Robert J. Marks II
Editor

# Advanced Topics in Shannon Sampling and Interpolation Theory

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With 91 Figures

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