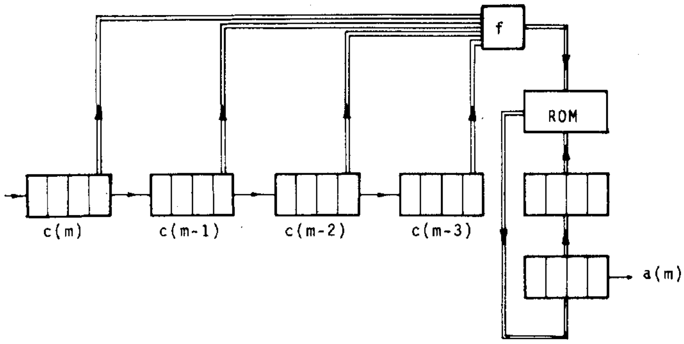


Fig. 1. Proposed implementation of convolution.

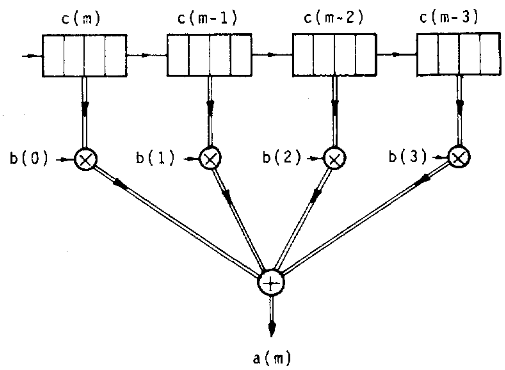


Fig. 2. Classical implementation of convolution.

eral and much more efficient than the classical architecture of Fig. 2. For the special case of GF($2^n$), the architecture can be simplified even further since addition may be performed with simple XOR gates and each clock in the ROM will process precisely one bit of each $c(m - k)$. Thus, the computation time of each period is a small fragment of the time required with the classical architecture of Fig. 2.

## IV. Conclusions

A novel and, in theory, optimal method for the computation of cyclic convolutions in Galois fields has been presented. A simple calculation will show, for example, that for $N = 3$, $p = 2$, $n = 4$, the present method requires three multiplications, while that in [5] requires four. As another example, if $N = 5$, $p = 2$, $n = 2$, the present method requires seven multiplications against the ten multiplications required in [5]. An important difference, however, is that the method in [5] requires no bit-by-bit calculation, while the present method does. This disadvantage can be eased by using special software or hardware such as proposed in Section III.

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# An FIR Estimation Filter Based on the Sampling Theorem

DMITRY RADBEL AND ROBERT J. MARKS, II

*Abstract*—**The estimation of noise-perturbed bandlimited stochastic signal samples by FIR filtering is considered. The mean-square error of the estimate is used as the criterion of performance. We contrast three types of filters: all-pass, a sampling-theorem-based filter, and the minimum mean-square error (Wiener) filter. Although the Wiener filter is linearly optimal, its design requires detailed knowledge of the processes' second-order statistics. The sampling theorem filter does not. For large signal-to-noise ratios and large filter orders, the two filters perform nearly identically asymptotically. Furthermore, we demonstrate that for a fixed filter order, there exists an optimal sampling rate which decreases with increasing signal-to-noise ratio.**

## I. Introduction

The classical Whittaker [1]-Shannon [2] sampling theorem states that if a signal has a maximum frequency of $W$ Hz, then it can be characterized exactly by an infinite number of the sampled values of the signal spaced equadistantly $1/2W$ seconds apart. In practical situations, we can estimate the signal from a large but finite amount of noise-contaminated samples.

A number of papers have appeared that deal with bounds of the truncation error magnitude for noiseless deterministic signals. Let us mention the fundamental papers of Yao and Thomas [3], Brown [4], and Piper [5]. Our paper treats the signal stochastically and deals with truncation (and noise) error using a mean-square error (rather than error magnitude) as a measure of the restoration performance. We explore the cleaning of data samples with FIR filters based on the sampling theorem and the Wiener filter. Although the latter is optimal in minimizing mean-square error (MSE), its implementation requires detailed knowledge of the processes' second-order statistics. The truncated sampling theorem (ST) approach

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does not. For large signal-to-noise ratios (SNR's) and filter order, the two algorithms perform nearly the same.

For a fixed filter order, we demonstrate the existence of a sampling rate which, for the ST estimate, minimizes the MSE. Sampling below this rate greatly increases the noise contribution to error through severe aliasing. Sampling above the rate, on the other hand, decreases the information contributed from each sample. We also show that, in certain instances, increasing the SNR of the samples does not contribute significantly to estimate improvement.

## II. Preliminaries

Let $f(t)$ denote a real zero-mean wide-sense stationary random process with autocorrelation

$$R_f(t-\tau) = E[f(t) f(\tau)]$$

where $E$ denotes the expectation value operator. Let $f(t)$ be bandlimited in the sense that there exists a $W$ such that

$$R_f(\tau) = \int_{-W}^{W} S_f(u) \exp(j2\pi u\tau)\, du$$

where the power spectral density (PSD) is

$$S_f(u) = \int_{-\infty}^{\infty} R_f(\tau) \exp(-j2\pi u\tau)\, d\tau.$$

Let $2B$ denote a sampling rate not less than $2W$ and define the sampling rate parameter

$$r = W/B \leqslant 1.$$

We form the estimate

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2B}\right) \operatorname{sinc}(2Bt - n) \tag{1}$$

where $\operatorname{sinc}(y) = \sin(\pi y)/(\pi y)$. Then $\hat{f}(t)$ and $f(t)$ are equal at all values of time in the mean-square sense:

$$E[\,|f(t) - \hat{f}(t)|^2\,] = 0. \tag{2}$$

Note, in particular, that since $\operatorname{sinc}(n) = \delta_n$ = Kronecker delta, we have a stronger equality at the sample point locations:

$$\hat{f}\left(\frac{m}{2B}\right) = f\left(\frac{m}{2B}\right). \tag{3}$$

Since $R_f(\tau)$ has a bandwidth $2W$, we are motivated to form an alternate estimate by passing (1) through a filter unity on $|u| \leqslant W$ and zero elsewhere. The result is

$$\hat{f}_r(t) \equiv \hat{f}(t) * 2W \operatorname{sinc}(2Wt)$$

$$= r \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2B}\right) \operatorname{sinc}(2Wt - rn). \tag{4}$$

The estimate here is also equal to the sampled process in the mean-square sense:

$$E[\,|f(t) - \hat{f}_r(t)|^2\,] = 0. \tag{5}$$

Unlike the unfiltered case in (1), the estimates at the sample point locations here are generally dependent on all the samples of the process.

## III. Signal Sample Estimation

Let $\xi(t)$ denote a real zero-mean wide-sense stationary noise waveform with autocorrelation

$$R_\xi(t-\tau) = E[\xi(t)\xi(\tau)]$$

and noise level

$$\overline{\xi^2} = R_\xi(0).$$

We assume that the noise and signal are uncorrelated:

$$E[\xi(t) f(\tau)] = 0.$$

As a consequence, the observed signal

$$g(t) = f(t) + \xi(t)$$

has autocorrelation

$$R_g(\tau) = R_f(\tau) + R_\xi(\tau)$$

and a signal-to-noise ratio of

$$\mathrm{SNR} = \overline{f^2}/\overline{\xi^2}$$

where the signal level is

$$\overline{f^2} = R_f(0).$$

Let $\{g(n/2B) | -\infty < n < \infty\}$ be input into a $(2N+1)$st-order FIR filter whose output values $\{\bar{f}(n/2B) | -\infty < n < \infty\}$ are estimates of the samples $\{f(n/2B) | -\infty < n < \infty\}$. In general,

$$\bar{f}\left(\frac{m}{2B}\right) = \sum_{n=-N}^{N} g\left(\frac{n}{2B}\right) h[m-n]$$

where $h[\cdot]$ is the filter's impulse response. Without loss in generality, we will examine the estimate at the origin:

$$\bar{f}(0) = \sum_{n=-N}^{N} g\left(\frac{n}{2B}\right) h[n] \tag{6}$$

where we have assumed that $h$ is even. The measure of the filter's performance will be the MSE:

$$\epsilon(N) = E[\,|f(0) - \bar{f}(0)|^2\,]. \tag{7}$$

Three choices of $h[\cdot]$ and their corresponding MSE's will be now considered and later contrasted.

### A. Sample Point Estimate (SPE)

Motivated by (3), we might choose

$$h_a[n] = \delta_n$$

for which $\bar{f}_a(0) = g(0)$. Indeed, the result is exact in the absence of noise. In the presence of noise,

$$\epsilon_a(N) = E[\,|f(0) - g(0)|^2\,]$$

$$= \overline{\xi^2}. \tag{8}$$

### B. Sampling Theorem (ST) Estimate

Motivated by (4) for $t = 0$, an alternate estimate is

$$\bar{f}_b(0) = r \sum_{n=-N}^{N} g\left(\frac{n}{2B}\right) \operatorname{sinc}(rn). \tag{9}$$

This corresponds to $h_b[n] = r \operatorname{sinc}(rn)$ and is essentially an FIR filter obtained by retaining $2N+1$ sample values of the impulse response of an ideal low-pass filter. From (5), the estimate is asympotically optimal in the absence of noise.

In the presence of noise, the error for the truncated estimate of the sample at the origin is, in general,

$$\epsilon_b(N) = E[|f(0) - r \sum_{n=-N}^{N} g\left(\frac{n}{2B}\right) \operatorname{sinc}(rn)|^2]$$

$$= R_f(0) - 2r \sum_{n=-N}^{N} R_f\left(\frac{n}{2B}\right) \operatorname{sinc}(rn)$$

$$+ r^2 \sum_{n=-N}^{N} \sum_{m=-N}^{N} R_g\left(\frac{n-m}{2B}\right)$$

$$\cdot \operatorname{sinc}(rn) \operatorname{sinc}(rm). \tag{10}$$

After some tedious but straightforward manipulation, the following more computationally attractive iterative form results:

$$\epsilon_b(N+1) = \epsilon_b(N) - 4r R_f\left(\frac{N+1}{2B}\right) \operatorname{sinc} r(N+1)$$

$$+ 2r^2 \operatorname{sinc}^2 r(N+1) \left[R_g(0) + R_g\left(\frac{2N+2}{2B}\right)\right]$$

$$+ 2r^2 \operatorname{sinc} r(N+1) \sum_{n=-N}^{N}$$

$$\cdot \left[R_g\left(\frac{n+N+1}{2B}\right) + R_g\left(\frac{n-N-1}{2B}\right)\right] \operatorname{sinc}(rn) \tag{11}$$

with initialization

$$\epsilon_b(0) = (1-r)^2 R_f(0) + r^2 R_\xi(0). \tag{12}$$

*C. FIR Wiener Filter Estimate*

Lastly, we consider the impulse response in (6) that, for fixed $N$, minimizes the error in (7). The minimum MSE estimate results in the familiar FIR Wiener filter [11]. The impulse response of this filter can be found as solution of the linear equations

$$R_f\left(\frac{m}{2B}\right) = \sum_{n=-N}^{N} h_c[n] R_g\left(\frac{m-n}{2B}\right), \quad |m| \leq N. \tag{13}$$

By exploiting the Toeplitz nature of the matrix of coefficients, several efficient recursive procedures can be used for solving this system of equations. We will mention the Levinson and Robinson algorithms [6] and Durbin's recursive procedure [7]. An algorithm to compute coefficients of the FIR Wiener filter was recently proposed by Manolakis *et al.* [8].

Due to symmetry (and by previous assumption), $h_c$ is even and thus can be found be solving $N + 1$ equations only.

Use of the solution of (13) yields

$$\overline{f_c}(0) = \sum_{n=-N}^{N} g\left(\frac{n}{2B}\right) h_c[n]. \tag{14}$$

The corresponding (minimum) MSE is

$$\epsilon_c(N) = R_f(0) - \sum_{n=-N}^{N} h_c[n] R_f\left(\frac{n}{2B}\right). \tag{15}$$

## IV. MSE OF ESTIMATES

For a given filter, the sources of estimation error are data noise and truncation. In this section, the MSE of the estimates is analyzed for each of the three estimates. The filters are assumed to be ideal.

*A. Sample Point Estimate*

The concept of truncation does not apply here. The MSE noise is given in (8).

*B. Sampling Theorem Estimate*

For the ST estimate, the resulting error in (10) is separable into components due to truncation and data noise:

$$\epsilon_b(N) = \epsilon_T(N) + \epsilon_D(N)$$

where subscript $T$ refers to truncation and $D$ to data noise. From (10),

$$\epsilon_T(N) = R_f(0) - 2r \sum_{n=-N}^{N} R_f\left(\frac{n}{2B}\right) \operatorname{sinc}(rn)$$

$$+ r^2 \sum_{n=-N}^{N} \sum_{m=-N}^{N} R_f\left(\frac{n-m}{2B}\right)$$

$$\cdot \operatorname{sinc}(rn) \operatorname{sinc}(rm).$$

Using (2), one can easily demonstrate that this error approaches zero as $N$ tends to infinity. The error due to the data noise is

$$\epsilon_D(N) = r^2 \sum_{n=-N}^{N} \sum_{m=-N}^{N} R_\xi\left(\frac{n-m}{2B}\right) \operatorname{sinc}(rn) \operatorname{sinc}(rm). \tag{16}$$

Note that (11) and (12) can similarly be decomposed to provide iterative relationships which are more computationally attractive. Asymptotically, we can write (16) as [10]

$$\epsilon_D(\infty) = \lim_{N\to\infty} \epsilon_D(N)$$

$$= \lim_{N\to\infty} \epsilon_b(N)$$

$$= r \sum_{k=-\infty}^{\infty} R_\xi\left(\frac{k}{2B}\right) \operatorname{sinc}(rk) \tag{17}$$

where, in (16), we let $k = n - m$, $N \to \infty$ and we have used the sampling-theorem-based identity

$$r \sum_{n=-\infty}^{\infty} \operatorname{sinc}(rn) \operatorname{sinc} r(k-n) = \operatorname{sinc}(rk).$$

Two closed form special cases of (17) are worthy of note. If the correlation duration of the noise is less than the sampling interval, the noise is sample-wise white:

$$R_\xi\left(\frac{n}{2B}\right) = \overline{\xi^2}\, \delta_n. \tag{18}$$

Then (17) becomes

$$\epsilon_D(\infty) = r\, \overline{\xi^2}. \tag{19}$$

The noise level is thus reduced by the sampling rate ratio. For a Laplace autocorrelation with parameter $\alpha$,

$$R_\xi(\tau) = \overline{\xi^2}\, e^{-\alpha|\tau|},$$

(17) becomes [14]

$$\epsilon_D(\infty) = \frac{2\overline{\xi^2}}{\pi} \, a \tan \left[\frac{\sinh\left(\frac{\alpha}{2B}\right) \tan\left(\frac{\pi r}{2}\right)}{\cosh\left(\frac{\alpha}{2B}\right) - 1}\right]. \tag{20}$$

We return to the truncated estimate analysis. For sample-wise white noise, (16) becomes

$$\epsilon_D(N) = r^2 \, \overline{\xi^2} \sum_{n=-N}^{N} \mathrm{sinc}^2(rn).$$

Obviously, $\epsilon_D(N+1) \geqslant \epsilon_D(N)$. The error due to data noise generally increases when the noise has roughly a uniform or predominately low-frequency dominated PSD. For low SNR, $\epsilon_b(0)$ may be smaller than $\epsilon_b(\infty)$. For example, for sample-wise white noise from (12) and (19), this happens when

$$\epsilon_b(0) = (1-r)^2 R_f(0) + r^2 R_\xi(0) < r R_\xi(0) = \epsilon_b(\infty) = \epsilon_D(\infty)$$

or

$$\mathrm{SNR} < \frac{r}{1-r}.$$

### C. FIR Wiener Filter

Even with efficient algorithms, the numerical solution of the FIR Weiner filter for large $N$ is difficult. The solution can be obtained in closed form for the noncausal linear filter [9]

$$\overline{f(0)} = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2B}\right) h_c[n]$$

where

$$h_c[n] = \int_{-W}^{W} A(u) e^{j\pi un/B} du$$

and

$$A(u) = \frac{S_f(u)}{S_g(u)} = \frac{S_f(u)}{S_f(u) + S_\xi(u)}.$$

The corresponding MSE is

$$\epsilon_c(\infty) = \int_{-W}^{W} \frac{S_f(u) \cdot S_\xi(u)}{S_f(u) + S_\xi(u)} du. \tag{21}$$

For white noise, the result is

$$\epsilon_c(\infty) = \frac{1}{1 + \frac{1}{\mathrm{SNR}}} \cdot r \, \overline{\xi^2}. \tag{22}$$

It is easy to see that for $\mathrm{SNR} >> 1$, (22) is very close to the corresponding limit of the ST estimation in (19). Note that $\epsilon_c(N)$ is not separable into signal and noise errors as is $\epsilon_b(N)$.

## V. Results

### A. Performance Comparison

Fig. 1 illustrates the contribution of the data noise to $\epsilon_b(N)$ for sample-wise white noise. The contribution of truncation is also shown for a signal with a uniform PSD:

$$S_f(u) = \frac{\overline{f^2}}{2W} \mathrm{rect}\left(\frac{u}{2W}\right)$$

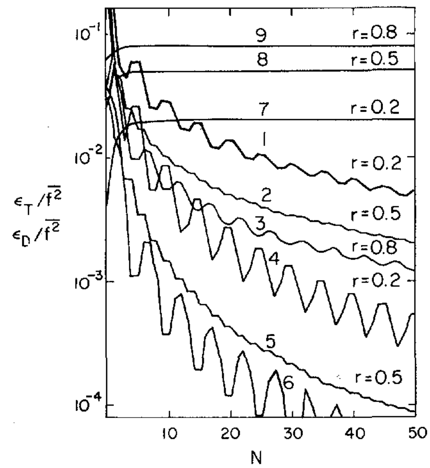


Fig. 1. Truncation and data noise errors in ST estiamte for SNR = 10. 1, 2, 3: truncation error for uniform PSD signal. 4, 5, 6: truncation error for triangular PSD. 7, 8, 9: data noise error for sample-wise white noise.

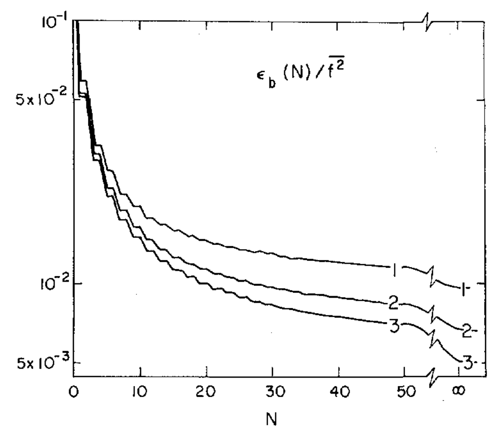


Fig. 2. Mean-square error in ST estimate for uniform PSD signal and Laplace autocorrelation noise with $r = 0.5$, SNR = 100. 1) $\alpha = 0.5$ W. 2) $\alpha = 6$ W. 3) Sample-wise white nosie ($\alpha = \infty$).

and triangular PSD:

$$S_f(u) = \frac{\overline{f^2}}{W}\left(1 - \frac{|u|}{W}\right).$$

Note that the triangular case converges quicker. Roughly, when compared to the uniform spectrum, the triangular spectrum contains a greater amount of low frequencies, and thus results in a "smoother" signal.

Also note that the truncation error approaches zero slower for higher sampling rates. Roughly, for high sampling rates, the $(N+1)$st sample is highly correlated with the $N$th sample. Thus, not much more information is gained for the estimate. Similarly, for fixed $N$, a large sampling rate corresponds to small intervals. As $N$ increases for fixed $r$, it thus takes more samples to get to the truncation error's asymptotic value of zero.

Interestingly, $\epsilon_T(N)$ is not a monotonically decreasing function of $N$. Rather, it has clear periodic maximums and minimums whose locations depend only on $r$.

Fig. 2 illustrates $\epsilon_b(N)$ for Laplace autocorrelation noise. The asymptotic values from (20) are also shown.

The MSE for both ST and FIR Wiener estimates is shown in Fig. 3. Clearly, the FIR Wiener filter gives better results by

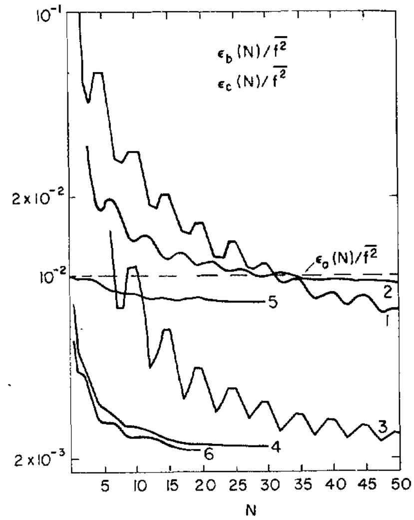


Fig. 3. Mean-square error in ST and FIR Wiener filter estimates for SNR = 100 and sample-wise white noise. 1, 3, 4, 6: $r = 0.2$; 2, 5: $r = 0.8$. 1, 2, 4, 5: signal with uniform PSD. 3, 6: signal with triangular PSD. 1, 2, 3 st: 4, 5, 6 Wiener.

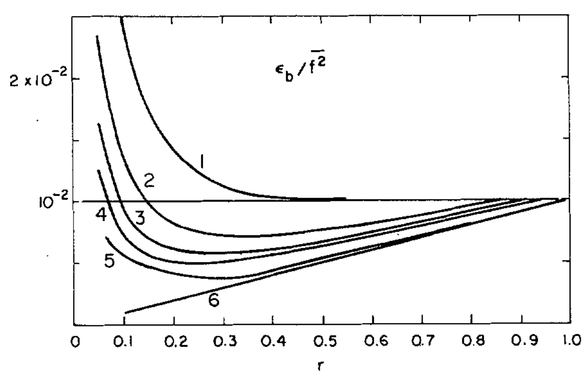


Fig. 4. Mean-square error versus $r$ for $N \leqslant N_{MAX}$, sample-wise white noise, SNR = 100. 1-4: signal with uniform PSD, $N_{MAX} = 20$, 40, 60, and 80, respectively. 5: signal with triangular PSD, $N_{MAX} = 20$. 6: nontruncated estimate ($N_{MAX} = \infty$).

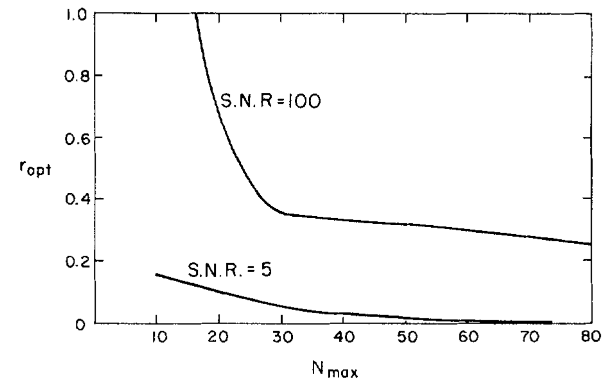


Fig. 5. Sampling theorem estimate: optimum sampling rate ratio versus $N_{MAX}$ for a uniform PSD signal and sample-wise white noise.

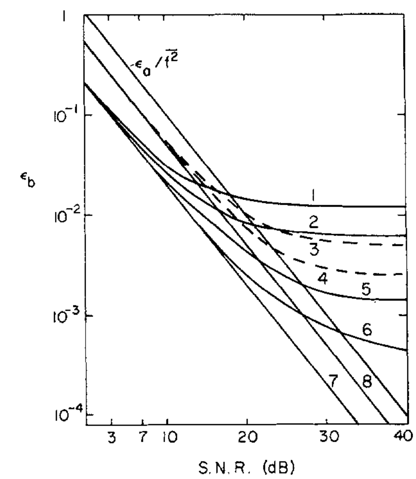


Fig. 6. Sampling theorem estimate: mean-square error versus SNR for sample-wise white noise. Signal with uniform PSD: 1) $r = 0.2$, $N_{MAX} = 20$. 2) $r = 0.2$, $N_{MAX} = 40$. 3) $r = 0.5$, $N_{MAX} = 20$. 4) $r = 0.5$, $N_{MAX} = 40$. Signal with triangular PSD: 5) $r = 0.2$, $N_{MAX} = 20$. 6) $r = 0.2$, $N_{MAX} = 40$. Nontruncated estimation: 7) $r = 0.2$. 8) $r = 0.5$.

design. The asymptotic performance is nearly reached with $N = 10$–15. Note that although the ST gives an asymptotically better estimate than does the SPE estimate, $\epsilon_a(N) < \epsilon_b(N)$ for low $N$.

### *B. Optimal Sampling Rates*

From previous observations, $\epsilon_T(N)$ is generally a decreasing function of $r$ with $N$ fixed. We also know that $\epsilon_D(N)$ usually exhibits just the opposite behavior: it increases [10]. Therefore, we can conclude that the total MSE

$$\epsilon_b(N) = \epsilon_T(N) + \epsilon_D(N) \tag{23}$$

may have a minimum at some sampling rate for fixed $N$. A plot of $\epsilon_b(N)$ versus $r$ is shown in Fig. 4 for various $N \leqslant N_{MAX}$. For $N_{MAX}$ sufficiently high, the curves have distinct minimums at points we will denote by $r_{opt}$. We can define $r_{opt}$ at a given $N_{MAX}$ as the sampling rate ratio which gives us minimum $\epsilon_b(N)$ with $N$ not exceeding $N_{MAX}$. The horizontal line is the SPE error. For this example, for the ST to give a better estimate than the SPE, $N$ must roughly exceed 20.

A sketch of $r_{opt}$ versus $N_{max}$ is shown in Fig. 5 for two SNR's for sample-wise white noise. Clearly, with $N$ fixed, higher SNR's have higher $r_{opt}$'s. This is understandable: a high SNR means that the truncation error dominates the total MSE in (23). As $N \to \infty$, $r_{opt}$ decreases.

This is not necessarily the case for a noise with a PSD of limited extent. Here, an increase in the sampling rate above some limit does not lead to a decrease in $\epsilon_D(\infty)$.

### *C. The Dependence of MSE on SNR*

An increase in SNR decreases the normalized error due to data noise $\epsilon_D(N)/\overline{f^2}$ and leaves the normalized truncation error $\epsilon_T(N)/\overline{f^2}$ the same. Clearly, after some point, a further increase in SNR will have an insignificantly small effect on the normalized MSE, $\epsilon_b(N)/\overline{f^2}$. Then, for a high SNR,

$$\epsilon_b(N) \approx \epsilon_T(N).$$

The family of curves $\epsilon_b(N)$ versus SNR for various $r$ and $N_{MAX}$ is shown in Fig. 6. Each curve follows the line for $N \to \infty$ at low SNR (when $\epsilon_D >> \epsilon_T$), and then flattens. At some SNR, the ST estimate becomes worse than SPE, i.e., the data filtering only worsens the quality of our estimation.

## VI. Conclusions

The FIR Wiener filter gives much better results than a simple FIR ST estimate. Its disadvantage lies in the fact that coefficients of the Wiener filter are parameterized by assumed statistical models for signal and noise. A change in signal and/or noise statistics requires filter recalculation. When the actual signal and noise PSD's deviate from nominal, the performance of the Wiener filter will deteriorate. It is desirable, then, to have a filter which performs well over classes of possible signal and noise PSD's. For a detailed discussion of robust signal processors, the interested reader is referred to papers by Kassam, Lim, and Poor [12], [13].

Conclusions regarding FIR ST estimate can be summarized as follows.

1) For fixed (or limited) $N$ and fixed SNR, there exists some optimal finite sampling rate which gives minimum MSE for the estimate. Sampling above or below this rate will increase MSE.

2) Asymptotically (high SNR, high $N$), the ST and optimal Weiner filter perform identically. Indeed, a minimum MSE solution for $N \to \infty$ and SNR $\to \infty$ is the sampling theorem [15].

3) Under certain conditions (high SNR, low $N$), the FIR ST estimate gives a worse estimate of the signal than input samples directly. $[\epsilon_b(N) > \epsilon_a(N)]$.

4) Unlike the noncausal $N \to \infty$ filtering, there is some point where a further increase in the SNR does not significantly improve the filter performance.

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# Image Design: Generation of a Prescribed Image Through a Diffraction-Limited System with High-Contrast Recording

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*Abstract*—**Image restoration involves the recovery of an image which has been distorted by a given imaging system. "Image design," on the other hand, aims at determining the input image which when distorted by an imaging system (e.g., a display device) becomes a desired pattern. The image design problem is encountered in the design of masks for microphotography, microlithography, laser printing, and aids for the visually impaired. In this correspondence, we solve the "image design" problem using linear programming techniques for the case of an imaging system modeled by a band-limited linear system followed by a noninvertible point nonlinearity.**

## I. Introduction

This paper investigates a new image processing problem, which may be called "image design" (or, perhaps, "image synthesis"). Consider a nonideal imaging system, e.g., a microfilming camera, a laser printing system, or an image display device. An "input" image is fed to the system and the "output" image is generated (displayed). The goal of image design is to determine the input image that generates a prescribed desired output image.

An example of some ad hoc image design procedures which have been adopted in the past by the microprinting and microphotographic industry are the corrections usually made in the original masks. These corrections are deliberately introduced to compensate for the distortions caused by the microcamera itself. Serifs are introduced around corners to remove rounding effects, and sharp local reductions of thickness are made in intersecting lines to prevent the formation of fillers [1], [2]. For instance, Kodak recommends the addition of triangular serifs to corners, the dimension of which were determined by a process of trial and error [2].

At first sight it appears that this problem of "image design" is mathematically identical to the well-known problem of image restoration, the difference between them being mainly one of motive. One subtle difference, however, has to do with the existence of a solution. In an image restoration problem, the measured output results from an actual, albeit unknown, input. In the absence of measured error or noise, at least one solution must exist. In an image design problem, on the other hand, it is possible that no input image is able to produce the desired output image. Therefore, in image design, a problem of great

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