Evaluation of stochastic-resonance-based detectors of weak harmonic signals in additive white Gaussian noise

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A thorough evaluation of stochastic resonance in the framework of statistical detection theory is presented both as a nonlinear signal preprocessor and as a detector. The pertinent receiver operating characteristics are compared with those of the known statistically optimum detector using extensive Monte Carlo simulations. Parameter optimization and computational budget aspects are discussed. [S1063-651X(98)08306-8]

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I. INTRODUCTION

The stochastic resonance (SR) concept was introduced in the early 1980s by Benzi [1] as a typical feature of bistable [2] systems driven by an additive [4] mixture of a time-harmonic [8] signal and a Gaussian white [12] noise. For these systems the output signal-to-noise ratio (SNR) is counterintuitively found to *increase* upon increasing the input noise level, up to a (broad) maximum.

SR was subsequently found to occur in Schmitt triggers (threshold and hysteresis systems) [22], monostable systems [23], threshold systems with a deterministic return to the rest state [24], level crossing detectors (LCDs) [25–29], so-called nondynamical nonthreshold detectors (random-pulse generators with exponentially input-dependent rates) [32–34], chaotic systems [36–38], *static* memoryless nonlinear devices [39], bistable systems with a fluctuating barrier height and no additive noise [40], and even *linear* systems with multiplicative noise [41]. As a result, in a few years SR emerged gradually as a paradigm whose universal character was shown [42–45] to be intimately related to the pervasiveness of the fluctuation-dissipation theorem [46].

The SR paradigm has drawn considerable attention in such diverse fields as climatology [47–50]; chemistry [51–53]; laser physics [54–56]; neuroscience, including single-neuron [57–60] and many-neuron models [61–63]; biophysics and physiology [64–75]; particle accelerators [76]; solid-state physics, including bistable magnetic systems [77,78], electron paramagnetic resonance [79,80], ferroelectrics [81] and ferromagnetics [82], fluorescence [83], Ising systems [84–87], Josephson junctions [88], superconducting quantum interference device loops [89,91], and kink-antikink systems [92], Landau-Ginzburg models [93], mean-field models [94,95], mesoscopic systems [96], spin waves [97], superparamagnetic particles [98,99], and tunnel diodes [100]; and even sociology [101], as witnessed by several topical meetings [102–104]. The pertinent literature is indeed still in the exponentially growing stage, as can be seen from the bibliographical database maintained by Gammaitoni *et al.* [151]. The possible use of SR in connection with (weak) signal detection experiments, with special reference to gravitational waves, was suggested since the infancy of SR [105]. A remarkable body of analytical, numerical, analog-simulation, and experimental results support the evidence of both power gain (at the signal frequency) and SNR gain in a variety of SR systems, including bistable (see, e.g., [106,107]), threshold (see, e.g., [27]), and static nonlinear devices [39].

It should be mentioned that some results in the technical literature should be taken with caution, the reported SNRs being either defined in nonstandard ways [27], badly computed [108], or almost irrelevant (see the discussion in [109]) to the weak signal detection issue [110,111]. Furthermore, for nonlinear systems both the power gain and the output SNR are inherently rather *ill defined* [107,113], except in the weak driving regime, where it has been rigorously proved that even in SR systems the output SNR is always less than the input one [115].

The fundamental question is whether SR could be used to construct *better* detectors, the benchmark being represented by the well-known (statistically optimum) Neyman-Pearson detectors (ONPDs) for signals embedded in additive Gaussian noise [116,117]. The answer to this question can be obtained by a comparison of the pertinent receiver operating characteristics (ROCs) [116,117].

Inchiosa and Bulsara pointed out this issue [114]. In contrast to earlier beliefs, they found that ROCs obtained from the output of (single as well as multiple, coupled) first-order bistable systems driven by time-harmonic signals in additive white Gaussian noise were always *worse* than those obtained from the input [118].

In this paper we go one step further by showing in Sec. III that applying the ONPD to the *output* of a *general* first-order nonlinear system consisting of a time-harmonic signal with known frequency and initial phase in additive white Gaussian noise is *strictly equivalent* to applying the ONPD to the plain input signal and noise mixture. Thus *no* gain should be expected from using a (first-order) SR signal preprocessor, although one might positively speculate that SR could be used to *recover* in part the SNR degradation due to *unwanted* but otherwise unavoidable nonlinearities along the signal processing chain [114].

However, it might be worth looking at SR systems not as signal preprocessors but as detectors where one searches for some *characteristic* SR signature depending on the presence

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of the signal. For the simplest SR systems studied in this paper, the chosen signature is the (time-harmonic) symmetry breaking of the output probability density function (PDF) (see Sec. IV) [119,122]. For these *symmetry-breaking detectors* we introduce and characterize in terms of ROCs both parametric and nonparametric detection strategies, in Sec. V. The main finding is that in the incoherent (*unknown* initial phase) case, the non-parametric detector admits a simple implementation whose performance is nearly as good as that of the ONPD, with a substantial saving in computational budget (see Sec. VI).

II. STOCHASTIC RESONANCE: HEURISTICS

This section is intended as an introduction to the subject and can be skipped by the more expert reader. We consider a one-dimensional nonlinear stochastic system, described by the Langevin equation

$$\dot{x}=-\frac{d}{dx}V(x)+A\sin(\omega_s t+\phi)+\epsilon n(t),\qquad x(0)=x_0, \tag{1}$$

where $V(x)$ is a quartic potential

$$V(x)=-a\frac{x^2}{2}+b\frac{x^4}{4},\quad a,b>0, \tag{2}$$

having two stable stationary points at $x_m^{\pm}=\pm\sqrt{a/b}$, an unstable one at $x=0$, and a barrier height

$$V_0=\frac{a^2}{4b}. \tag{3}$$

The above quartic bistable potential Langevin equation will henceforth be referred to as QPLE. The noise $n(t)$ in Eq. (1) is assumed as stationary, zero mean, white, and Gaussian, with

$$E[n(t)n(t+\tau)]=\delta(\tau). \tag{4}$$

The PDF $p(x,t)$ of the stochastic process $x(t)$ in Eq. (1) is ruled by the Fokker-Planck equation [126,127]

$$\frac{\partial p(x,t)}{\partial t}=\frac{\partial}{\partial x}\left\{\left[\frac{dV(x)}{dx}-A\sin(\omega_s t+\phi)\right]p(x,t)\right\}+\frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}p(x,t),\qquad p(x,0)=\delta(x-x_0). \tag{5}$$

In the general case ($A\neq 0$, $\epsilon\neq 0$) this equation cannot be solved in the closed form [128]. In the absence of noise ($\epsilon=0$) and with a *subthreshold* ($A<V_0$) signal, $x(t)$ oscillates forever in a neighborhood of either stable stationary point (no transition allowed). In the presence of noise only ($A=0$), the response $x(t)$ fluctuates in the neighborhood of $x_m^{\pm}$, jumping at random times between the two wells. The first-passage time from either well to the unstable equilibrium point is a random variable whose mean (Kramers time) is given exactly by the double integral [126]

$$T_K=\frac{2}{\epsilon^2}\left|\int_0^{x_m^{\pm}}dy\,\exp[2V(y)/\epsilon^2]\times\int_{-\infty}^{y}\exp[-2V(z)/\epsilon^2]dz\right|, \tag{6}$$

which can be evaluated under suitable approximations [139,140]. The simple estimate

$$T_K=\frac{\sqrt{2}\pi}{a}\exp\left[\frac{2V_0}{\epsilon^2}\right] \tag{7}$$

provides a nice approximation in the useful range of parameters [141].

In the presence of both signal and noise, jumping between the two potential wells is possible even for a subthreshold input signal. It can be heuristically realized that, under suitable conditions, jumping may *lock* to the time-harmonic forcing term. It was initially suggested [1] that such a behavior would result from the matching of the forcing period $T_s=2\pi/\omega_s$ to twice the Kramers time

$$T_s\sim 2T_K. \tag{8}$$

The original SR definition (8), which resembles a *classical* resonance, has suffered several criticisms. Gammaitoni and co-workers introduced *alternative* definitions, more directly interpretable as a bona fide resonance [142,143].

III. OPTIMUM DETECTION AT THE OUTPUT OF A SR DEVICE

Here we consider the strict detection problem (the sought signal is completely known) and formulate the optimum detection strategy for continuous observation of the output of a general first-order nonlinear system consisting of a time-harmonic signal with known frequency and initial phase in additive stationary white ($E[n(t)n(t+\tau)]=\delta(\tau)$) noise (a class to which the QPLE belongs), viz.,

$$\dot{x}=a(x)+s(t)+\epsilon n(t),\quad t\in[0,T],\qquad x(0)=x_0. \tag{9}$$

Introducing time discretization with step size Δ, one has

$$\begin{gathered}x_{k+1}-x_k=\Delta[a(x_k)+s_k+\epsilon n_k],\quad k=0,1,\ldots,N-1,\\ N=\lfloor T/\Delta\rfloor,\\ x_k=x(k\Delta),\quad s_k=s(k\Delta),\quad n_k=n(k\Delta).\end{gathered} \tag{10}$$

The ONPD [116,117] for the set of observations x_k, $k=0,1,\ldots,N-1$, is based on the *maximum likelihood ratio* test, viz.,

$$\begin{gathered}l(x_0,\ldots,x_{N-1})\geq\Gamma\Rightarrow\text{signal present},\\ l(x_0,\ldots,x_{N-1})<\Gamma\Rightarrow\text{no signal present},\end{gathered} \tag{11}$$

where

$$l(x_0,\ldots,x_{N-1})=\frac{p(x_0,\ldots,x_{N-1}|\text{signal})}{p(x_0,\ldots,x_{N-1}|\text{no signal})}, \tag{12}$$

$p(\ )$ being the joined PDF and Γ a suitable threshold. Using the Euler approximation [144] for the involved PDFs, one has [127]

$$p(x_0,\ldots,x_{N-1}|\text{no signal}) =(2\pi\Delta\epsilon^2)^{-N/2}\prod_{k=0}^{N-1}\exp\left\{-\frac{[x_{k+1}-x_k-a(x_k)\Delta]^2}{2\Delta\epsilon^2}\right\}, \tag{13}$$

$$p(x_0,\ldots,x_{N-1}|\text{signal})=(2\pi\Delta\epsilon^2)^{-N/2} \times\prod_{k=0}^{N-1}\exp\left\{-\frac{[x_{k+1}-x_k-a(x_k)\Delta-s_k\Delta]^2}{2\Delta\epsilon^2}\right\}. \tag{14}$$

Hence

$$\ln[l(x_0,\ldots,x_N)]=\frac{1}{\epsilon^2}\sum_{k=0}^{N-1}\{s_k[x_{k+1}-x_k-a(x_k)\Delta]-\tfrac{1}{2}s_k^2\Delta\}. \tag{15}$$

Letting $\Delta\rightarrow 0$ and taking Eq. (10) into account, Eq. (11) can be recast as

$$\frac{1}{\epsilon^2}\int_0^T s(t)z(t)dt\geqslant\ln\Gamma+\frac{1}{2}\frac{E_s}{\epsilon^2}\Rightarrow\text{signal present},$$
$$\frac{1}{\epsilon^2}\int_0^T s(t)z(t)dt<\ln\Gamma+\frac{1}{2}\frac{E_s}{\epsilon^2}\Rightarrow\text{no signal present}, \tag{16}$$

where

$$z(t)=s(t)+\epsilon n(t),\quad E_s=\int_0^T s^2(t)dt. \tag{17}$$

The above decision rule reproduces the ONPD (coherent correlator [117]) applied to the *input* signal plus noise mixture. Thus, if the process is observed continuously, insertion of a SR filter (or, more generally, a first-order nonlinear system) does *not* improve the detection performance.

IV. PDF SYMMETRY BREAKING

In the absence ($A=0$) of the time-harmonic driving signal, the steady-state solution of Eq. (5) is [126]

$$p_{ss}(x)=C\exp\left[\frac{\frac{ax^2}{2}-\frac{bx^4}{4}}{\epsilon^2/2}\right], \tag{18}$$

shown in Fig. 1, where C is a normalization constant. The two peaks centered at $x_m^{\pm}$ have equal widths, which increase with the noise rms value ϵ. Switching the sinusoidal forcing term on produces a *modulated-in-time symmetry breaking* of the PDF. This is most easily described in the asymptotic *adiabatic* limit $T_s\gg T_K$, where Eq. (5) has the solution [126]

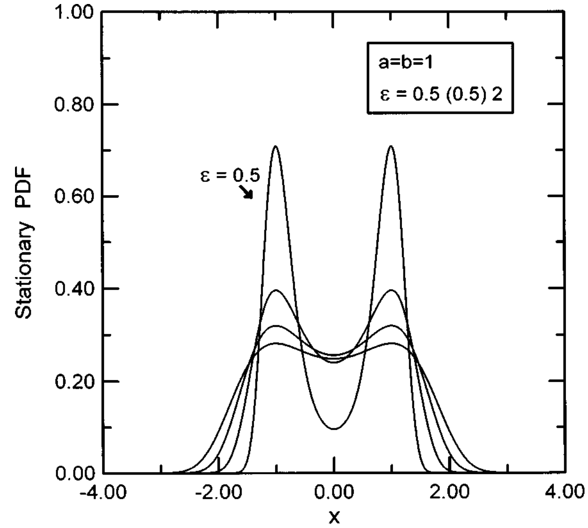

FIG. 1. Stationary (steady-state) output PDF relevant to the QPLE, in the absence of a signal.

$$p(x,t)\sim C(t)\exp\left[\frac{\frac{ax^2}{2}-\frac{bx^4}{4}+Ax\sin(\omega_s t+\phi)}{\epsilon^2/2}\right], \tag{19}$$

$C(t)$ now being a (time-dependent) normalization factor. Floquet theory confirms that an oscillating symmetry breaking of the PDF occurs at the same frequency of the driving signal for *all* values of the ratio T_s/T_K [134].

In the following subsection we shall discuss in terms of ROCs detection algorithms based on the possibly simplest PDF symmetry-breaking indicators, viz.,

$$P_+(t)=\text{Prob}\{x(t)>0\}=\int_0^\infty p(x,t)dx \tag{20}$$

and

$$E(t)=E[x(t)]=\int_{-\infty}^{\infty}x\,p(x,t)dx. \tag{21}$$

In the absence of a signal $P_+=0.5$, $E=0$. Typical time evolutions of P_+ and E, obtained from numerical Monte Carlo [144] simulations, are displayed in Figs. 2 and 3. Irrespective of the initial conditions, after a short transient *both* P_+ and E *lock* to the signal, but for some (constant) phase lag [145], which vanishes in the adiabatic limit.

QPLE parametrization, resonant features, and optimization

Before proceeding further, to optimize performance, it is convenient to introduce dimensionless parameters and variables as follows: The normalized angular frequency is

$$\overline{\omega}_s=\omega_s T_K, \tag{22}$$

The normalized potential-barrier height is

$$\overline{V}_0=\frac{V_0}{\epsilon^2}, \tag{23}$$

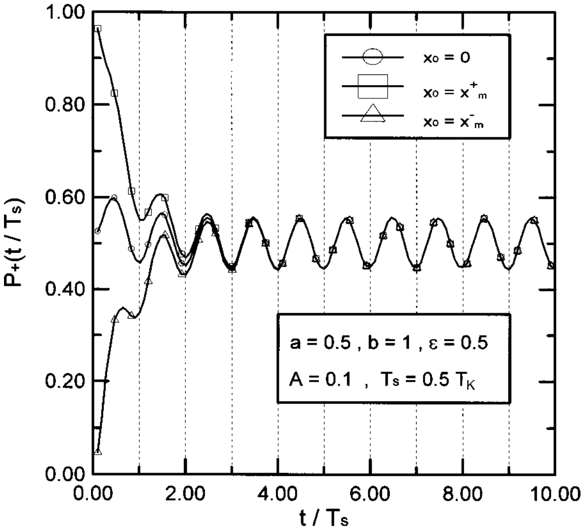

FIG. 2. Time behavior of the asymmetry indicator P_+, in the presence of an input harmonic signal, for three different initial conditions. The vertical dotted lines correspond to the zeros of the input signal.

and the signal-to-noise ratio $\mathcal{R}$ is

$$\mathcal{R}=\sqrt{\frac{E_s}{\epsilon^2}}=\sqrt{\frac{NT_sA^2}{2\epsilon^2}}, \tag{24}$$

E_s being the signal energy and N the number of observed cycles,

$$\overline{x}=\frac{x}{\epsilon\sqrt{T_K}},\quad \overline{t}=\frac{t}{T_K},\quad \overline{a}=aT_K=\sqrt{2}\pi\ \exp\,(2\overline{V}_0).$$

Equation (1) can be recast as

$$\dot{\overline{x}}=\overline{a}\overline{x}-\frac{\overline{a}^2}{4\overline{V}_0}\overline{x}^3+\mathcal{R}\sqrt{\frac{\overline{\omega}_s}{N\pi}}\sin(\overline{\omega}_s\overline{t}+\phi)+n(\overline{t}),$$

$$\overline{x(0)}=\overline{x}_0. \tag{25}$$

The (steady-state) values of $\max_t[P_+(t)]$ and $\max_t[E(t)]$, are displayed in Figs. 4 and 5, respectively, as functions of the normalized (angular) frequency ω_sT_K, for several values of $\mathcal{R}$ at fixed $\overline{V}_0=0.25$. Both show a resonant behavior, being maximum at

$$\overline{\omega}_s\sim\pi,\quad \text{i.e.,}\quad T_s\sim 2T_K, \tag{26}$$

which is the original Benzi SR condition (8). The numerical results merge smoothly into their adiabatic limits as $\overline{\omega}_s$ tends to zero, as seen in Figs. 4 and 5. The phase lag between the maxima of $P_+(t)$, $E(t)$, and the input signal turns out to be

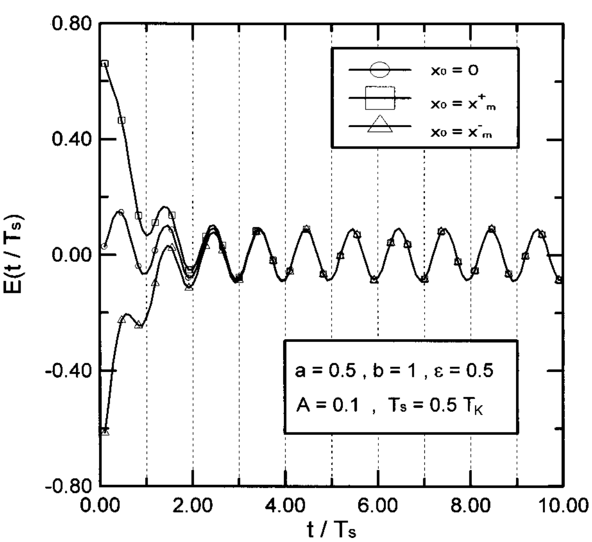

FIG. 3. Time behavior of the asymmetry indicator E, in the presence of an input harmonic signal, for three different initial conditions. The vertical dotted lines correspond to the zeros of the input signal.

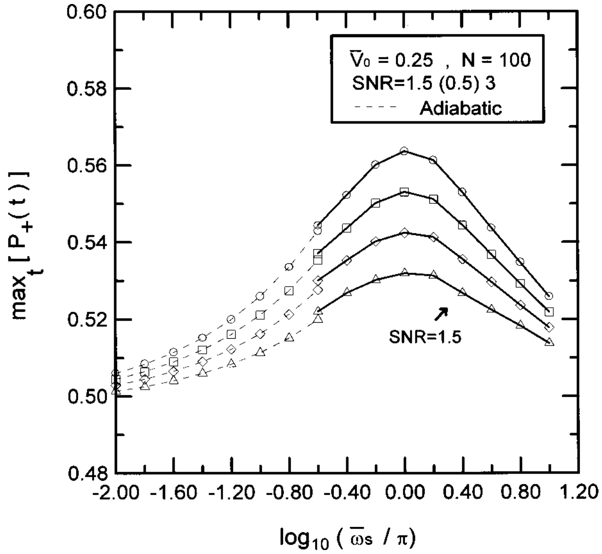

FIG. 4. $\max_t[P_+(t)]$ frequency response of the SR detector for various values of the SNR. Dashed curves refer to the adiabatic approximation.

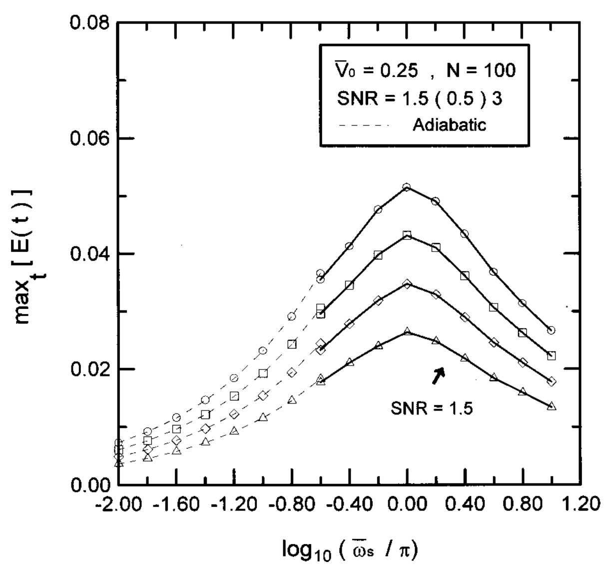

FIG. 5. $\max_t[E(t)]$ frequency response of the SR detector for various values of the SNR. Dashed curves refer to the adiabatic approximation.

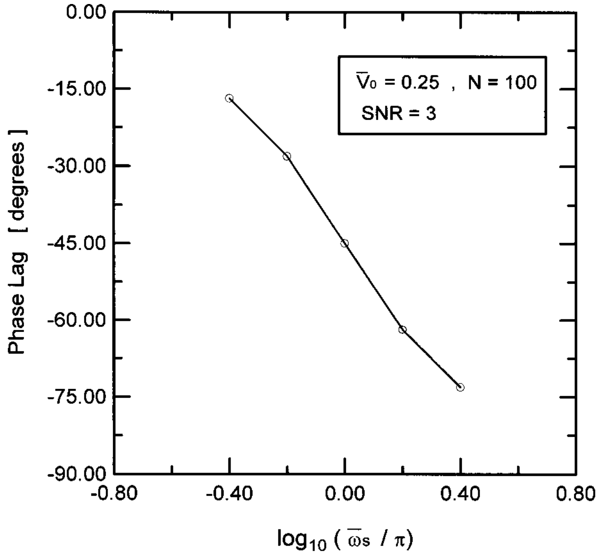

FIG. 6. Phase lag of $\max_t[P_+(t)]$,$\max_t[E(t)]$ vs the normalized angular frequency.

the same, is almost independent of the $\mathcal{R}$ value (see Fig. 6), and vanishes in the adiabatic limit. Changing $\bar{V}_0$ affects only the resonance peak height, as shown in Figs. 7 and 8.

For the best PDF symmetry-breaking detection, one should look for the largest possible value of $\max_t[P_+]$ and $\max_t[E]$. Accordingly, for a given signal frequency ω_s and noise power spectral density ϵ^2, one should enforce Eq. (26) and determine $\bar{V}_0$ according to Figs. 7 and 8. Then, using Eqs. (3) and (7), the optimum potential parameters a,b can be readily obtained.

V. PDF SYMMETRY-BREAKING DETECTORS

In this section we develop a quantitative analysis of QPLE-related PDF symmetry-breaking detectors in terms of their ROCs. We assume that the system parameters have been optimized as shown in the preceding section.

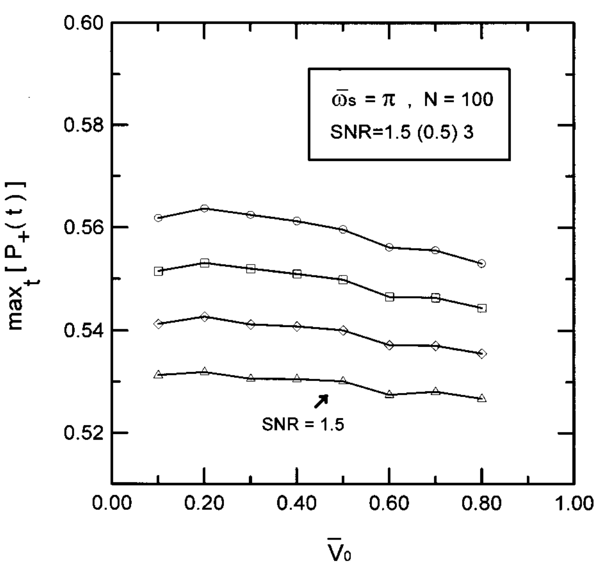

FIG. 7. $\max_t[P_+(t)]$ at resonance ($\overline{\omega_s}=\pi$) vs the normalized potential barrier height.

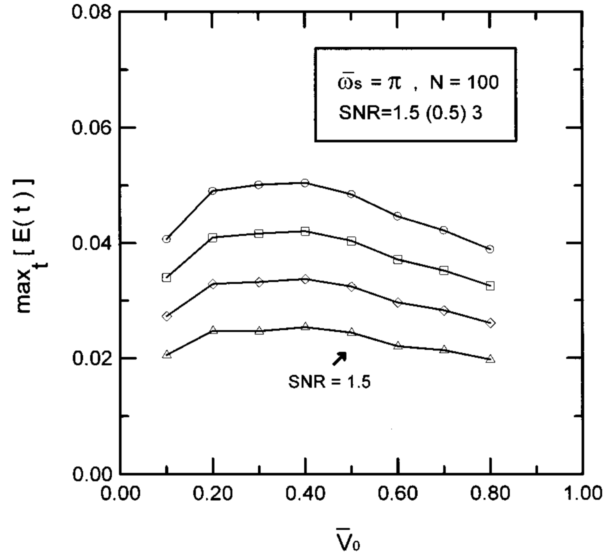

FIG. 8. $\max_t[E(t)]$ at resonance ($\overline{\omega_s}=\pi$) vs the normalized potential barrier height.

The QPLE *output* $x(t)$ is sampled throughout an interval $(0,NT_s)$ at

$$t_k=\frac{\frac{(2k+1)\pi}{2}-\psi}{\omega_s},\quad k=0,1,2,...,2N-1,\tag{27}$$

i.e., at times when the asymmetry effect induced by the signal presence is maximum, ψ being the (known) phase lag introduced by the SR processor, and the time series [147]

$$x_k=(-1)^k x(t_k),\quad k=0,1,2,...,2N-1,\tag{28}$$

is formed.

We shall first consider the case where both the angular frequency ω_s and the initial phase ϕ of the signal are known (coherent detection). We assume $\phi=0$ for simplicity.

A. Parametric strategy: Sample mean

In the parametric strategy we use the first moment (mean) μ of the samples x_k as a likelihood ratio

$$\mu=\frac{1}{2N}\sum_{k=0}^{2N-1}x_k.\tag{29}$$

One accordingly adopts the decision γ_1 (asymmetrical distribution, signal present) if and only if

$$\mu>\Gamma\tag{30}$$

and the decision γ_0 (symmetrical distribution, no signal present) if and only if

$$\mu\leqslant\Gamma,\tag{31}$$

$\Gamma\in\mathbb{R}$ being a suitable threshold. The detector's performance is described by the *false-alarm* and *false-dismissal* probabilities

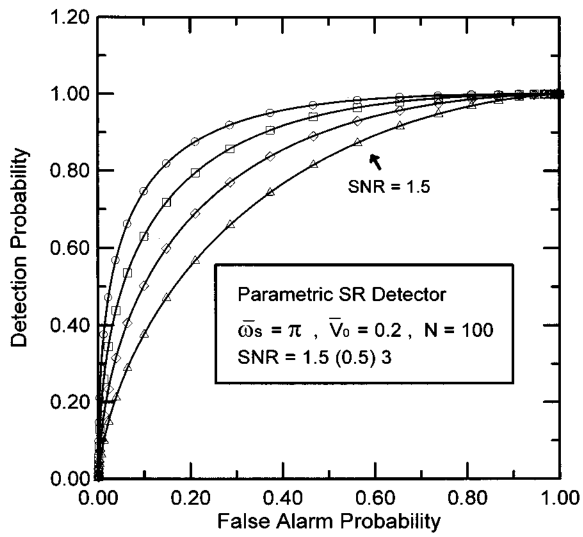

FIG. 9. Receiver operating characteristics of the parametric SR detector.

$$\alpha = \text{Prob}\{\mu > \Gamma | \text{no signal}\}, \quad \beta = \text{Prob}\{\mu \leq \Gamma | \text{signal}\}. \tag{32}$$

The stochastic variables x_k can be assumed as statistically independent [148] and are identically distributed. Hence [117]

$$\alpha = 1 - F\left(\frac{\Gamma}{\sqrt{2N}\sigma_0}\right), \tag{33}$$

$$\beta = F\left(\frac{\Gamma - 2NE_1}{\sqrt{2N}\sigma_1}\right), \tag{34}$$

where

$$E_1 = E[x_k | \text{signal}], \quad \sigma_1^2 = \text{Var}[x_k | \text{signal}],$$

$$\sigma_0^2 = \text{Var}[x_k | \text{no signal}],$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt. \tag{35}$$

In the Appendix we show that in the adiabatic assumption, this strategy is equivalent to the (optimum, strobed) Neyman-Pearson ratio test performed on the output of the SR processor. This supports the heuristic conclusion that the same strategy might be nearly optimum in the *nonadiabatic* regime as well. The input SNR parametrized ROCs ($1-\beta$ vs α) of the above-described SR detector are shown in Fig. 9. A comparison with the ROCs of the ONPD applied to the *input* signal and noise mixture (coherent correlator), displayed in Fig. 10, shows an average uniform loss of approximately 3 dB. As the coherent correlator can be cheaply implemented using an analog multiplier, the above parametric SR symmetry-breaking detector offers *no* apparent advantage over direct ONPD application to the input signal and noise mixture.

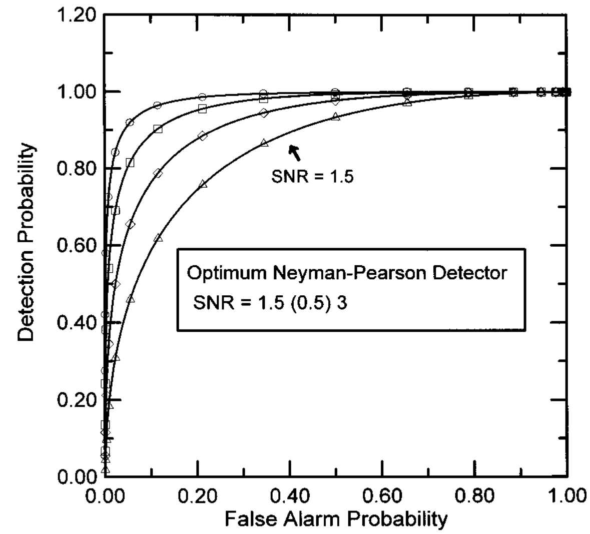

FIG. 10. Receiver operating characteristics of the ONPD.

B. Nonparametric strategy: Sign counting

The second conceivable strategy is nonparametric (see [117] for an abstract definition). We now use the total number N_+ of positive samples as a likelihood ratio

$$N_+ = \sum_{k=0}^{2N-1} U(x_k), \tag{36}$$

where $U(\ )$ is Heaviside's step function. One adopts the decision γ_1 (asymmetrical distribution, signal present) if and only if

$$N_+ > \Gamma \tag{37}$$

and the decision γ_0 (symmetrical distribution, no signal present) if and only if

$$N_+ \leq \Gamma, \tag{38}$$

where Γ is a (positive) integer threshold (unilateral symmetry-testing hypothesis [117]). The detector's performance is described by the false-alarm and false-dismissal probabilities

$$\alpha = \text{Prob}\{N_+ > \Gamma | \text{no signal}\} = I_{1/2}(\Gamma + 1, 2N - \Gamma), \tag{39}$$

$$\beta = \text{Prob}\{N_+ \leq \Gamma | \text{signal}\} = 1 - I_{P_+}(\Gamma + 1, 2N - \Gamma), \tag{40}$$

where $P_+ = \text{Prob}\{x_k > 0\}$ and $I_p(x,y)$ is the incomplete Beta function [146]. The corresponding ROCs, parametrized in terms of the input SNR, are shown in Fig. 11.

It is seen that the nonparametric (sign-counting) decision rule performs only *slightly* worse (less than 1 dB, on average) than the parametric (mean) one. On the other hand, the sign-counting strategy is computationally *very cheap* since it requires only binary and/or integer arithmetics. In the next section we will show that the sign-counting detector can be implemented with little additional burden for the noncoherent case as well.

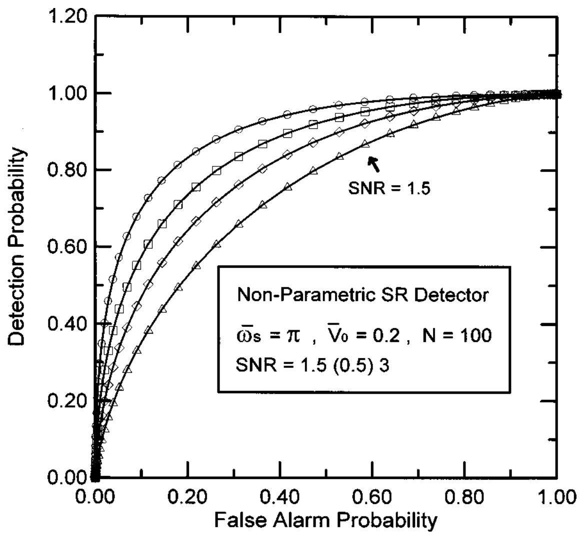

FIG. 11. Receiver operating characteristics of the nonparametric SR detector.

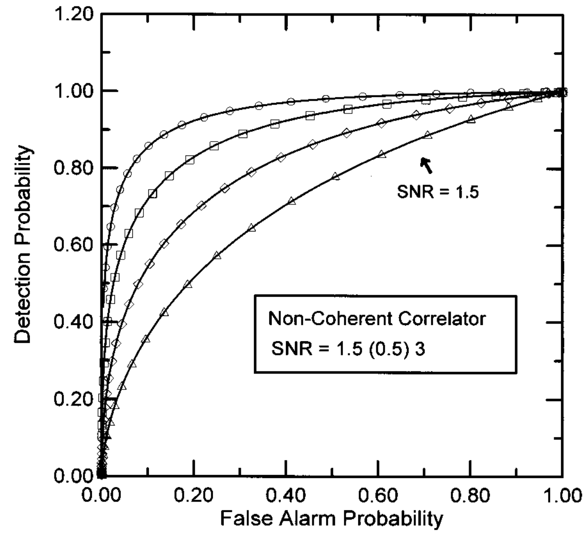

FIG. 12. Receiver operating characteristics of the NCC.

C. Nonparametric noncoherent detection

The possibly simplest (nonparametric) detector for the case where the initial phase of the signal is unknown (noncoherent detector) is obtained by taking the largest sign count (36) among those corresponding to N_s different (equispaced) starting times in a half period of the sought signal, viz., using the likelihood ratio

$$N_+ = \max_{m\in(0,N_s)} \sum_{k=0}^{2N-1} U\left[(-1)^k x\left(t_k + \frac{mT_s}{2N_s}\right)\right]. \qquad (41)$$

For sufficiently large N_s, we might expect the *same* performance as in the known initial-phase case, i.e., some 4 dB worse than the known initial-phase ONPD (coherent correlator).

The ONPD for signals with unknown initial phase is the noncoherent correlator [116,117], (NCC), whose ROCs shown in Fig. 12 display a loss of about 3 dB as compared to that of the coherent correlator. It is concluded that the noncoherent detector (41) is *nearly as good* as the NCC, while being computationally much cheaper (only binary and integer arithmetics required). The likelihood ratio (41) can be computed using, e.g., the *parallel* scheme shown in Fig. 13, which uses a single SR processor plus a sampling gate, a shift register, and N_s accumulators.

VI. CONCLUSIONS AND RECOMMENDATIONS

We presented a thorough evaluation in terms of ROCs of QPLE-SR-based detectors of weak time-harmonic signals in white Gaussian noise, where the signal-induced PDF asymmetry is observed. This complements the study by Inchiosa

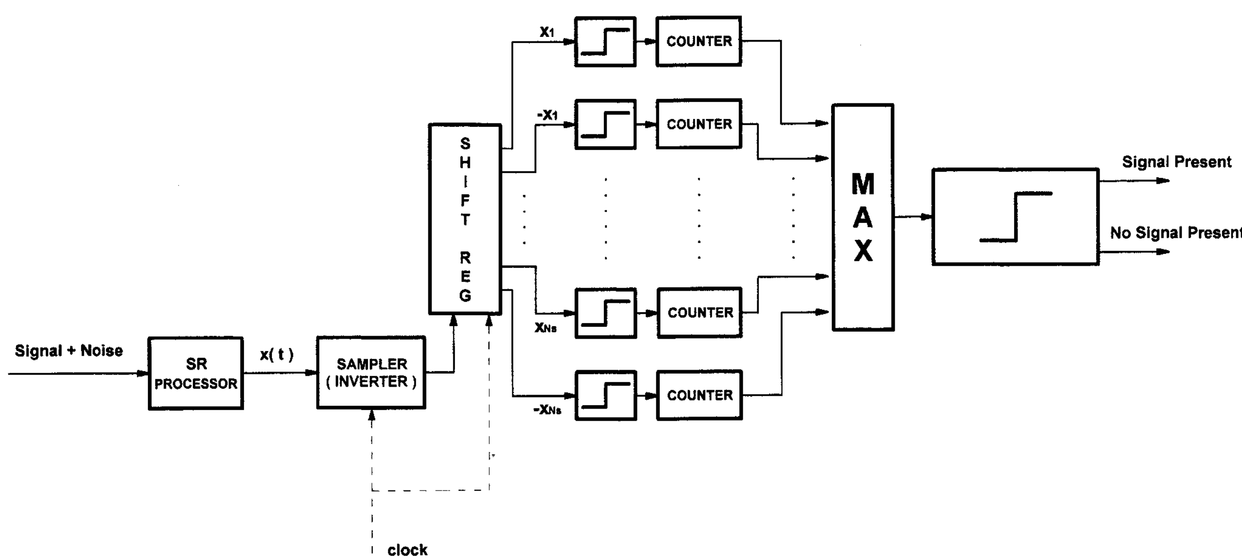

FIG. 13. Block diagram of the noncoherent SR detector, with the oversampling factor N_s.

and Bulsara [114], who worked out the case where the output PSD is observed instead. We further showed that (for continuous observation) insertion of a SR preprocessor provides *no gain* over the straightforward ONPD of the plain signal and noise mixture.

Several directions for future research are suggested. SR-detector performance in non-Gaussian and/or nonstationary noise should be investigated. A ROC analysis should be carried out for detectors where the escape-time distribution is observed. Better analog-to-digital implementations could be studied.

We stress that SR detectors should be judged not only on the basis of ROCs, but also in terms of their computational complexity, in view of getting reasonable trade-offs between statistical (detection) and computational (implementation) efficiency. In this connection, the applicability of SR-based detectors to chirping or Doppler-modulated signals, which is relevant in the context of gravitational wave physics and for which the optimum Neyman-Pearson approach is computationally quite demanding [149,150], might deserve further study.

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APPENDIX: OPTIMUM PDF–SYMMETRY-BREAKING DETECTION STRATEGY IN THE ADIABATIC REGIME

In the adiabatic regime ($T_s \gg T_K$) the random variables $x_k = x(t_k)$ are independent and identically distributed and their joint PDF can be accordingly written, using Eq. (19), as follows:

$$w(x_0,\ldots,x_{2N-1}|\text{signal}) = (C_1)^{2N} \prod_{k=0}^{2N-1} \exp\left[\frac{\frac{ax_k^2}{2} - \frac{bx_k^4}{4} + Ax_k \sin(\omega_s t_k)}{\epsilon^2/2}\right], \tag{A1}$$

$$w(x_0,\ldots,x_{2N-1}|\text{no signal}) = (C_0)^{2N} \prod_{k=0}^{2N-1} \exp\left[\frac{\frac{ax_k^2}{2} - \frac{bx_k^4}{4}}{\epsilon^2/2}\right], \tag{A2}$$

C_0, C_1 being suitable normalization constants. The (logarithm of the) likelihood ratio is therefore:

$$\ln\left\{\frac{w(x_0,\ldots,x_{2N-1}|\text{signal})}{w(x_0,\ldots,x_{2N-1}|\text{no signal})}\right\} = \left(\frac{C_1}{C_0}\right)^{2N} \sum_{k=0}^{2N-1} x_k \sin(\omega_s t_k) \propto \sum_{k=0}^{2N-1} (-1)^k x_k, \tag{A3}$$

which, is just the sample mean (sufficient statistic).

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