hilbert Space H . Let P_1 and P_2 be the respective projection operators associated with these sets. Let P be the composite operator defined by $Px = (P_1 \circ P_2)x = P_1(P_2x)$, for all $x \in H$. Then P is nonexpansive.

Proof: Let x_1 and x_2 be arbitrary elements of H . P_1 and P_2 are nonexpansive, therefore,

$$\begin{aligned} \|Px_1 - Px_2\| &= \|P_1(P_2x_1) - P_1(P_2x_2)\| \\ &\leq \|P_2x_1 - P_2x_2\| \leq \|x_1 - x_2\|. \end{aligned}$$

2.4. Fixed Points

This section is primarily concerned with the structure of the fixed point set of the composite operator P . The onus

for almost all of the restrictions on the general applicability of the iterative signal synthesis technique must be ascribed to the results of this section, but, as will be seen, these restrictions imbue the fixed-point set of P with some very desirable properties.

The following theorem provides a sufficient condition for the existence of a fixed-point set for P .

Theorem 2: Let X be a uniformly convex Banach Space, C a bounded, closed convex subset of X , and T a nonexpansive mapping of C into itself. Then T has a fixed point in C [17].

The proof of this theorem is beyond the scope of this discussion. What is salient, though, is the requirement that C be closed and bounded. As regards the composite operator $P = P_1 \circ P_2$, the restriction is that the convex set C_1 , in which the fixed points of P will lie, must be closed and bounded. The qualification that C_1 be bounded is, in a purely abstract setting, relatively severe. In a physical setting, however, there is always an ultimate power or energy constraint—corresponding closely, for instance, to a bound on the L_2 norm—upon a signal; and the limitation imposed by Theorem 2 upon C_1 is not a restriction in the sense of a physically realizable signal. The restrictions on C_2 remain those of closure and convexity, and C_2 may well be unbounded, as in the case of a subspace. In all following discussion, therefore, C_1 will be considered to be bounded as well as closed.

The following lemma, due to Helmut Schaefer [13], concerns the structure of the set of fixed points of a nonexpansive operator, and leads to Corollary 1 which will be used in the proof of the convergence properties of the sequence $\{P_\lambda^n(Px)\}$.

Lemma 4: Let C be a bounded, closed convex set in a Hilbert Space H . Let T be a nonexpansive mapping of H into C , with fixed point set F_T in C . Then F_T is a convex subset of C .

Proof: Suppose that F_T has at least two distinct elements, and let u_1 and u_2 be two such elements. Let u_0 be any element of the set

$$\{x: x \in H, x = \lambda u_1 + (1 - \lambda)u_2, 0 < \lambda < 1\}$$

the line segment joining u_1 and u_2 ; and define

$$\begin{aligned} Z'_1 &= u_0 - u_1 & Z_1 &= Tu_0 - u_1 \\ Z'_2 &= u_2 - u_0 & Z_2 &= u_2 - Tu_0. \end{aligned}$$

Thus

$$Z'_1 + Z'_2 = Z_1 + Z_2.$$

Furthermore,

$$\|Z_1\| = \|Tu_0 - u_1\| = \|Tu_0 - Tu_1\| \leq \|u_0 - u_1\| = \|Z'_1\|.$$

Similarly, $\|Z_2\| \leq \|Z'_2\|$, yielding

$$\|Z_1 + Z_2\| \leq \|Z_1\| + \|Z_2\| \leq \|Z'_1\| + \|Z'_2\|. \quad (2)$$

Substituting for u_0 ,

$$\begin{aligned} \|Z'_1\| + \|Z'_2\| &= \|(1 - \lambda)(u_2 - u_1)\| + \|\lambda(u_2 - u_1)\| \\ &= \|u_2 - u_1\| \\ &= \|Z'_1 + Z'_2\| = \|Z_1 + Z_2\|. \end{aligned}$$

Combining this with (2),

$$\|Z_1 + Z_2\| = \|Z_1\| + \|Z_2\| = \|Z'_1\| + \|Z'_2\| = \|Z'_1 + Z'_2\|. \quad (3)$$

Since $\|Z_i\| \leq \|Z'_i\|$, $i=1,2$, it must be the case that $\|Z_i\| = \|Z'_i\|$, $i=1,2$. For any two elements, x_1 and x_2 , of H it is true that

$$\begin{aligned} \|x_1 + x_2\|^2 &= \|x_1\|^2 + 2\operatorname{Re}(x_1|x_2) + \|x_2\|^2 \\ &\leq \|x_1\|^2 + 2|(x_1|x_2)| + \|x_2\|^2 \\ &\leq \|x_1\|^2 + 2\|x_1\| \cdot \|x_2\| + \|x_2\|^2 \end{aligned}$$

with equality if and only if $x_2 = \gamma x_1$, where γ is an element of the scalar field over which H is defined.

By virtue of (3), then, $Z_2 = \gamma Z_1$ and $Z'_2 = \gamma' Z'_1$, for some nonzero scalars γ and γ' . From the definitions of the Z_i 's and Z'_i 's,

$$Z_1 + \gamma Z_1 = Z'_1 + \gamma' Z'_1$$

or

$$Z_1 = \frac{1 + \gamma'}{1 + \gamma} Z'_1.$$

But $\|Z_1\| = \|((1 + \gamma')/(1 + \gamma))Z'_1\| = \|Z'_1\| \neq 0$, therefore, $\gamma = \gamma'$ and $Z_i = Z'_i$, $i=1,2$. It is now readily observable that $u_0 \in F_T$ as

$$Z_1 - Z_2 = 2Tu_0 - (u_1 + u_2) = 2u_0 - (u_1 + u_2) = Z'_1 - Z'_2$$

which yields $Tu_0 = u_0$. Since a particular value of λ was not specified in the definition of u_0 , F_T is proven to be both closed and convex.

The following corollary of Lemma 4 characterizes subsets of F_T in terms of the limit point of the sequence $\{T^n x\}$, where x is any element of H and T is the operator of Lemma 4. The corollary will be invoked in Theorem 5, the convergence theorem for the sequence $\{P_\lambda^n(Px)\}$. By the nonexpansivity of T , the sequence $\{\|T^n x - u\|\}$ is monotone nonincreasing for any $u \in F_T$, since

$$\|T^n x - u\| = \|T^n x - Tu\| \leq \|T^{n-1}x - u\|, \quad n=1,2,\dots$$

Thus for any $u \in F_T$, there exists the non-negative limit

$$d(u) = \lim_{n \rightarrow \infty} \|T^n x - u\|$$

for each $x \in H$. Consider the following subset of F_T , for some fixed x in H ,

$$F_d = \{u \in F_T : d(u) \leq d\}$$

for some fixed nonnegative constant d .

Corollary 1: Let T , C and H be as in Lemma 4. Let F_d , a subset of F_T , be that of the preceding definition, where x is any fixed element of H [18].

A) For any $d \geq 0$, F_d is a bounded, closed convex subset of F_T , nonempty if d is large enough.

B) Let Δ be the smallest value of d such that F_d is nonempty, then the set F_Δ consists of a single element.

Proof of A:

As C is closed and bounded, F_T will be nonempty by virtue of Theorem 2. F_T and F_d , subsets of C , are clearly bounded. Suppose that F_d consists of more than one

element for some $d > 0$, and let u_1 and u_2 be any two distinct elements thereof. Consider, for any $\alpha \in (0, 1)$,

$$\begin{aligned} &\|T^n x - (\alpha u_1 + (1 - \alpha)u_2)\| \\ &= \|\alpha(T^n x - u_1) + (1 - \alpha)(T^n x - u_2)\| \\ &\leq \alpha\|T^n x - u_1\| + (1 - \alpha)\|T^n x - u_2\|, \quad \text{all } n. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \|T^n x - (\alpha u_1 + (1 - \alpha)u_2)\| \leq d. \quad (4)$$

It was established in Lemma 4 that F_T was convex, thus $(\alpha u_1 + (1 - \alpha)u_2) \in F_T$ and it is thus an element of F_d , by (4). Hence, F_d is convex and closed; and statement A is verified.

Proof of B:

Case 1, $\Delta = 0$:

If $\Delta = 0$, $u \in F_\Delta$ if and only if $\lim_{n \rightarrow \infty} \|T^n x - u\| = 0$, i.e., $\{T^n x\}$ converges strongly to u . For a given $x \in H$, there can be only a single distinct element of F_T satisfying this condition, and $F_\Delta = \{u\}$.

Case 2, $\Delta > 0$:

In this case, the proof is by contradiction. Suppose that F_Δ consists of more than one distinct element, and let u_1 and u_2 be any two such elements. Then $\|u_1 - u_2\| > 0$. By the uniform convexity of H , there exists a $\delta = \delta(\alpha) > 0$ for all $\alpha \in (0, 1)$ such that

$$\begin{aligned} &\|\alpha(T^n x - u_1) + (1 - \alpha)(T^n x - u_2)\| \\ &\leq (1 - \delta) \max\{\|T^n x - u_1\|, \|T^n x - u_2\|\}, \quad \text{all } n \end{aligned}$$

since $u_1 - u_2 = (T^n x - u_2) - (T^n x - u_1)$. Hence,

$$\|T^n x - (\alpha u_1 + (1 - \alpha)u_2)\| \leq (1 - \delta)\Delta, \quad \text{as } n \rightarrow \infty.$$

Thus $d(\alpha u_1 + (1 - \alpha)u_2) < \Delta$, $0 < \alpha < 1$. F_Δ is convex, and so $(\alpha u_1 + (1 - \alpha)u_2) \in F_\Delta$, but the existence of some $u \in F_\Delta$ for which $d(u) < \Delta$ contradicts the definition of F_Δ . Whence, F_Δ consists of a single element, and corollary is established.

The goal of the iterative signal synthesis technique, as stated in Section I, is to generate ordered pairs of points, each pair containing one element from C_1 and one from C_2 , such that the distance between the elements of each pair is a minimum. The following theorem addresses this issue. One should note that the statement of the theorem assumes that the Hilbert Space H is defined over the scalar field of real numbers. This is the second restriction imposed upon the synthesis technique in Section 2.4., as H , if it is interpreted as a function space, must then be composed entirely of real-valued functions or of imaginary-valued functions, but may not contain complex-valued functions. The synthesis technique may still be employed to generate complex-valued functions, however, if an appropriate product space— $H_1 \times H_2$, where H_1 consists of real-valued functions and H_2 consists of imaginary-valued functions—can be constructed with a real-valued inner product.

There are two additional concepts which will be required in the proof of the theorem. The first is that of the distance between the sets C_1 and C_2 ,

$$d(C_1, C_2) = \min_{\substack{x_1 \in C_1 \\ x_2 \in C_2}} \|x_1 - x_2\|$$

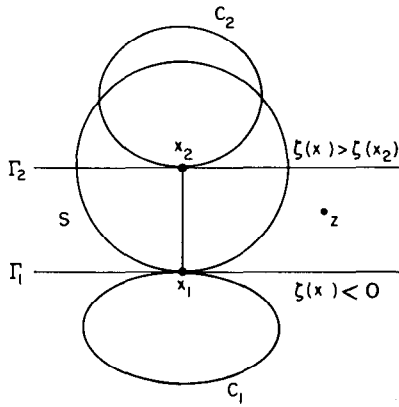


Fig. 2. Distance between the members of each element of D_p .

which is well defined since C_1 and C_2 are closed. F_p , the set of fixed points of the composite operator $P = P_1 \circ P_2$, is a subset of C_1 ; and the set of dual fixed points of P is defined as

$$D_p = \{(x_1, x_2) : x_1 \in F_p, x_2 = P_2 x_1\}.$$

In light of Lemma 2, for each $(x_1, x_2) \in D_p$, x_2 is uniquely determined by x_1 . Each element of F_p corresponds to a single element—an ordered pair—of D_p .

Theorem 3: Let H be a Hilbert Space defined over the scalar field of real numbers; and let C_1 and C_2 be two disjoint, closed convex sets in H . Furthermore, let C_1 be bounded. Define P_1 and P_2 to be the respective projection operators for C_1 and C_2 , and define the composite operator $P = P_1 \circ P_2$. Denote the set of fixed points of P by F_p and its set of dual fixed points by D_p . Then, for each $(x_1, x_2) \in D_p$,

$$\|x_1 - x_2\| = d(C_1, C_2).$$

Proof: By Theorem 2, D_p and F_p are not empty. Referring to Fig. 2, let x_1 be an element of F_p and (x_1, x_2) be the corresponding element of D_p . x_1 may be assumed to be the origin, ϕ , with no loss of generality, since that assumption only implies a translation in H .

There exists a hyperplane $\Gamma_1 = \{x \in H : \zeta(x) = 0\}$ containing x_1 , where ζ is a linear functional on H , such that $(x_2 - x_1)$ is orthogonal to Γ_1 and $\zeta(x_2) > 0$ [19]. Now, C_1 lies entirely in the closed halfspace $\{x \in H : \zeta(x) \leq 0\}$, a claim which will be substantiated through a proof by contradiction.

Suppose that there exists a $z \in C_1$ such that $\zeta(z) > 0$, and consider the closed sphere

$$S = \{x \in H : \|x_2 - x\| \leq \|x_2 - x_1\|\} \\ = \{x \in H : \|x_2 - x\| \leq \|x_2\|\}$$

since $x_1 = \phi$. S is the collection of all points in H which are no further from x_2 than x_1 . Since C_1 is convex, it contains the segment $(1 - \lambda)x_1 + \lambda z = \lambda z$, for $0 \leq \lambda \leq 1$. It will be shown that a subset of this segment lies within S , violating the condition that $x_2 = P_2 x_1$.

If, in fact, some subset of that segment is contained in the interior of S , it will be the case that

$$\|x_2 - x_1\| > \|x_2 - ((1 - \lambda)x_1 + \lambda z)\|.$$

Equivalently,

$$\|x_2\|^2 > \|x_2 - \lambda z\|^2 \\ = \|x_2\|^2 - 2(x_2|\lambda z) + \|\lambda z\|^2$$

or

$$2\lambda(z|x_2) > \lambda^2\|z\|^2 \\ \lambda < \frac{2(z|x_2)}{\|z\|^2}. \tag{5}$$

Since $\|z\|^2 > 0$, values of λ such that $(1 - \lambda)x_1 + \lambda z$ lays in the interior of S will exist if $(z|x_2) > 0$. Consider the projection of z onto the subspace spanned by x_2 , and denote this element as αx_2 . An application of the Projection Theorem yields [19]

$$(z - \alpha x_2|x_2) = 0, \quad \text{hence } \alpha = \frac{(z|x_2)}{\|x_2\|^2}$$

is a maximal proper subspace, and since x_2 is orthogonal to Γ_1 by construction

$$z - \alpha x_2 = x_\gamma, \quad \text{for some } x_\gamma \in \Gamma_1.$$

Furthermore, ζ is a linear functional, so

$$\zeta(z) = \zeta(\alpha x_2 + x_\gamma) = \zeta(\alpha x_2) = \frac{(z|x_2)}{\|x_2\|^2} \zeta(x_2). \tag{6}$$

The range of ζ is the real numbers, $\zeta(x_2)$ and $\zeta(z)$ are strictly positive, thus (6) implies that $(z|x_2) > 0$. By (5) there must then exist points along the segment $(1 - \lambda)x_1 + \lambda z$, contained in C_1 , in the interior of S and hence closer to x_2 than x_1 —violating the definition of P_2 . Whence, C_1 is contained in the halfspace $\{x \in H : \zeta(x) \leq 0\}$.

Now consider the hyperplane

$$\Gamma_2 = \Gamma_1 + x_2 = \{x \in H : \zeta(x) = \zeta(x_2)\}.$$

Proceeding as above, and noting that $x_1 = P_1 x_2$, it may shown that C_2 is contained in the halfspace $\{x \in H : \zeta(x) \geq \zeta(x_2)\}$.

Γ_1 and Γ_2 are parallel by construction, thus

$$d(C_1, C_2) = d(\Gamma_1, \Gamma_2) = \|x_1 - x_2\|$$

where x_1 was an arbitrary element of F_p , and the proof of Theorem 3 is complete.

2.5. Asymptotically Regular Operators

At this juncture, aspects of the structure of the composite operator P and of its set of fixed points F_p have been established, and the discussion now turns to the convergence behavior of the sequence of iterates $\{P_\lambda^n(Px)\}$, for an arbitrary $x \in H$. The requisite arguments rely to a significant extent upon the concepts of asymptotic regularity and weak convergence.

An operator T in a normed linear space X is said to be asymptotically regular if, for each $x \in X$, $\|T^{n+1}x - T^n x\| = \|(I - T)T^n x\|$ converges to zero as $n \rightarrow \infty$ [20]. Thus the sequence $\{T^n x\}$ is convergent for each $x \in X$. The following theorem asserts that any nonexpansive operator in a uniformly convex Banach Space can be “converted” to an asymptotically regular, nonexpansive operator [20], [13].

Theorem 4: Let X be a uniformly convex Banach Space, T a nonexpansive, self-mapping of X with a nonempty set F_T of fixed points. For any fixed $\lambda \in (0, 1)$, define $S_\lambda = \lambda I + (1 - \lambda)T$. Then the following statements are true.

- A) F_T is the set of fixed points of S_λ .
- B) S_λ is nonexpansive.
- C) S_λ is asymptotically regular.

Proof of A:

Suppose that $u \in X$ is a fixed point of S_λ . Then $S_\lambda u = \lambda u + (1 - \lambda)Tu = u$, or $(1 - \lambda)Tu = (1 - \lambda)u$. So $Tu = u$, therefore, $u \in F_T$, and statement A is verified.

Proof of B:

Let x_1 and x_2 be arbitrary elements of X , and consider $\|S_\lambda x_1 - S_\lambda x_2\|$.

$$\begin{aligned} \|S_\lambda x_1 - S_\lambda x_2\| &= \|\lambda(x_1 - x_2) + (1 - \lambda)(Tx_1 - Tx_2)\| \\ &\leq \lambda\|x_1 - x_2\| + (1 - \lambda)\|Tx_1 - Tx_2\| \\ &\leq \lambda\|x_1 - x_2\| + (1 - \lambda)\|x_1 - x_2\| \end{aligned}$$

by the nonexpansivity of T . Therefore, $\|S_\lambda x_1 - S_\lambda x_2\| \leq \|x_1 - x_2\|$ for arbitrary x_1 and x_2 in X and statement B is established.

Proof of C:

Define $x_n = S_\lambda^n x$ for some $x \in X$. Then

$$\begin{aligned} \|S_\lambda^{n+1}x - S_\lambda^n x\| &= \|S_\lambda x_n - x_n\| = \|(\lambda - 1)x_n + (1 - \lambda)Tx_n\| \\ &= (1 - \lambda)\|x_n - Tx_n\|. \end{aligned}$$

Since $\lambda \in (0, 1)$, the fact that $\|x_n - Tx_n\|$ converges to zero as $n \rightarrow \infty$ will suffice to establish the asymptotic regularity of S_λ . Let u be an arbitrary element of F_T , and note that

$$\|x_{n+1} - u\| = \|S_\lambda x_n - S_\lambda u\| \leq \|x_n - u\|, \quad \text{all } n \geq 1$$

by the nonexpansivity of S_λ . Clearly, the sequence $\{\|x_n - u\|\}_{n=1}^\infty$ is monotone nonincreasing with n , and it therefore converges to some nonnegative limit d_0 . Hence,

$$\|x_{n+1} - u\| \rightarrow d_0 \quad \text{and} \quad \|x_n - u\| \rightarrow d_0, \quad \text{as } n \rightarrow \infty.$$

Suppose that $d_0 > 0$. Suppose further that $\|x_n - Tx_n\|$ does not converge to zero. Then there exists an $\epsilon > 0$ such that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = \lim_{n \rightarrow \infty} \|(x_n - u) - (Tx_n - u)\| > \epsilon > 0.$$

By Lemma 1, statement B, however, there exists a $\delta > 0$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\lambda(x_n - u) + (1 - \lambda)(Tx_n - u)\| \\ \leq (1 - \delta) \left[\lim_{n \rightarrow \infty} \|x_n - u\| \right] \end{aligned}$$

since $\|Tx_n - u\| = \|Tx_n - Tu\| \leq \|x_n - u\|$, where the λ is that in the definition of S_λ . Inasmuch as $(x_{n+1} - u) = \lambda(x_n - u) + (1 - \lambda)(Tx_n - u)$, the convergence of $\|x_n - Tx_n\|$ to some value other than zero implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u\| \leq (1 - \delta) \left[\lim_{n \rightarrow \infty} \|x_n - u\| \right]$$

which is false, as $\|x_{n+1} - u\|$ and $\|x_n - u\|$ must converge to the same limit as $n \rightarrow \infty$. Therefore, by contradiction, the convergence of $\|x_n - Tx_n\|$ to zero as $n \rightarrow \infty$ is established, whence S_λ is asymptotically regular, completing the proof.

It is worth noting that in a slightly different context, the choice of λ in an operator of the form $\lambda I + (1 - \lambda)T$ has been shown to have a direct effect on the rate of convergence of the sequence $\{(\lambda I + (1 - \lambda)T)^n x\}$ [10], [21], and is likely to have a similar effect here.

2.6. Weak Convergence

The last general notion which must be introduced prior to establishing the convergence properties of $\{P_\lambda^n(Px)\}$ is that of weak convergence. As a preface, the Reisz-Fréchet Representation Theorem [19] will be stated: "If ζ is a bounded linear functional on a Hilbert Space H , there exists a unique vector $y \in H$ such that for all $x \in H$, $\zeta(x) = (x|y)$. Furthermore, we have $\|\zeta\| = \|y\|$ and every y determines a unique bounded linear functional in this way." Since the dual of a normed linear space X is the space of all bounded linear functionals defined on X , the foregoing theorem may be interpreted as saying that a Hilbert Space is its own dual in the sense of the correspondence between bounded linear functionals defined on that space and elements of that space. As will become apparent, weak convergence involves considerations in both a normed space and its dual; and, if that space is a Hilbert Space, the Reisz-Fréchet Theorem allows the criteria for weak convergence to be phrased in terms of a single space.

Let X' denote the dual of a normed linear space X . A sequence $\{x_n\}$ in a normed linear space X is said to be weakly convergent if a finite $\lim_{n \rightarrow \infty} \zeta(x_n)$ exists for each $\zeta \in X'$; $\{x_n\}$ is said to converge weakly to an element $x_\infty \in X$ if $\lim_{n \rightarrow \infty} \zeta(x_n) = \zeta(x_\infty)$ for each $\zeta \in X'$ [12]. Restated in terms of the Reisz-Fréchet Theorem: A sequence $\{x_n\}$ in a Hilbert Space H is said to be weakly convergent if a finite $\lim_{n \rightarrow \infty} (x_n|y)$ exists for all $y \in H$; $\{x_n\}$ is said to converge weakly to an element $x_\infty \in H$ if $\lim_{n \rightarrow \infty} (x_n|y) = (x_\infty|y)$ for all $y \in H$.

While strong convergence, i.e., convergence in the norm, of a sequence $\{x_n\}$ to an element x_∞ in a normed space implies weak convergence of $\{x_n\}$ to x_∞ , the converse is not true; and examples of sequences which converge weakly to some weak limit x_∞ , but not strongly to that limit, may be constructed [19]. The following example demonstrates, however, that the notion of the weak limit is meaningful in practical situations.

This example is set in the real $L_2(-\infty, \infty)$ Hilbert Space. Consider a weakly convergent sequence $\{x_n\}_{n=1}^\infty$ with weak limit x_0 , all in $L_2(-\infty, \infty)$; and suppose that the limiting behavior of x_n , as $n \rightarrow \infty$, on the compact interval $[0, T]$, $0 < T < \infty$, is of primary interest. A countably infinite set of orthonormal basis functions, $\{e_i\}$, for $L_2[0, T]$ is then selected (for example, the basis functions for the Fourier Series, $\{(1/T), (2/T)\cos(2\pi t/T), (2/T)\sin(2/\pi t/T), \dots, (2/T)\cos(2n\pi t/T), (2/T)\sin(2n\pi t/T), \dots\}$). The functions $e_i(t)$ are then extended into $L_2(-\infty, \infty)$ by defining the new functions,

$$e_i^{(2)} = \begin{cases} e_i, & t \in [0, T] \\ 0, & t \in (-\infty, 0) \cup (T, \infty). \end{cases}$$

Now, the $L_2(-\infty, \infty)$ inner product of any element in $L_2(-\infty, \infty)$ with any one of the $e_i^{(2)}$'s will equal the $L_2[0, T]$ inner product of that element with the corresponding e_i . Furthermore, with respect to the $L_2(-\infty, \infty)$ inner product, $\lim_{n \rightarrow \infty} (x_n | e_i^{(2)}) = (x_0 | e_i^{(2)})$ for all i . It is well known that a generalized Fourier Series of the form $\sum_{i=1}^{\infty} (x | e_i) e_i$, where $x \in L_2[0, T]$ and the $L_2[0, T]$ inner product is employed, converges strongly to x on $[0, T]$. Since x_0 , restricted to $[0, T]$, is in $L_2[0, T]$, strong convergence of $\{x_n\}$ to x_0 on $[0, T]$ has been established. In general, weak convergence in $L_2(-\infty, \infty)$ implies strong convergence on any compact (i.e., bounded and closed) subset of the real line.

In a somewhat analogous manner to the definition of a complete normed linear space—a Banach Space—in terms of Cauchy sequences, a normed linear space is said to be sequentially weakly complete if every weakly convergent sequence from that space converges weakly to an element of that space. It may be shown that all reflexive spaces, and thus all uniformly convex spaces, are sequentially weakly complete [12]; and this result will be employed in the proof of Theorem 5, in which the convergence properties of $\{P_\lambda^n(Px)\}$ are established.

An immediate result of the definition of a weakly convergent sequence and of a weak limit is that of the following lemma.

Lemma 5: If in a Hilbert Space H the sequence $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to x_0 , then for any $x \neq x_0$ [18],

$$\liminf_{n \rightarrow \infty} \|x_n - x\| > \liminf_{n \rightarrow \infty} \|x_n - x_0\|.$$

Proof: By definition of the norm, both limits are finite. Consider,

$$\begin{aligned} \|x_n - x\|^2 &= \|x_n - x_0 + x_0 - x\|^2 \\ &= \|x_n - x_0\|^2 + 2\operatorname{Re}(x_n - x_0 | x_0 - x) + \|x_0 - x\|^2 \end{aligned}$$

Since x_0 is the weak limit of $\{x_n\}_{n=1}^{\infty}$, $(x_n | x_0 - x) \rightarrow (x_0 | x_0 - x)$ as $n \rightarrow \infty$, and $(x_n - x_0 | x_0 - x) \rightarrow 0$ as $n \rightarrow \infty$. $\|x_0 - x\| > 0$ and the Lemma is established.

One more lemma will be required for the proof of Theorem 5; and, towards this end, the notion of a demiclosed operator will now be introduced. Let C be a closed convex set in a Banach Space X , and A a mapping from C into X . If for any sequence $\{x_n\}$ in C with weak limit x_0 in C , the strong convergence (i.e., in norm) of the sequence $\{Ax_n\}$ to some $y_0 \in X$ implies that $Ax_0 = y_0$, then the operator A is said to be demiclosed [18].

Lemma 6: In a Hilbert Space H , for every nonexpansive mapping T of C , a closed convex set, into H , the mapping $(I - T)$ is demiclosed [18].

Proof: Let $\{x_n\}$ be a weakly convergent sequence in C with weak limit $x_0 \in C$, and let the sequence $\{x_n - Tx_n\}$ converge to some $y_0 \in H$. By the nonexpansivity of T ,

$$\liminf_{n \rightarrow \infty} \|x_n - x_0\| \geq \liminf_{n \rightarrow \infty} \|Tx_n - Tx_0\|.$$

Noting that $Tx_n = x_n - y_0$ in the limit as $n \rightarrow \infty$, the

preceding inequality may be rewritten as

$$\liminf_{n \rightarrow \infty} \|x_n - x_0\| \geq \liminf_{n \rightarrow \infty} \|x_n - y_0 - Tx_0\|$$

and since $\{x_n\}_{n=1}^{\infty}$ converges weakly to x_0 , $x_0 = y_0 + Tx_0$ or $(I - T)x_0 = y_0$, by virtue of Lemma 5. Therefore, $(I - T)$ is demiclosed.

2.7. Convergence Theorem

Our final theorem may now be stated and proven.

Theorem 5: Let C be a bounded closed convex set in a Hilbert Space H , and let T be a nonexpansive, asymptotically regular mapping of C into itself with a nonempty set of fixed points F_T . Then, for any $x \in C$, the sequence of iterates $\{T^n x\}$ is weakly convergent to an element of F_T [18].

Proof: The sequence $\{T^n x\}$ will first be shown to be weakly convergent. Let u be an element of F_T , by the nonexpansivity of T it is true that

$$\|T^n x\| - \|u\| \leq \|T^n x - u\| \leq \|T^{n-1} x - u\| \leq \dots \leq \|x - u\|.$$

Thus $\|T^n x\| \leq \|x - u\| + \|u\|$ for all n . Let ζ be any bounded linear functional defined on H , and let $y \in H$ be the corresponding element of H in the sense of the Reisz-Fréchet Theorem. Then $\|y\| = \|\zeta\| < \infty$, and

$$\begin{aligned} \lim_{n \rightarrow \infty} |\zeta(T^n x)| &= \lim_{n \rightarrow \infty} |(T^n x | y)| \\ &\leq \lim_{n \rightarrow \infty} \|T^n x\| \cdot \|y\| \\ &\leq (\|x - u\| + \|u\|) \cdot \|y\| \\ &< \infty \end{aligned}$$

where the Cauchy-Schwarz Theorem has been invoked. Hence $\{T^n x\}$ is weakly convergent and has its weak limit in H , since Hilbert Spaces are weakly sequentially complete.

Recalling the definitions of $d(u)$ and F_Δ from Corollary 1, let $F_\Delta = \{u_0\}$ which implies that $d(u_0) = \Delta$, where the x of $d(\cdot)$ is that of the statement of this Theorem. Through a proof by contradiction, $\{T^n x\}$ will be shown to converge weakly to u_0 .

Suppose that $\{T^n x\}$ converges weakly to some $z \in H$, $z \neq u_0$. T is asymptotically regular, by definition, and the sequence $\{T^n x - T^{n+1} x\} = \{(I - T)T^n x\}$ must therefore converge strongly to the origin as $n \rightarrow \infty$. By Lemma 6, $(I - T)$ is demiclosed, so it must be the case that $(I - T)z = \phi$ as well. Hence, $z \in F_T$. The result of Lemma 5 is then

$$\Delta = d(u_0) = \lim_{n \rightarrow \infty} \|T^n x - u_0\| > \lim_{n \rightarrow \infty} \|T^n x - z\| = d(z)$$

since the limit infimum of a sequence equals the limit of the sequence if that limit exists (the existence of the above limits was established in the discussion preceding Corollary 1). This inequality violates the definition of Δ , generating the necessary contradiction and completing the proof.

2.8. Synopsis

Each of the results of the preceding seven sections was presented in the most general context possible, with the intent of providing points of departure for further study

and development of the iterative synthesis technique of this paper and of other similar techniques. Those results will now be reiterated explicitly in terms of signal synthesis in the presence of inconsistent constraints.

The setting is a Hilbert Space H defined over the scalar field of real numbers. C_1 and C_2 are disjoint, closed convex sets in H —each one composed of the class of signals satisfying some given constraint, or set of constraints. Furthermore, C_1 is bounded. Lemma 2 guarantees that the respective nearest point maps P_1 and P_2 exist, and that they yield unique points in these sets for every $x \in H$.

The composite operator $P = P_1 \circ P_2$ is then nonexpansive by Theorem 1; and Theorem 2 assures that the fixed-point set F_P of P , and hence the set of dual fixed points D_P , are not empty. Theorem 3 warrants that $\|x_1 - x_2\| = d(C_1, C_2)$ for all $(x_1, x_2) \in D_P$.

Theorem 4 states that any operator of the form $P_\lambda = \lambda I + (1 - \lambda)P$, $\lambda \in (0, 1)$ is a nonexpansive asymptotically regular operator with F_P as its set of fixed points. Theorem 5 is thus applicable, and the sequence of iterates $\{P_\lambda^n x\}$ will converge weakly to an element of F_P for any $x \in C_1$. More generally, $\{P_\lambda^n(Px)\}$ will converge weakly to an element of F_P for any $x \in H$, as $Px \in C_1$.

It is worth noting that P_1 , P_2 and P are not, at any point in this section, assumed to be linear—this synthesis technique will always be appropriate, so long as C_1 and C_2 are of correct form.

III. EXAMPLE

As an example of signal synthesis in the presence of an inconsistent set of constraints, the technique developed in Section II will now be applied to the generation of a data window for spectral estimation. A data window is a time function which is necessarily of finite duration, and whose spectral energy is, ideally, concentrated about the origin in the frequency domain. By virtue of its finite extent in time, the window will be of infinite bandwidth; and, in practice, the design of such a window may be viewed as an attempt to attain a narrow, high amplitude, spectral concentration about the frequency origin, while simultaneously minimizing that window's spectral energy for nonzero frequencies—in the parlance of spectral estimation, the desire is to maximize resolvability while minimizing leakage.

The two disjoint closed convex sets from $L_2(-\infty, \infty)$, to which the synthesis technique of this paper will be applied, have been selected as follows. C_2 will be the set of all time signals which are identically zero outside of the interval $[-T/2, T/2]$, where T is the duration of the desired data window. C_1 will be the set of all signals whose Fourier transforms are real, nonnegative, bandlimited to some frequency Ω , and have a prescribed area ρ . Elements of C_1 will be real, even time functions, and their projections into C_2 will thus be even as well. Denoting the Fourier transform of some $f(t) \in L_2(-\infty, \infty)$ as $F_1(\omega) + jF_2(\omega)$, where F_1 and F_2 are real-valued functions of ω , Youla has shown that the projection of $f(t)$ into C_1 is given by $\mathcal{F}^{-1}\{r_\Omega(\omega)(F_1(\omega) + c)^+\}$ [10], where \mathcal{F}^{-1} denotes the inverse

Fourier transform,

$$r_\Omega(\omega) = \begin{cases} 1, & \omega \in [-\Omega, \Omega] \\ 0, & \omega \notin [-\Omega, \Omega] \end{cases}$$

$$(G(\omega))^+ = \begin{cases} G(\omega), & G(\omega) \geq 0 \\ 0, & G(\omega) < 0 \end{cases}$$

and where the real constant c is chosen such that

$$\int_{-\Omega}^{\Omega} (F_1(\omega) + c)^+ d\omega = \rho.$$

The projection of some $f(t) \in L_1(-\infty, \infty)$ into C_2 is clearly $r_{T/2}(t)f(t)$, where

$$r_{T/2}(t) = \begin{cases} 1, & t \in \left[-\frac{T}{2}, \frac{T}{2}\right] \\ 0, & t \notin \left[-\frac{T}{2}, \frac{T}{2}\right] \end{cases}$$

It should be noted, before proceeding with this example, that there are many available techniques for estimating the spectral content of a signal. The generation of such an estimate by the transformation of the product of an observed segment of the signal and a data window is but one method. The intent of this discussion is not an in-depth analysis of spectral estimation—the reader is referred to a paper by Kay and Marple [22] for an overview of the subject, and to a paper by Harris [23] on the topic of data windows.

An iterative technique, such as the one of this paper, must be implemented on a digital computer to be of practical worth. Time and frequency axes, as well as function amplitudes, must be discretized. Since the sets C_1 and C_2 were defined in terms of $L_2(-\infty, \infty)$, instead of $l_2(-\infty, \infty)$, all the usual caveats concerning approximations to the continuous world by the discrete one must be heeded. An attempt has been made to minimize aliasing effects by setting the Nyquist Frequency to be more than seventy times the value of Ω in the definition of C_1 , and any integrals which must be approximated will be calculated using the trapezoidal rule. It is assumed that a discrete approximation to the projection operator corresponding to C_1 will be appropriate, although, for reasons of practicality, the area ρ will now become a range of areas $\rho \pm \Delta$, where $\Delta \leq 10^{-5}\rho$. Perhaps the most important aspect of this discrete approximation is that the use of a finite-length Discrete Fourier Transform (DFT) will implicitly generate periodic functions.

The specific example to be presented is that of the design of a 64 point data window for spectral estimation. It is assumed that all signals of interest have negligible frequency content above 10 Hz—the selected Nyquist Frequency, F_N . The sampling frequency, F_S , will be 20 Hz, yielding a sampling period T_S of 0.05 s and a spacing of $\Delta_f^{(1)} = F_S/64 = 0.3125$ Hz for the frequency samples of the associated DFT.

The data window will be designed on a 1024-point sequence with the same spacing in time, T_S , as the 64 point

sequence, and, hence, with a spacing of $\Delta_f^{(2)} = F_s/1024 = 0.01953125$ Hz for the frequency samples of the associated DFT. Now, $\Delta_f^{(2)}/\Delta_f^{(1)} = 16$; and the advantage of this factor of 16 density increase in frequency samples for the design sequence, over the sequence whose length is that of the data window, will be that C_1 may be meaningfully defined with values of Ω less than $\Delta_f^{(1)}$.

In order to provide the most realistic emulation of the continuous case, the time origin for the 1024-point sequence was defined to be the 513th point of that sequence. Any sequence of 1024 points which is real and displays even symmetry about the 513th point will have a real and even—about the frequency origin—DFT. With the time origin so defined, the span of the truncation window $r_{T/2}(t)$ will be an odd number of points. In what follows, then, the C_2 functions which have been generated by the iterative synthesis technique will have been produced by a 65-point rectangular window of height one, centered at point 513. The data window, whose performance will be subsequently evaluated, will be composed of the final C_2 sequence from the point whose time abscissa is $-T/2$ up to, but excluding, the point whose time abscissa is $T/2$ yielding a sequence of length 64.

A modular, Fortran 77, program named CONVEX was developed to carry out iterations of the form $\{P_\lambda^n\}$. The program was supplied with the projection routines for the C_1 and C_2 of this section. C_1 was parameterized as follows: $\rho = 1.3 \pm 1.3 \times 10^{-5}$, and $\Omega = 7\Delta_f^{(2)} = 7/16\Delta_f^{(1)} = 0.13671875$ Hz. The single parameter for C_2 , $T/2$, was set to $32T_s = 1.6$ s. The fixed points of P were specified to lie in C_1 , thus $P = P_1 \circ P_2$; and the value of λ , in the definition of P_λ , was chosen to be 10^{-3} . For this example, CONVEX was instructed to carry out 50 iterations and to make the results of the 3rd and 50th iterations available for inspection.

To aid in the evaluation of the convergence properties of the sequence of iterates, the following measure of convergence was defined

$$\epsilon(n) = \frac{\|P_\lambda^n(Px) - P_\lambda^{n-1}(Px)\|}{\|P_\lambda(Px) - Px\|}, \quad n = 1, 2, \dots$$

Since P_λ is nonexpansive, ϵ will be a monotone non-increasing function of n . Moreover, the asymptotic regularity of P_λ would lead one to expect ϵ to be a decreasing function for sufficiently large n .

The starting point for the iterations, the “ x ” of $\{P_\lambda^n(Px)\}$, is displayed in Fig. 3. This function is neither an element of C_1 nor of C_2 . In the process of developing CONVEX, it was observed that the point to which the sequence of iterates converged was essentially invariant with respect to its starting point.

Figs. 4 and 5 are plots of $\epsilon(n)$ versus n on two different ordinate scales. The technique is observed to converge rapidly. With these two sets, convergence is typically achieved within forty iterations for a wide range of time—bandwidth products generated by particular parameterizations of C_1 and C_2 . The oscillatory behavior of ϵ ,

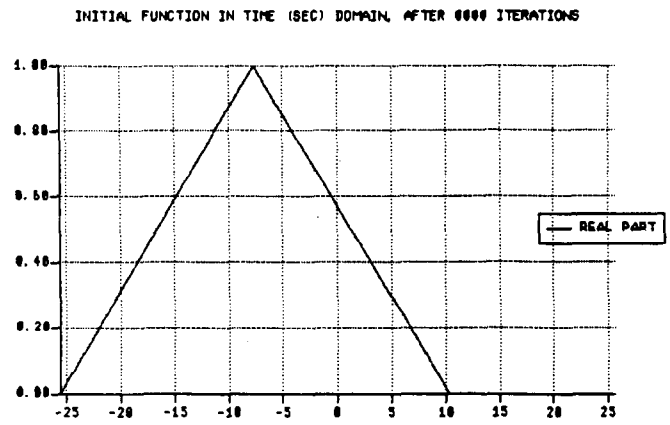


Fig. 3.

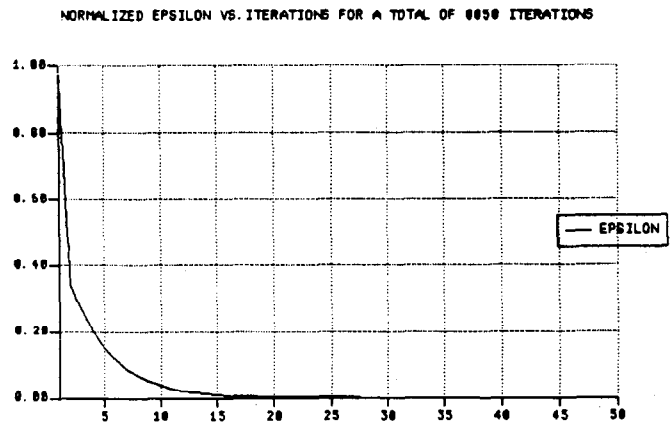


Fig. 4.

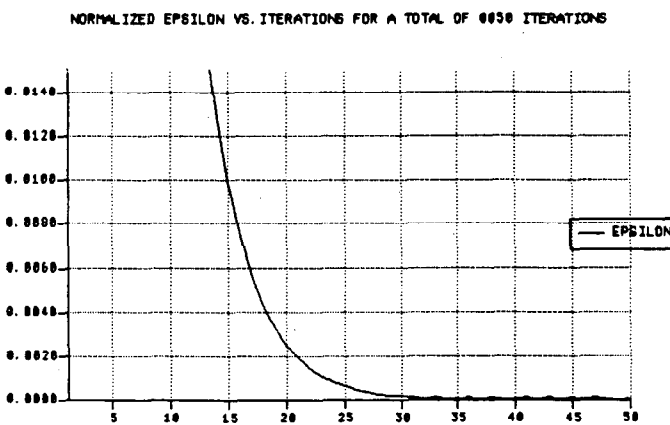


Fig. 5.

above $n = 32$, may be ascribed to the fact that the ρ for C_1 was defined to be a range of values, rather than a single value.

Figs. 6 and 7 depict the C_1 function after three iterations in the time and frequency domains, respectively. In the time domain, the function is observed to be radically different from the initial function of Fig. 3, and to be periodic—a result of time discretization and the use of the DFT in the associated projection routine. The time function also displays even symmetry about the time origin, as

SET 1 FUNCTION IN TIME (SEC) DOMAIN AFTER 0003 ITERATIONS

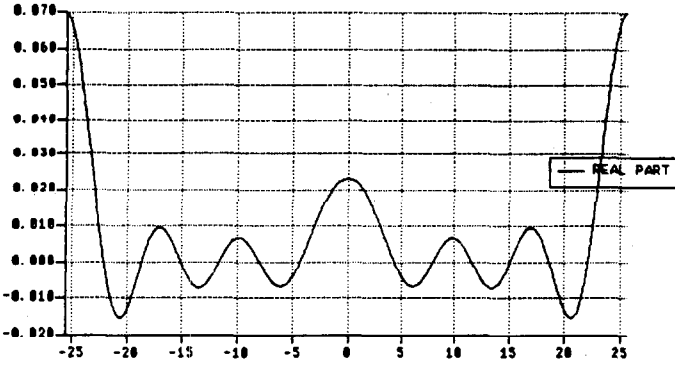


Fig. 6.

SET 2 FUNCTION IN FREQUENCY (HZ) DOMAIN AFTER 0003 ITERATIONS

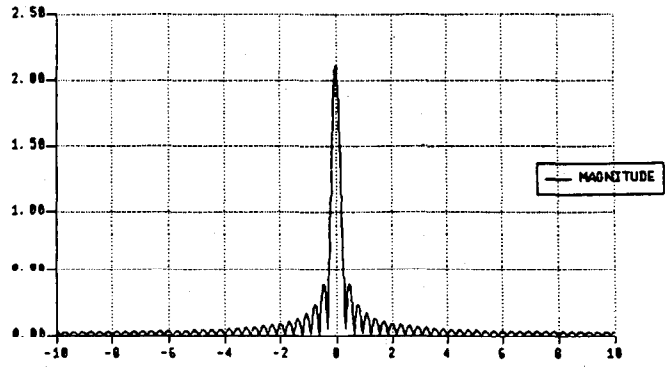


Fig. 9.

SET 1 FUNCTION IN FREQUENCY (HZ) DOMAIN AFTER 0003 ITERATIONS

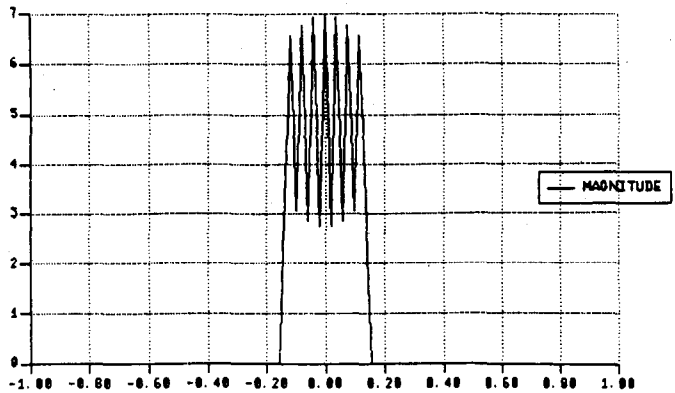


Fig. 7.

SET 1 FUNCTION IN TIME (SEC) DOMAIN AFTER 0050 ITERATIONS

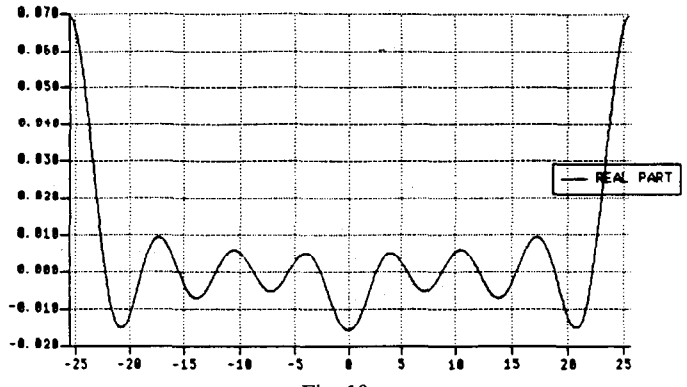


Fig. 10.

SET 2 FUNCTION IN TIME (SEC) DOMAIN AFTER 0003 ITERATIONS

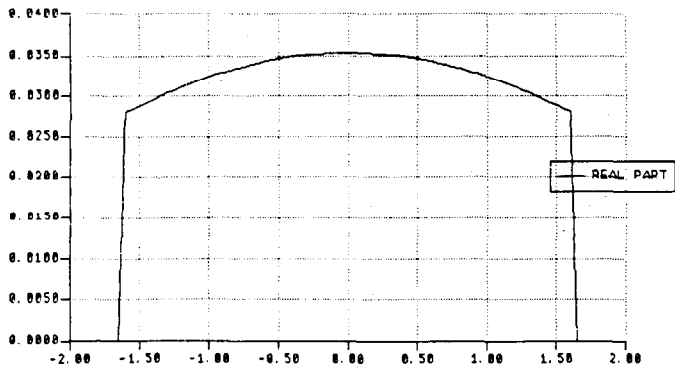


Fig. 8.

SET 1 FUNCTION IN FREQUENCY (HZ) DOMAIN AFTER 0050 ITERATIONS

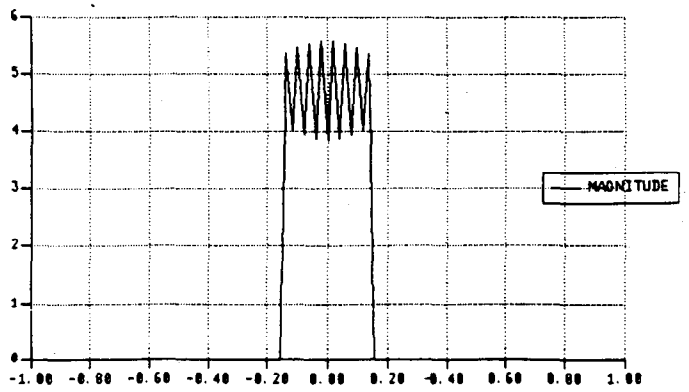


Fig. 11.

is expected. Fig. 7 shows that the function is composed of 6 cosinusoids and a constant. Although the inclusion of a component at $7\Delta_f^{(2)}$ Hz is allowed, its presence is difficult to observe in either the time or frequency plots, as the time plot displays a total of twelve peaks and dips. One should note that the definition of C_1 implies that Fig. 7 is the entire DFT of Fig. 6, since elements of C_1 have real, positive transforms.

The C_2 function, after three iterations, is displayed in Fig. 8, while its DFT appears in Fig. 9. The time function is the product of $r_{T/2}(t)$ and the C_1 function after two iterations. The function is evenly symmetric about the

origin and real. Its DFT is, therefore, real. The appearance of Fig. 9 is typical of the result of the application of a time truncation operator to a bandlimited signal.

Figs. 10 and 11 represent the C_1 function, after fifty iterations, in the time and frequency domains, respectively. The most noticeable difference between this time function and the C_1 function after three iterations is the inversion of the central lobe about the time points -4 s and 4 s. The function now has a total of 14 peaks and dips, and, as is to be expected, the frequency corresponding to $7\Delta_f^{(2)}$ Hz is now in evidence in the frequency plot. The ordering of the relative amplitudes of the various frequency components

SET 1 FUNCTION IN TIME (SEC) DOMAIN, AFTER 0050 ITERATIONS

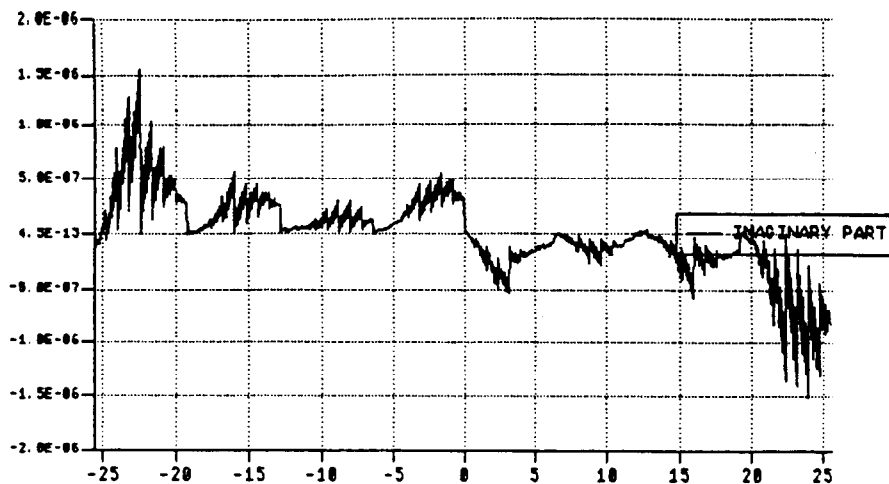


Fig. 12.

SET 2 FUNCTION IN TIME (SEC) DOMAIN, AFTER 0050 ITERATIONS

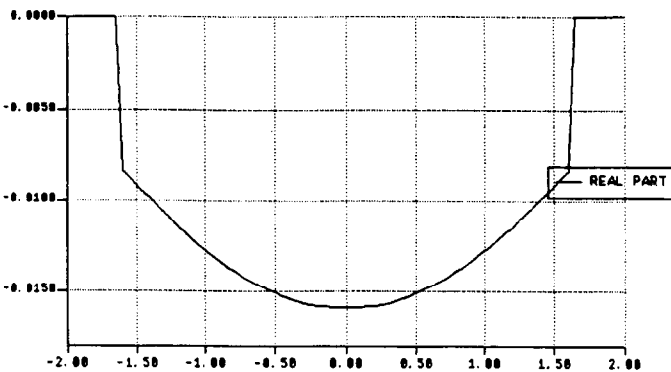


Fig. 13.

SET 2 FUNCTION IN FREQUENCY (HZ) DOMAIN, AFTER 0050 ITERATIONS

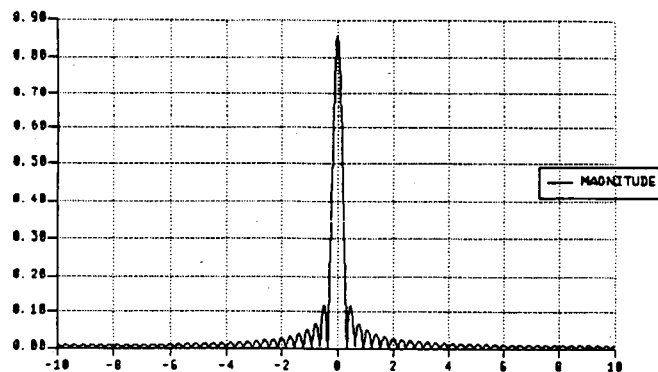


Fig. 14.

has also been reversed from that at the third iteration; and the relative amplitudes of these components has equalized somewhat.

As evidence of the propriety of the discrete implementation of the iterative technique in the program CONVEX, Fig. 12—the imaginary part of the C_1 function after 50 iterations—has been included in this section. The peak amplitude in Fig. 12 is more than 90 dB below the peak of the corresponding real function, Fig. 10, and is well within the limits of computational “noise”. The fact that this data appears to have some structure is not surprising in light of the iterative nature of the synthesis technique.

Figs. 13 and 14 depict the C_2 function, after 50 iterations, in the time and frequency domains, respectively. The primary differences between this time function and the C_2 function after three iterations are its negativity—corresponding to the inversion of the central lobe in the fiftieth iterate in C_1 —and its increased curvature. In the frequency domain, the key change is the increased ratio of the main lobe amplitude to the amplitudes of the sidelobes. For the third C_2 iterate, the ratio of the main lobe’s amplitude to that of the first sidelobe is approximately 6.0, while for the 50th iteration this ratio is about 7.2.

Fig. 15 is the magnitude of the DFT of the 50th C_2 iteration, on an expanded abscissa; and Fig. 16 is the magnitude of the DFT of a 65-point rectangular window, whose amplitude in time was normalized to yield the same main frequency lobe amplitude. The main frequency lobe of the C_2 function appears to be slightly wider than that of the rectangular window. The side lobe amplitudes of the C_2 function are, however, approximately 2/3 of the values of those of the rectangular window.

The efficacy of the fiftieth C_2 iterate as a data window will now be evaluated. As stated earlier, the window will be formed by retaining the values of this C_2 function whose time abscissas run from $-T/2$ up to, but excluding, the point corresponding to time $T/2$. The performance of this window will be compared with that of the rectangular window, the Hanning window and the Hamming window [24]. These particular windows were selected because they form a representative collection of frequently used windows. The rectangular window is known for its good resolvability—narrow main lobe—and poor leakage properties—high side lobe amplitudes. The Hanning window has just the opposite properties—low leakage, but poor resolvability. The Hamming window attempts to attain a balance of

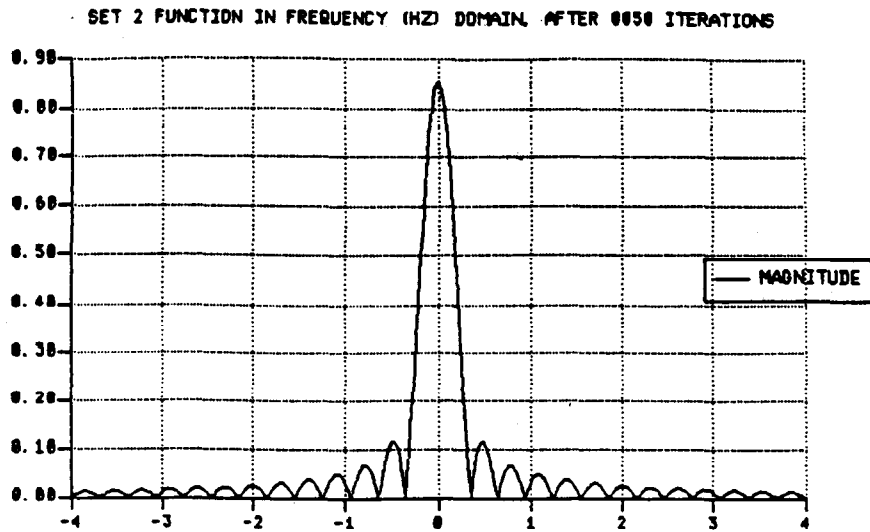


Fig. 15.

FREQUENCY (HZ) DOMAIN
Transform of Rectangular Pulse of Same Time Duration as Fiftieth C_2 Iterate

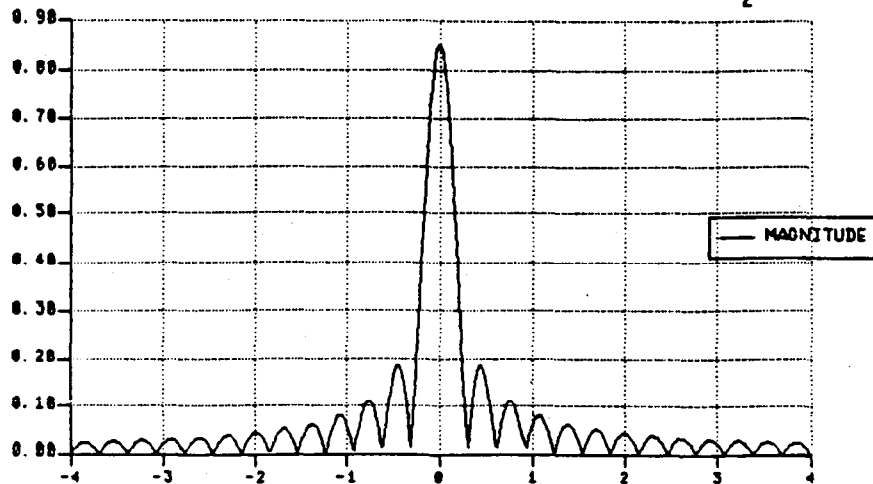


Fig. 16.

leakage and resolvability and is, in fact, a weighted average of the Hanning and rectangular windows. For a sequence of length N , the Hanning and Hamming windows are, respectively, defined to be

$$\frac{1}{2} \left[1 - \cos \left(\frac{2\pi n}{N-1} \right) \right], \quad 0 \leq n \leq N-1$$

and

$$0.54 - 0.46 \cos \left(\frac{2\pi n}{N-1} \right), \quad 0 \leq n \leq N-1.$$

To aid in comparisons, the windows were normalized so that their main spectral lobes were of equal amplitude.

The four windows will first be compared on the basis of leakage performance. By applying the windows to a single sinusoid of frequency $16.5 \Delta_f^{(1)} = 5.15625$ Hz, with unit amplitude, the main spectral lobe for each window will be centered between adjacent frequency sample points of the 64-point DFT, and the peaks of the side lobes will be approximately aligned with the sample points. Figs. 17–20 depict this example for the generated window, the rectan-

TRANSFORM IN FREQUENCY (HZ) DOMAIN OF DATA WINDOWED BY ITERATIVE RESULT
Leakage Performance

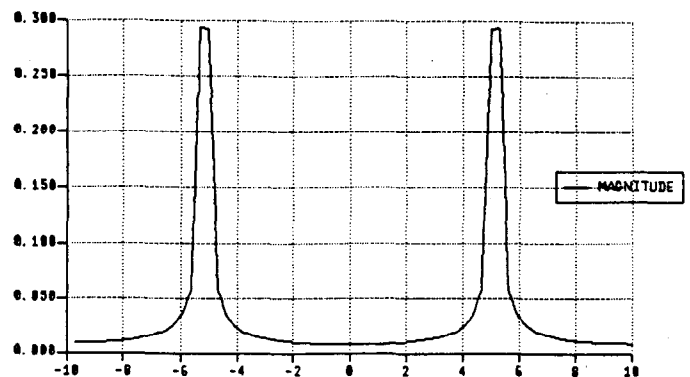


Fig. 17.

gular window, the Hanning and Hamming windows, respectively. A comparison of the peak amplitudes among the four plots reveals that the Hanning window most effectively confines the energy of the sinusoid, followed by the Hamming, the generated and the rectangular windows,

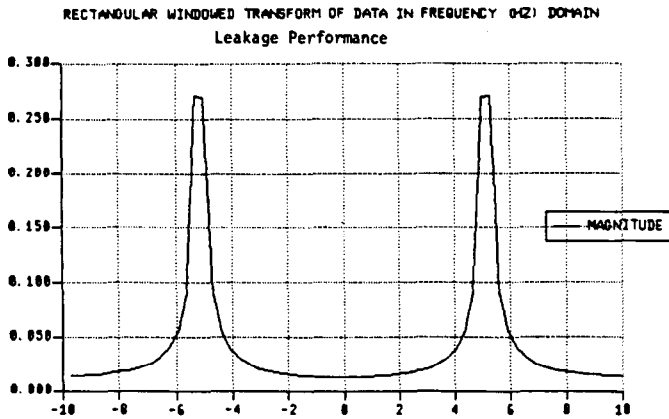


Fig. 18.

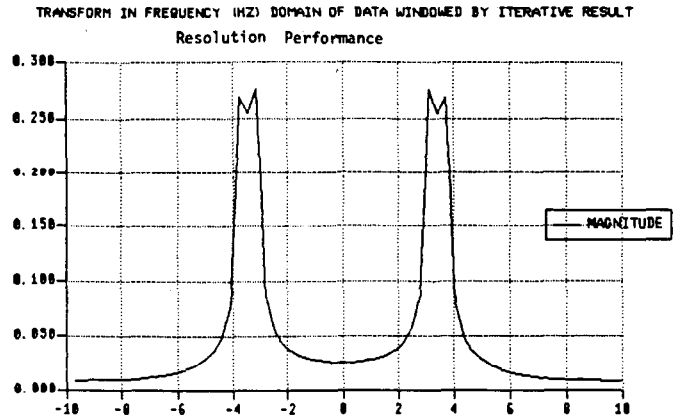


Fig. 21.

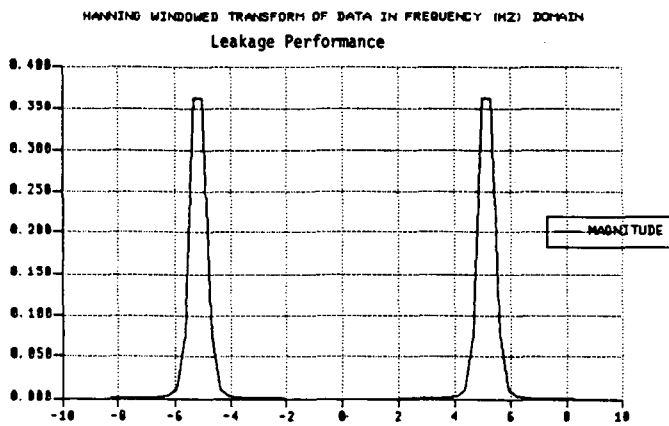


Fig. 19.

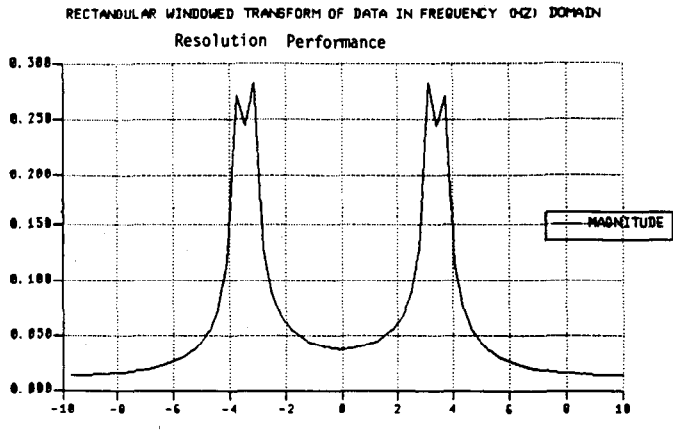


Fig. 22.

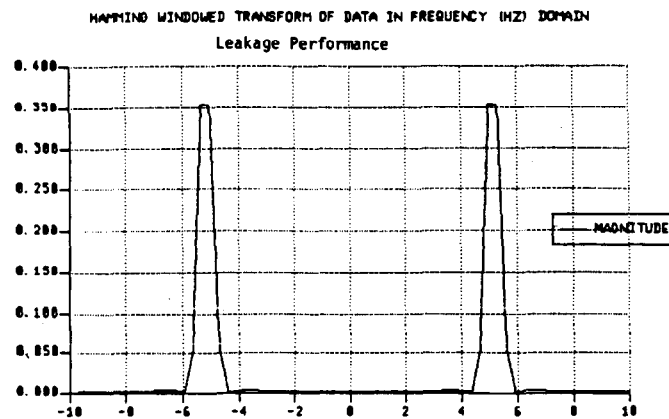


Fig. 20.

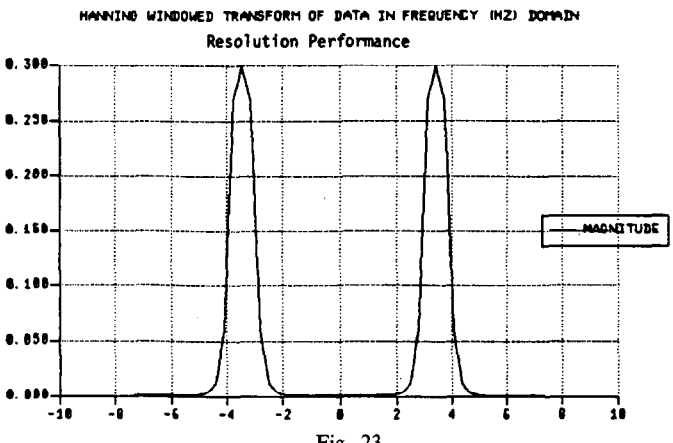


Fig. 23.

respectively. On the basis of the lobe widths at the 0.050 amplitude level, the generated window may be observed to have better performance, with respect to leakage in the vicinity of the main lobe, than the rectangular window. In this respect, the generated window may also be seen to have performance equal to that of the Hanning window, and nearly so good as that of the Hamming window.

The resolvability of the various windows is contrasted in Figs. 21-24 (generated, rectangular, Hanning and Hamming, respectively). The windows were applied to two closely spaced, unit amplitude, sinusoids—one at $10.62\Delta_f^{(1)} = 3.31875$ Hz and the other at $11.38\Delta_f^{(1)} = 3.55625$ Hz. The generated window may be observed to perform nearly so

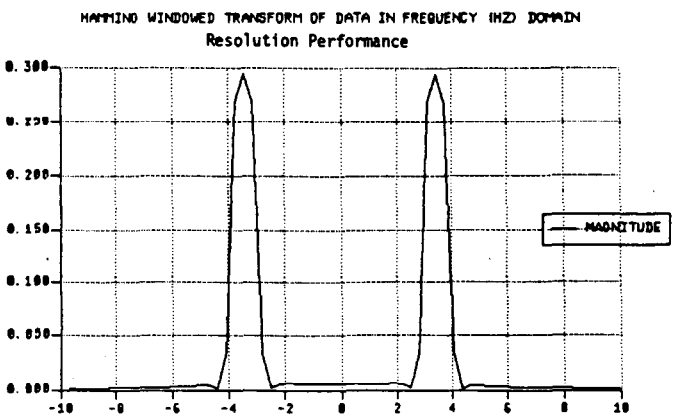


Fig. 24.

TRANSFORM IN FREQUENCY (HZ) DOMAIN OF DATA WINDOWED BY ITERATIVE RESULT
Combined Resolution and Leakage Performance

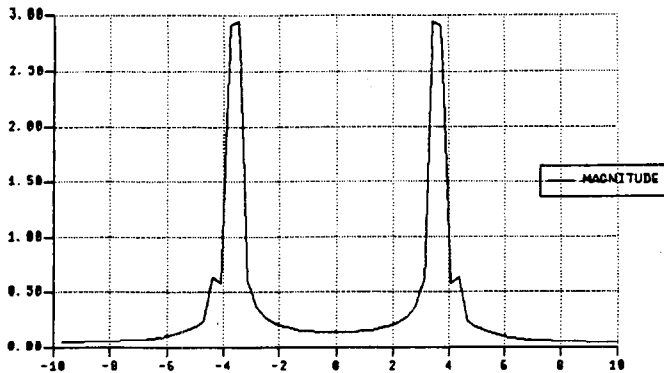


Fig. 25.

HAMMING WINDOWED TRANSFORM OF DATA IN FREQUENCY (HZ) DOMAIN
Combined Resolution and Leakage Performance

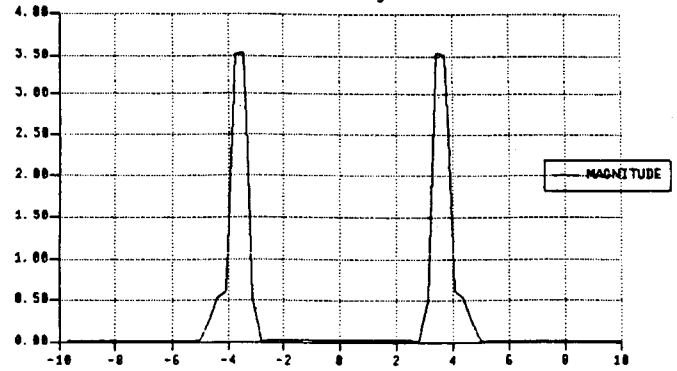


Fig. 28.

RECTANGULAR WINDOWED TRANSFORM OF DATA IN FREQUENCY (HZ) DOMAIN
Combined Resolution and Leakage Performance

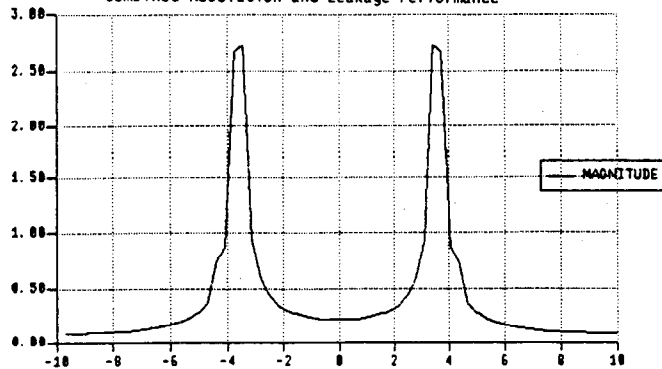


Fig. 26.

HANNING WINDOWED TRANSFORM OF DATA IN FREQUENCY (HZ) DOMAIN
Combined Resolution and Leakage Performance

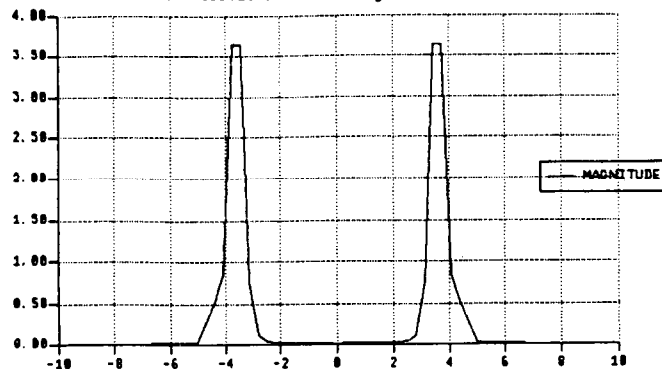


Fig. 27.

well as the rectangular window in this respect, while the Hanning and Hamming windows both fail to distinguish the two frequencies.

The third comparison among the windows combines the considerations of leakage and resolvability. The windows were applied to a data sequence comprised of an amplitude 10 component at $11.5\Delta_f^{(1)} = 3.59375$ Hz, and another sinusoid of amplitude 1.25 at $14\Delta_f^{(1)} = 4.375$ Hz. The results of this test appear in Figs. 25–28 (generated, rectangular, Hanning and Hamming, respectively). It is apparent that the window generated by the iterative synthesis technique is the only one of the four to clearly distinguish the presence of a discrete lower amplitude frequency compo-

nent. Of the remaining three, the performance of the Hamming window is the most appealing, as would be expected.

It is hoped that this example has demonstrated the utility of the synthesis technique of this paper. By satisfying a constraint of finite time duration, while nearly satisfying a constraint upon the spectral content of the signal, a data window was generated whose ability to resolve proximal frequencies of greatly varying amplitudes is superior to that of three commonly employed windows.

IV. REMARKS

In the course of any research endeavor, the treatment of certain germane topics will be precluded by limitations upon the available time and space. The intent of Section IV is to remark upon some of those topics and to mention aspects of the synthesis technique of this paper which may benefit from further research.

The reader may recall that Theorem 3 of Section II established that for each $(x_1, x_2) \in D_p$, $\|x_1 - x_2\| = d(C_1, C_2)$ —with the assumption that the underlying Hilbert Space was defined over the scalar field of real numbers. It would clearly be desirable to extend the Theorem to Hilbert Spaces which are defined over the scalar field of complex numbers and, hence, to make possible the synthesis of complex-valued functions within the context of a single space. In its present form, Theorem 3 necessitates a product space structure, of the type mentioned in the discussion preceding that theorem, in order that it be applicable to complex-valued functions. To effect such an extension, it will be, in the authors' opinion, necessary to change the manner in which the Theorem is proven. The current proof relies, to a significant extent, upon the sign of the inner product of certain elements in the space—a notion which becomes tenuous when the inner product is allowed to assume complex values. Theorem 3 is the only result of Section II which is dependent upon the scalar field over which the underlying space is defined.

It is the authors' belief that the utility of the synthesis technique of this paper will be conditional upon the construction of a more extensive collection, than that which is currently available, of convex sets corresponding to meaningful signal constraints, and of their corresponding nearest point maps. Although the presentation of this

paper was phrased in terms of signal synthesis, the technique may find application in the more general class of constrained optimization problems, in which the severity of the various constraints generates an inconsistency. Some examples might include the design of control systems and the design of manufacturing processes. As in the case of signal synthesis, however, the identification of convex sets corresponding to meaningful design constraints, and of their associated nearest point maps, will be a prerequisite.

The discussions in this paper were confined to the case of two, disjoint, closed convex sets. Consider the case of N , mutually disjoint, closed convex sets $\{C_1, \dots, C_N\}$, with respective projection operators P_1, \dots, P_N . Suppose that C_1 is bounded, and form the composite operator $P = P_1 \circ \dots \circ P_N$. Then, by Theorems 2 and 4 of Section II, respectively, the set of fixed points of P is nonempty and $P_\lambda = \lambda I + (1 - \lambda)P$, $0 < \lambda < 1$, is asymptotically regular. Furthermore, by Theorem 5, the sequence $\{P_\lambda^n(Px)\}_{n=1}^\infty$ is weakly convergent to a fixed point of P for an arbitrary element x of the underlying Hilbert Space. Thus the formal mathematical extension of the technique of this paper, to any finite number of mutually disjoint, closed convex sets, is feasible. The questions which must be resolved, in this case, are those of a physical interpretation for the results of an application of the technique. Specifically, the relationship of the fixed points of P to the elements of the remaining $N - 1$ sets must be ascertained. In the case of two disjoint sets, if x_1 was an element of the set of fixed points of $P = P_1 \circ P_2$, it was demonstrated that $\|x_1 - P_2 x_1\|$ was minimized. In the general case, preliminary results have shown that

$$\|x_1 - P_N x_1\| + \|P_N x_1 - (P_{N-1} \circ P_N) x_1\| + \dots \\ + \|(P_2 \circ \dots \circ P_N) x_1 - (P_1 \circ \dots \circ P_N) x_1\|$$

where x_1 is a fixed point of $P = P_1 \circ \dots \circ P_N$, is not necessarily equal to the minimal length of a closed polygonal path with a single vertex in each set.

ACKNOWLEDGMENT

The authors are indebted to Prof. Victor Klee, of the Mathematics Department at the University of Washington, Seattle, for suggesting the scheme of the proof for Theorem 3 of Section II. They are also grateful for the assistance provided by Prof. Mark Damborg, of the University of Washington's Electrical Engineering Department, in the formulation of the results of Section II.

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