Restoration of Continuously Sampled Band-Limited Signals from Aliased Data

ROBERT J. MARKS, II, MEMBER, IEEE

Abstract—A continuously sampled signal is obtained by periodically placing a signal to zero. A straightforward closed form method is presented for restoration of continuously sampled bandlimited signals—even when the data is aliased. The sampled signal is simply multiplied by a periodic function specified by the duty cycle of the degradation and the severity of aliasing. This product is then placed through a filter with bandwidth equal to that of the signal. The filter acts as an interpolator and the original signal is restored.

Introduction

The restoration problem under consideration is as follows: A bandlimited signal is periodically set to zero. Given the signal's bandwidth, we wish to reconstruct the original signal even when the data are aliased. Such analysis, for example, is useful for the restoration of spatially multiplexed images stored on a continuous medium.

On the surface we are seemingly confronted with a paradox. On one hand, aliased data are commonly assumed to be degraded beyond recovery. On the other hand, knowledge of a band-limited signal over any arbitrarily small interval is sufficient to specify the signal everywhere. This follows from well known analyticity arguments [1]–[3]. We will demonstrate that the former assumption is incorrect for the problem at hand. Indeed, a number of well known techniques can be applied to this problem.

One technique involves taking a sample of the sampled signal and its first M-1 derivatives in each known interval. If T is the sample period, then the image can be recovered in M/T exceeds or equals the Nyquist rate [3]–[5].
A second solution involves application of a sampling theorem using interlaced samples [3], [5]-[6] or, more generally, from irregularly spaced sample points [3], [5]-[10]. As long as there exists at least one sample per Nyquist interval, the signal can be recovered. Clearly, we simply need to sample the object at a sufficiently dense rate over those intervals where the signal is known.

The above restoration schemes require discrete sampling of the degraded signal. Known data are not used. One analog restoration technique requires cumbersome evaluation of an integral equation directly analogous to Slepian and Pollak's classical analysis [1]-[3]. An alternate analog technique is a straightforward modification of Gerchberg's iterative algorithm [3], [11]-[15] which can be placed in closed form [15]-[18].

Continuously sampled signals can also be restored to an approximation by using linear or logarithmic filtering [19]. Indeed, if the sampling rate is sufficiently fast, the data can be unaliased and exact deterministic interpolation is possible in the spirit of the conventional sampling theorem [5]. Rader [20] has presented a restoration technique for undersampled periodic functions.

In this paper, we present an algorithm for restoring continuously sampled band-limited signals from aliased data. The continuously sampled signal is multiplied by a periodic function specified by the severity of aliasing and the periodic degradation's duty cycle. This product is low pass filtered. The result in the absence of noise is the original signal.

**PRELIMINARIES**

Consider a finite energy band-limited signal \( f(x) \) with bandwidth \( 2W \). That is

\[
f(x) = \int_{-W}^{W} F(u) \exp(i2\pi ux) \, du
\]

where

\[
F(u) = \int_{-\infty}^{\infty} f(x) \exp(-i2\pi ux) \, dx.
\]

Define the periodic pulse train with unit period by

\[
r_{\alpha}(x) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{x-n}{\alpha}\right)
\]

where

\[
\text{rect}(\xi) = \begin{cases} 1; & |\xi| \leq \frac{1}{2} \\ 0; & |\xi| > \frac{1}{2} \end{cases}
\]

and

\[ \alpha < 1 \]

is the pulse train's duty cycle. The continuously sampled band-limited image is defined by

\[
g(x) = f(x) r_{\alpha}(x/T)
\]

where \( T \) is the pulse train's period. The interpolatory restoration problem is to determine \( f(x) \) from knowledge of \( g(x) \), \( r_{\alpha}(x/T) \) and \( 2W \).

**A RESTORATION TECHNIQUE FOR FIRST ORDER ALIASING**

The degradation process described by (2) is illustrated by the top three functions in Fig. 1. The corresponding operation in the frequency domain, shown in the bottom three functions in Fig. 1, is

\[
G(u) = F(u) * TR_{\alpha}(Tu)
\]

where the upper case letters denote the Fourier transforms of the corresponding functions in (2) and the asterisk denotes convolution. Expanding (1) in a Fourier series followed by transformation gives

\[
TR_{\alpha}(Tu) = \sum_{n=-\infty}^{\infty} c_{n} \delta \left( u - \frac{n}{T} \right)
\]

where

\[
c_{n} = \alpha \text{sinc} \alpha n = c_{-n}
\]

and \( \text{sinc} \xi = \sin(\pi \xi)/(\pi \xi) \). Thus (3) can be written as

\[
G(u) = \sum_{n=-\infty}^{\infty} c_{n} F \left( u - \frac{n}{T} \right).
\]

Clearly, if the sampling rate \( 1/T \) exceeds \( 2W \), the replicated spectra do not overlap and \( F(u) \) can be regained from \( G(u) \) by a simple low pass filter [5]. We are interested in the aliased case. If one spectra overlaps the right half zero-order spectra as in Fig. 2(a), we have first order aliasing. If two overlap, as in Fig. 2(b), we have second order aliasing, etc. In general, the order of aliasing is

\[
M = \langle 2WT \rangle
\]

where \( \langle x \rangle \) denotes "the greatest integer less than or equal to \( x \)."

Fig. 1. Illustration of the degradation of \( f(x) \) to \( g(x) \) (a) in \( x \); (b) in the frequency domain.

The degradation process described by (2) is illustrated by the top three functions in Fig. 1. The corresponding operation in the frequency domain, shown in the bottom three functions in Fig. 1, is

\[
G(u) = F(u) * TR_{\alpha}(Tu)
\]

where the upper case letters denote the Fourier transforms of the corresponding functions in (2) and the asterisk denotes convolution. Expanding (1) in a Fourier series followed by transformation gives

\[
TR_{\alpha}(Tu) = \sum_{n=-\infty}^{\infty} c_{n} \delta \left( u - \frac{n}{T} \right)
\]

where

\[
c_{n} = \alpha \text{sinc} \alpha n = c_{-n}
\]

and \( \text{sinc} \xi = \sin(\pi \xi)/(\pi \xi) \). Thus (3) can be written as

\[
G(u) = \sum_{n=-\infty}^{\infty} c_{n} F \left( u - \frac{n}{T} \right).
\]

Clearly, if the sampling rate \( 1/T \) exceeds \( 2W \), the replicated spectra do not overlap and \( F(u) \) can be regained from \( G(u) \) by a simple low pass filter [5]. We are interested in the aliased case. If one spectra overlaps the right half zero-order spectra as in Fig. 2(a), we have first order aliasing. If two overlap, as in Fig. 2(b), we have second order aliasing, etc. In general, the order of aliasing is

\[
M = \langle 2WT \rangle
\]

where \( \langle x \rangle \) denotes "the greatest integer less than or equal to \( x \)."
Fig. 2. Illustration of (a) first order aliasing; (b) second order aliasing.

Fig. 3. Removing the first order spectrum by subtracting two weighted and shifted versions of the degraded spectrum.

sufficient to restore the signal since the Fourier transform of a real function is Hermitian:

\[ F(u) = F^*(-u). \]

We can then show

\[ f(x) = 2 \, \text{Re} \int_0^W F(u) \exp(j2\pi ux) \, du \]

where Re denotes “the real part of.” Substituting (4) and “simplifying” gives

\[ f(x) = \frac{2}{c_0 - c_1 c_{-1}} \left[ g(x) \left( c_0 - c_1 \exp(j2\pi x/T) \right) \right] * \left[ W \, \text{sinc}(Wx) \, \exp(j\pi Wx) \right]. \]

Note that \( g(x) \) is multiplied by a periodic function and filtered. The corresponding restoration algorithm pictured in Fig. 4 follows as

\[ f(x) = \frac{2}{c_0^2 - c_1 c_{-1}} \left[ \left\{ g(x) \left[ c_0 \cos \pi Wx - c_1 \cos \left( W - \frac{2}{T} x \right) \right] * W \, \text{sinc} Wx \right\] \cos \pi Wx \right. \\
+ \frac{2}{c_0^2 - c_1 c_{-1}} \left[ \left\{ g(x) \left[ c_0 \sin \pi Wx - c_1 \sin \left( W - \frac{2}{T} x \right) \right] * W \, \text{sinc} Wx \right\] \sin \pi Wx. \]

A GENERAL INTERPOLATION TECHNIQUE

For \( M \)th order aliasing, there are \( 2M \) unwanted spectra interfering with the desired zero order spectrum. In this section, we demonstrate a general method for eliminating the unwanted spectra by application of the technique presented in the previous section.

Consider Fig. 5 in which \( 2M + 1 \) shifted versions of \( G(u) \) are shown, i.e.,

\[ \{ G(u - m/T) \, | \, m = -M, -M + 1, \cdots, M \}. \]

The interfering component spectra in each shifted \( G \) are shown not overlapping for presentation clarity. We now simply need to weight the \( m \)th shifted \( G \) by a coefficient \( b_m \) so that

\[ \sum_{m=-M}^{M} b_m G(u - m/T) \, \text{rect} \left( \frac{u}{2W} \right) = F(u). \]

With attention again to Fig. 5, this is equivalent to summing the weights of the component spectra in each column to give zero for the interfering spectra and unity for the zero order spectra. That is, find the \( b_m \)'s which satisfy

\[ \sum_{m=-M}^{M} b_m e^{-inm} = \delta_n; |n| \leq M \]

where \( \delta_n \) denotes the Kronecker delta. Viewing this as a matrix operation:

\[ \begin{bmatrix}
  c_0 & c_{-1} & \cdots & c_{-M} & \cdots & c_{-2M} \\
  c_1 & c_0 & \cdots & c_{-M+1} & \cdots & c_{-2M+1} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  c_M & c_{M-1} & \cdots & c_0 & \cdots & c_{-M} \\
  c_{2M} & c_{2M-1} & \cdots & c_M & \cdots & c_0 \\
  b_M & b_{-M+1} & \cdots & b_0 & \cdots & b_{-M} 
\end{bmatrix}
= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

it is clear the \( b_m \)'s can be solved for by solution of a Toeplitz set of equations [21].

Inverse transforming (5) gives the spatial domain restoration formula

\[ f(x) = [g(x) \, \theta_M(x/T)] \ast 2W \, \text{sinc} 2Wx \]

where \( \theta_M(x) \) is the trigonometric polynomial
Fig. 5. Illustration of the methodology of restoring $M$th order aliased data by summing $2M + 1$ shifted and weighted versions of the degraded spectrum.

$$\theta_M(x) = \sum_{m=-M}^{M} b_m \exp(-j2\pi mx). \quad (9)$$

Note, however, that since $g(x) = g(x) r_\alpha(x/T)$ we only require knowledge of $\theta_M$ where $r_\alpha$ is unity. Thus we define the periodic function

$$\psi_M(x) = \theta_M(x) r_\alpha(x). \quad (10)$$

Expanding in a Fourier series gives

$$\psi_M(x) = \sum_{n=-\infty}^{\infty} d_n \exp(j2\pi nx).$$

The coefficients are

$$d_n = \int_{-1/2}^{1/2} \psi_M(x) \exp(-j2\pi nx) \, dx$$

$$= \int_{-\alpha/2}^{\alpha/2} \theta_M(x) \exp(-j2\pi nx) \, dx$$

$$= \alpha \sum_{m=-M}^{M} b_m \text{sinc} \alpha(n - m)$$

where, in the last step we have used (9). From (6), we conclude

$$d_n = \begin{cases} d_n; & |n| \leq M \\ \alpha \sum_{m=-M}^{M} b_m \text{sinc} \alpha(n - m); & |n| > M. \end{cases}$$

Note that the $d_n$'s are also the weights of the remaining spectra after restoration. Plots of $\psi_M(x)$ for $\alpha = 0.5$ are shown in Fig. 6. Plots of $\psi_2(x)$ for various duty cycles are shown in Fig. 7.

In lieu of (8), the restoration algorithm pictured in Fig. 8 now becomes

$$f(x) = [g(x) \psi_M(x/T)] * 2W \text{sinc} \, 2Wx. \quad (11)$$

Fig. 6. Plots of $\psi_M(x) = \psi_M(x)$ for $\alpha = 0.5$ and $M = 1, 2, 3, 4,$ and $5$. The vertical scale is linear for $|\psi_M| \leq 1$ and logarithmic otherwise.

Fig. 7. Plots of $\psi_2(x)$ for various $\alpha$. The vertical scale is linear for $|\psi_2| \leq 1$ and is logarithmic otherwise.

Fig. 8. Restoration of $M$th order aliased data. The periodic function, $\psi_M(x/T)$, is parameterized by the degradation duty cycle $\alpha$, degradation period $T$, and the order of aliasing $M$. The filter has bandwidth $2W$. 
MARKS: CONTINUOUSLY SAMPLED BAND-LIMITED SIGNALS

NOTES

1) Taking the limit at \( T \to \infty \) holding \( \alpha T \) constant gives us the classic band-limited signal extrapolation problem [1]-[3]. This problem is ill posed [12], [22], [23]. We thus expect greater and greater noise sensitivity as \( \alpha \to 0 \).

2) If \( N>M \), then \( \theta_M \) could be used in (8) in lieu of \( \theta_M \) and \( \psi_M \) in lieu of \( \psi_M \). We simply eliminate the obvious question of the existence of an \( \theta_M(x) \) or a \( \psi_M(x) \) for arbitrary aliasing restoration arises. Consider, however, the product in (10) as \( N \to \infty \). The Fourier coefficient of \( \psi_M \) is given by the discrete convolution the Fourier coefficients of \( \theta_M \) and \( r_\alpha \):

\[
d_n = \sum_{m=-\infty}^{\infty} b_m c_{n-m}.
\]

From (6), \( d_n = \delta_n \). It follows that \( \psi_M(x) \to 1 \) and \( \theta_M(x) \to 1/r_\alpha(x) \). Thus, in the limit as \( M \to \infty \), \( \theta_M(x) \) becomes unbounded over those intervals where \( r_\alpha(x) \) is zero.

3) The sensitivity and instability noted in items (1) and (2) manifest themselves in the algorithm through a worsening of the corresponding matrix in (7). That is, as \( \alpha \to 0 \) or \( M \to \infty \), the C matrix becomes more and more ill conditioned [23].

4) The restoration algorithm in (8) is applicable to any periodic degradation of \( f(x) \) (under one condition). Simply use the Fourier coefficients of the periodic degradation as the \( c_n \)'s in (6) to determine the \( b_m \)'s. The condition is that the corresponding matrix in (7) is not singular. Such is the case, for example, in the Whittaker–Shannon sampling theorem [3], [24] where the Whittaker–Shannon sampling theorem is

\[
\sum_{p=-\infty}^{\infty} \delta(t-pT).
\]

The corresponding \( c_n \)'s are all equal.

5) Consider replacing \( r_\alpha(x/T) \) by \( 1/r_\alpha(x/T) \). The restoration algorithm then becomes a possible continuous solution to the classic interpolation problem of recovering \( f(x) \) from \( f(x) = 1 - \text{rect} [x/(1 - \alpha)T] \). Part of the known data, however, is obviously not used. Note also here that as \( T \to \infty \) with \( (1 - \alpha)T \) held constant, we obtain the classic interpolation problem.

6) Since \( c_n = c_{-n} \) in (7), it follows that \( b_n = b_{-n} \). Thus, (6) simplifies to

\[
b_0c_n + \sum_{m=1}^{M} b_m (c_{n-m} + c_{n+m}) = \delta_n; \quad 0 \leq n \leq M.
\]

The order of the corresponding matrix to be inverted for finding the \( b_m \)'s is thus reduced from \( 2M + 1 \) to \( M + 1 \). Then, from (9)

\[
\theta_M(x) = b_0 + 2 \sum_{m=1}^{M} b_m \cos 2\pi mx.
\]

Alternately, we could eliminate only those spectra overlapping the right half of the zero order spectrum and, as before, reconstruct the image from Hermitian symmetry. For \( M \)th order aliasing, \( M \) spectra need to be eliminated to the right of the zero order spectra and \( \langle M/2 \rangle \) to the left. The corresponding matrix is thus of order \( M + \langle M/2 \rangle + 1 \). If we further take advantage of the fact that \( b_n = b_{-n} \), the matrix reduces to order \( M + 1 \), identical in dimension to our previous result.

7) One of the referees kindly pointed out a similarity between our algorithm and a technique for eliminating crosstalk over a linear time-invariant channel [25]. Quoting from the review:

"Let \( p(t) \) denote a single finite-duration pulse and suppose that the channel response to \( p(t) \) as input is \( h(t) \). It is assumed that \( h(t) \) is known and essentially time limited. For any choice of constants \( c_r \), \( r = 0 \to \infty \), and prescribed sampling interval \( T > 0 \),

\[
g(t) = \sum_{r=0}^{\infty} c_r h(t - rT)
\]

is the channel response to the input

\[
f(t) = \sum_{r=0}^{\infty} c_r p(t - rT).
\]

Over the first interval \( 0 \leq t \leq T \), \( g(t) = c_0 h(t) \); hence, \( c_0 \) is uniquely determined. Over the second interval \( T < t \leq 2T \),

\[
g(t) - c_0 h(t) = c_1 h(t - T)
\]

so that \( c_1 \) is determined, etc."

ACKNOWLEDGMENT

The author expresses his deep appreciation to D. Kaplan for generating the data for Figs. 6 and 7.

REFERENCES

Fast Algorithms for Linear Prediction and System Identification Filters with Linear Phase

S. LAWRENCE MARPLE, JR., SENIOR MEMBER, IEEE

Abstract—A general finite impulse response (FIR) filter can be used as a linear prediction filter, if given only an input sample sequence, or as a system identification model, if given the input and output sequences from an unknown system. With known correlation, the coefficients of the FIR filter that minimize the mean square error in both applications are found by solution of a set of normal equations with Toeplitz structure. Using only data samples, the coefficients that yield the least squared error in both applications are found by solution of a set of normal equations with near-to-Toeplitz structure. Computationally efficient (fast) algorithms have been published to solve the coefficients from both types of normal equation structures. If the FIR filter is constrained to have linear phase, then the impulse response must be a system with linear prediction and system identification when the FIR filter is specialized to have linear phase.

When used for linear prediction analysis, the FIR filter output represents an estimate of the current input sample value in terms of a linearly weighted sum of past (or future) sample values. If the autocorrelation function for the input process is known, the FIR filter coefficients that will yield the minimum mean square linear prediction error (MMSE) are determined by solving a set of normal equations with Toeplitz structure. A computationally efficient algorithm for solving these normal equations is the Levinson recursion [1], which is one of the most well known fast algorithms in digital signal processing. If one only uses available data samples, the FIR filter coefficients that will yield the least squared linear prediction error (LSE) are determined by solution of a set of normal equations with structure that is near-to-Toeplitz in some sense. A fast algorithm also exists for solving these normal equations [2].

When used for a system identification application, the FIR filter output represents an estimate of the current output sample value from an unknown system in terms of a linearly weighted sum of past and/or future sample values from the input to the unknown system. If the autocorrelation function of the input process and the cross-correlation function between the input and output process are known, the FIR filter coefficients that will minimize the mean square error between

Robert J. Marks, II (S'71–M'72–S'79–M'80) was born in Sutton, WV, on August 25, 1950. He received the B.S. and M.S. degrees in electrical engineering from the Rose-Hulman Institute of Technology, Terre Haute, IN, in 1972 and 1973, respectively, and the Ph.D. from Texas Tech University, Lubbock, TX, in 1977. From 1973 to 1975, he served as a Research Engineer with the U.S. Department of the Navy. Presently, he is an Associate Professor of Electrical Engineering at the University of Washington, Seattle. Areas of interest include signal processing, statistical communication theory, and optical processing. He has published over thirty refereed journal and proceedings papers in these areas.

Dr. Marks is a member of OSA, SPIE, Eta Kappa Nu, and Sigma Xi. In 1982, he was awarded the Outstanding IEEE Branch Counsellor Award by IEEE.

INTRODUCTION

The finite impulse response (FIR) filter has served as the foundation for linear prediction signal analysis and has also frequently been used as a system identification model. This paper presents four fast algorithms for linear prediction

Manuscript received June 26, 1981; revised June 1, 1982.

The author is with The Analytic Sciences Corporation, McLean Operation, McLean, VA 22102.
