Alternating Projection onto Fuzzy Convex Sets

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Abstract— Alternating projections onto convex sets (POCS) is powerful tool for signal and image restoration. However, if POCS is among three or more nonintersecting convex sets, the result is not unique and POCS is generally not useful. This, however, can be overcome by allowing solutions that are in some sense, 'close' to each convex set. Such relaxation can be achieved through fuzzification of the sets into fuzzy convex sets. By performing POCS among the α-cuts of fuzzified sets, good solutions can be obtained. We propose morphological dilation as a fuzzification procedure. Fuzzy POCS is illustrated through application to the problems of time-bandwidth product minimization, signal extrapolation and solution of simultaneous equations.

Introduction

Alternating projections onto convex sets (POCS) [1] is a remarkably powerful method of signal recovery and synthesis. A (crisp) set, A, is convex if $\vec{x}_1 \in A$ and $\vec{x}_2 \in A$ implies that $\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2 \in A$ for all $0 \le \lambda \le 1$. In other words, the line segment connecting $\vec{x}_1$ and $\vec{x}_2$ is totally subsumed in A. Examples of sets of signals that are convex are the sets of bandlimited signals, duration limited signals, bounded signals, signals with energy less than one, signals with unit area, and complex signals with a specified phase.

The projection onto a convex set is illustrate in Figure 1. For a given $\vec{y} \notin A$, the projection onto A is the unique vector $\vec{x} \in A$ such that the mean square distance between $\vec{x}$ and $\vec{y}$ is minimum. If $\vec{y} \in A$, then the projection onto A is $\vec{y}$.

Here is the remarkable result of POCS. Given two or more convex sets with nonempty intersection, alternately projecting among the sets will converge to a point included in the intersection [2]. This is illustrated in Figure 2. If two convex sets do not intersect, then convergence is to a limit cycle that is a mean square solution to the problem. Specifically, the cycle is between points in each set that are closest in the mean square sense to the other set [3]. This is illustrated in Figure 3.

POCS breaks down in the important case where three or more convex sets do not intersect [4]. POCS converges to greedy limit cycles that are dependent on the ordering of the projections and do not display any desirable optimality properties. This is illustrated in Figure 4.

This third case, however, can be successfully addressed by fuzzy POCS. The problem becomes one of finding a solution that is, in some sense, equally close to each of the convex sets. The concept of 'close' suggests a fuzzification of the nonintersecting convex sets to fuzzy convex sets [5]. Even if three or more crisp sets do not intersect, α-cuts of their fuzzification can. In some cases, there exists an α such that intersection occurs at a single point. This is illustrated in Figure 5.

Fuzzy Convex Sets

The fuzzy set A_f (f for fuzzy) on the universal set E is defined by the membership function $\mu_A(\cdot)$ which maps E to the real value $[0,1]$. The set A_f can be written as

$$A_f = \{\vec{x}/\mu_A(\vec{x}) \ \mid \ \vec{x} \in E\}$$

Let A_f^α denote the crisp set corresponding to an α-cut of A_f.

$$A_f^\alpha = \begin{cases} \{\vec{x} \mid \mu_A(\vec{x}) \ge \alpha, \vec{x} \in E\} & ; \quad \alpha \ne 0 \\ E & ; \quad \alpha = 0 \end{cases} \tag{1}$$

The fuzzy set A_f is convex if all of its α-cuts ($0 \le \alpha \le 1$) are convex. Equivalently [6], the fuzzy set A_f is convex if and only if for every $0 \le \lambda \le 1$,

$$\mu_A[\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2] \ge \min\{\mu_A(\vec{x}_1), \mu_A(\vec{x}_2)\}$$

Fuzzified Convex Sets and Their Projections

Two methods of fuzzification of crisp convex sets of signals to fuzzy convex sets are useful in fuzzy POCS. If the crisp convex set is parameterized, the fuzzy convex set, in many cases, can be generated by fuzzification of parameter set. There is a homomorphism between the signal and

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parameter sets. The parameter set typically exists on an interval (*e.g.* $0 \leq$ Bandwidth $\leq \Omega$ for a set of bandlimited functions and $0 \leq$ Energy $\leq E$ for a set of signals with energy less than or equal to E) and is therefore typically convex. Fuzzification is achieved by dilation of this set with a convex dilation kernel. If the dilation kernel is convex, then the result is an α-cut of a fuzzy convex set. The degree of membership of a signal in the fuzzy signal set is equal to that of the membership of the parameter in the fuzzified parameter set. If, on the other hand, the crisp set of functions is not parameterized, fuzzification can be achieved through the direct morphological dilation of the signal in the set. By choosing convex dilation kernels of increasing dimension, α-cuts of the fuzzified convex set can be generated.

We now illustrate with some specific examples.

Bandlimited Signal

The set of the bandlimited signals with bandwidth Ω is

$$A_1 = \{x(t) \mid X(\omega) = 0 \quad \text{for} \quad |\omega| > \Omega\}$$

where the Fourier transform is

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Clearly, A_1 is convex. Let Ω_α be a nondecreasing function of α for $0 \leq \alpha \leq 1$ and $\Omega_1 = 0$. The dilation kernel [1] [7, 8] used to generate the α-cut of the fuzzification is

$$H_1^\alpha = \{\omega \mid |\omega| \leq \Omega_\alpha\}$$

The α-cut of the fuzzified set is

$$A_{1_f}^\alpha = \{x(t) \mid X(\omega) = 0 \quad \text{for} \quad |\omega| > \Omega + \Omega_\alpha\}$$

and the projection onto the convex α-cut is

$$\mathcal{P}_1^\alpha x \Leftrightarrow \begin{cases} 0 & ; \quad |\omega| > \Omega + \Omega_\alpha \\ X(\omega) & ; \quad \text{otherwise} \end{cases}$$

where $\Leftrightarrow$ denotes a Fourier transform pair.

Timelimited Signal

The convex set of timelimited signals is

$$A_2 = \{x(t) \mid x(t) = 0 \quad \text{for} \quad |t| > \tau\}$$

where 2τ is the centered interval over which the signal can be nonzero. The dilation kernel for the α-cut of the fuzzification operator is

$$H_2^\alpha = \{t \mid |t| \leq \tau_\alpha\}$$

[1] The dilation of the set $C \subset E$ with dilation kernel $D \subset E$ is $C \oplus D = \{\vec{x} \mid \text{there exist } \vec{y} \text{ such that } \vec{y} \in C \text{ and } \vec{x} - \vec{y} \in D\}$

where τ_α is a nondecreasing function α for $0 \leq \alpha \leq 1$ and $\tau_1 = 0$. The α-cut of the fuzzified set and the corresponding projection are

$$A_{2_f}^\alpha = \{x(t) \mid x(t) = 0 \quad \text{for} \quad |t| > \tau + \tau_\alpha\}$$

and

$$\mathcal{P}_2^\alpha x = \begin{cases} 0 & ; \quad |t| > \tau + \tau_\alpha \\ x(t) & ; \quad \text{otherwise} \end{cases}$$

Signals with Bounded Error

For a given signal $p(t)$, the convex set of signals with a bound of $KR(t)$ is

$$A_3 = \{x(t) \mid |x(t) - p(t)| \leq KR(t)\}$$

Using the dilation kernel

$$H_3^\alpha = \{k \mid |k| \leq K_\alpha\}$$

where K_α is a nondecreasing function α for $0 \leq \alpha \leq 1$ and $K_1 = 0$. The convex α-cut of the fuzzified set is

$$A_{3_f}^\alpha = \{x(t) \mid |x(t) - p(t)| \leq R_\alpha(t)\}$$

where $R_\alpha(t) = (K + K_\alpha)R(t)$. The projection onto α-cut of the fuzzy set is

$$\mathcal{P}_3^\alpha x = \begin{cases} p(t) + q(t) & ; \quad |x(t) - p(t)| > R_\alpha(t) \\ x(t) & ; \quad \text{otherwise} \end{cases}$$

where $q(t) = R_\alpha(t)[x(t) - p(t)]/|x(t) - p(t)|$.

Fuzzification by Signal Dilation

When the constraint set is not specified by a parameter or parameter set, then α-cuts of the fuzzification can be performed by direct dilation of each signal in the set. Let $g(\vec{x})$ be the fuzzification dilation kernel which maps E to the real value $[0, 1]$. Then the fuzzification of the crisp set, A, to the fuzzy set A_f is defined by

$$A_f = \{\vec{x}/\mu(\vec{x}) \mid \vec{x} \in E\} \tag{2}$$

where

$$\mu(\vec{x}) = \max_{\vec{y}}\{g(\vec{x} - \vec{y}) \mid \vec{y} \in A\}$$

Then we have the following theorem.

Theorem 1 *Let the crisp set A be convex and let $G = \{\vec{x}/g(\vec{x}) \mid \vec{x} \in E\}$ be a convex dilation kernel. Then A_f in equation (2) is a fuzzy convex set.*

The proof of the above theorem is in Appendix A.

We specify

$$g(\vec{x}) = m(\| \vec{x} \|)$$

Let $m(0) = 1$ and $m(z)$ be a monotonic decreasing function for $z \geq 0$. Then the α-cut of the fuzzification kernel, $g(\vec{x})$, is always a (convex) sphere. Let $m(R_\alpha) = \alpha$. We then have the following theorem for the projection onto A_f^α

Theorem 2 *Let $\vec{x}_0 \not\in A_f^\alpha$ and $\vec{x}_p^\alpha = \mathcal{P}^\alpha \vec{x}_0$. Then*

$$\vec{x}_p^\alpha = \vec{x}_p^1 + \frac{R_\alpha}{\| \vec{x}_0 - \vec{x}_p^1 \|}(\vec{x}_0 - \vec{x}_p^1)$$

where $\vec{x}_p^1$ is the projection onto the crisp set. The proof of is in Appendix B.

Method of Alternating Projections onto Fuzzy Convex Sets

The optimal POCS solution is achieved by the maximum value of α that results in a non empty intersection of all signal sets. In certain cases, this intersection can be at a single point.

To find the optimal solution, we start at a large value of α and iterate. If the iteration does not converge, α is decreased and the iteration is repeated. If convergence does occur, a search can be performed between the current and previous values of α for the optimal solution.

Examples

In this section, we will give the examples of signal synthesis and restoration based on fuzzy POCS.

Example 1 : Time-Bandwidth Product

Our problem is to find a one dimensional signal $x[n]$ which is positive, bandlimited, timelimited signal and has a specified area $V = \sum_n x[n]$. There exists no signal that satisfies all of these constraints. To apply fuzzy POCS, we will keep constant area and positivity sets crisp. The sets of bandlimited and timelimited signals, though, will be fuzzified, The convex crisp sets are

$$\begin{aligned} A_1 &= \{x[n] \mid X[k] = 0 \text{ for } k \neq 0\} \\ A_2 &= \{x[n] \mid x[n] = 0 \text{ for } n \neq 0\} \\ A_4 &= \{x[n] \mid x[n] \geq 0\} \\ A_5 &= \left\{ x[n] \mid \sum_{n=-L/2}^{L/2-1} x[n] = V \right\} \end{aligned}$$

where L is the length of the $x[n]$ and $X[k]$ is the discrete Fourier transform of $x[n]$. The α-cuts of the fuzzified sets are

$$A_{1_f}^\alpha = \left\{ x[n] \mid X[k] = 0 \text{ for } |k| > -\sqrt{\xi}\Theta_\alpha \right\}$$

$$A_{2_f}^\alpha = \left\{ x[n] \mid x[n] = 0 \text{ for } |n| > -\frac{\Theta_\alpha}{\sqrt{\xi}} \right\}$$

where $\Theta_\alpha = \frac{L}{4}\log\alpha$ and ξ parameterize the relative importance between bandlimitedness and timelimitedness. Figure 6 using $V = L = 1024$ shows the results for various values of ξ. Figure 6a, 6b and 6c result when ξ is 4, 1 and 0.25, respectively. In Figure 6, the solid lines show $x[n]$ and the broken lines show the Gaussian function which has the same peak value, same mean, and the same variance as the signal, $x[n]$. When ξ is large, then timelimitedness is more important than bandlimitedness as shown in Figure 6. Not surprisingly, fuzzy POCS yields a result quite close to the Gaussian curve. The Gaussian is the function which displays the minimum time-bandwidth product [9].

Example 2 : Bandlimited Signal Extrapolation

This example is motivated by the celebrated Papoulis Gerchberg algorithm [10, 11]. The problem is estimation of a high bandwidth signal, $p[n]$, with a signal of lower bandwidth. Assume the signal is $p[n] = \text{sinc}(2Bn)$. Let A_1 be the set of signals with frequency components no greater than $\frac{B}{2}$. Note that $x[n] \not\in A_1$. In our simulation, $B = 1/64$. The crisp convex sets are

$$\begin{aligned} A_1 &= \{x[n] \mid X[k] = 0 \text{ for } |k| > L/128\} \\ A_3 &= \{x[n] \mid x[n] = p[n]\} \end{aligned}$$

where $p[n] = \text{sinc}(n/32)$. The α-cuts of the fuzzified set are

$$A_{1_f}^\alpha = \{x[n] \mid X[k] = 0 \text{ for } |k| \leq \Phi_\alpha\}$$

$$A_{3_f}^\alpha = \left\{ x[n] \mid |x[n] - p[n]| \leq \frac{\Psi[n]}{\sqrt{\xi}}\log\alpha \right\}$$

where

$$\Phi_\alpha = -\frac{\sqrt{\xi}L}{4}\log\alpha + \frac{L}{128},$$

$$\Psi[n] = \frac{1 - \exp(-|2n|/L)}{\log\alpha_{min}},$$

and ξ parameterized the relative importance of the two constraints. L is the length of $x[n]$, and α is varied from 1 to $\alpha_{min} = 1/3$. Figure 7a, 7b and 7c correspond to values of $\xi = 4$, 1 and 0.25 respectively. In our case, if ξ is large, then the known signal is more important than the signal being bandlimited. Figure 8 shows $|X[k]|$ for each case.

Example 3 : Solution of a Set of Overdetermined Linear Equations

This example outlines solution of the overdetermined linear equation, $\underline{Q}\vec{x} = \vec{y}$, by POCS, assuming that $\underline{Q}^T\underline{Q}$ is nonsingular. The minimum mean square error solution is

$$\vec{x}_a^* = (\underline{Q}^T\underline{Q})^{-1}\underline{Q}^T\vec{y}$$

In other words, for a given $\underline{Q}$ and $\vec{y}$,

$$\min_{\vec{x}}\{\| \underline{Q}\vec{x} - \vec{y} \|\} = \| \underline{Q}\vec{x}_a^* - \vec{y} \| \tag{3}$$

where $\|\cdot\|$ is l_2 norm.

For the POCS solution, however, the result is a quite different. Let

$$\underline{Q} = [\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_N]^T$$

and $\vec{y} = [y_1, y_2, \cdots, y_N]^T$. Then the linear equation can be written as [12]

$$\vec{q}_i^T \vec{x} = y_i \quad \text{for} \quad i = 1, 2, \cdots, N$$

The crisp solution set for the i^{th} equation is

$$B_i = \{\vec{x} \mid |\vec{q}_i^T \vec{x} - y_i| \leq 0\}$$

Using the fuzzification with a dilation kernel $h(\alpha) = e^{-\alpha}$ on the singleton value of $\vec{q}_i^T \vec{x} - y_i$, the α-cut of the fuzzification is

$$B_{i_f}^{\alpha} = \{\vec{x} \mid |\vec{q}_i^T \vec{x} - y_i| \leq -\log\alpha\} \tag{4}$$

We now seek the maximum value of α (minimum of $-\log\alpha$) which satisfies (4). In other words,

$$\min_{\vec{x}}\{\max_i[|\vec{q}_i^T \vec{x}_p^* - y_i|]\}$$

$$= \min_{\vec{x}}\{\| \underline{Q}\vec{x}_p^* - \vec{y} \|_\infty\}$$

where $\|\cdot\|_\infty$ is l_∞ norm of the metric space. Thus, in contrast to minimum mean square (l_2 norm) solution in (3), we obtain a minimum L_∞ norm solution using fuzzy POCS.

When we use the nonparameterized method for fuzzification with $g(\vec{x}) = m(\| \vec{x} \|)$ and $m(z) = e^{-z}$, the α-cut of the fuzzification is

$$C_{i_f}^{\alpha} = \{\vec{x} \mid |\vec{q}_i^T \vec{x} - y_i| \leq - \| \vec{q}_i \| \log\alpha\} \tag{5}$$

We now seek the maximum value of α which satisfies (5). In other words,

$$\min_{\vec{x}}\left\{\max_i\left[\frac{|\vec{q}_i^T \vec{x}_q^* - y_i|}{\| \vec{q}_i \|}\right]\right\}$$

$$= \min_{\vec{x}}\{\| \underline{D}_Q^{-1}\underline{Q}\vec{x}_q^* - \underline{D}_Q^{-1}\vec{y} \|_\infty\}$$

where

$$\underline{D}_Q = diag[\| \vec{q}_1 \|, \| \vec{q}_2 \|, \cdots, \| \vec{q}_N \|]$$

Notes

1. Consider the case, illustrated in Figure 3, where two crisp convex sets do not intersect. Let the limit cycle be between points $\vec{y}_A$ and $\vec{y}_B$. If both sets are fuzzified using the same convex dilation kernel, then an optimal solution using fuzzy POCS is the point $(\vec{y}_A + \vec{y}_B)/2$.
2. Our procedure using fuzzy POCS begins with small convex sets. The sets grow until intersection occurs. Alternately, initialization can be initiated with large α-cuts of convex sets. The sets are reduced in size until iteration breaks into a small limit cycle.

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References

[1] H. Stark, editor, **Image Recovery: Theory and Application**, (Academic Press, Orlando, 1987).

[2] D.C. Youla and H. Webb, "Image restoration by method of convex set projections: Part I - Theory", *IEEE Trans. Med. Imaging*, vol MI-1, pp.81-94, 1982.

[3] M.H. Goldburg and R.J. Marks II "Signal synthesis in the presence of an inconsistent set of constraints", *IEEE Transactions on Circuits and Systems*, vol. CAS-32 pp. 647-663 (1985).

[4] D.C. Youla and V. Velasco, "Extensions of a result on the synthesis of signals in the presence of inconsistent constraints", *IEEE Transactions on Circuits and Systems*, vol. CAS-33, pp.465-468 (1986).

[5] M.R. Civanlar and H.J. Trussel, "Digital signal restoration using fuzzy sets", *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. ASSP-34, p.919 (1986).

[6] L. Zadeh, "Fuzzy Sets", **Information Control**, vol.8, pp.338-353, 1965. Reprinted in J.C. Bezdek & S.K. Pal, **Fuzzy Models for Pattern Recognition**, (IEEE Press, 1992).

[7] J. Serra, **Image analysis and mathematical morphology**, (Academic Press, New York, 1982).

[8] P. Maragos and R. W. Schafer, "Morphological filter - Part I: Their set-theoretic analysis and relations to linear shift-invariant filters", *IEEE Trans. on Acoustic, Speech, and Signal Processing*, vol.ASSP-35, no.8, 1987. pp.1153-1169.

[9] J.B. Thomas, **An Introduction to Statistical Communication Theory**, (Wiley, NY, 1969).

[10] A. Papoulis, "A new algorithm in spectral analysis and bandlimited signal extrapolation", *IEEE Transactions on Circuits and Systems*, vol.CAS-22, pp.735-742 (1975).

[11] R.J. Marks II, **An Introduction to Shannon Sampling and Interpolation Theory**, (Springer-Verlag, 1991).

[12] S. Kuo and R.J. Mammon, "Resolution enhancement of tomographic images using the row action projection method", *IEEE Transactions on Medical Imaging*, vol.10, no, pp.593-601 (1992).

Appendices

Proof of Theorem 1

Let G^α denote an α-cut of G

$$G^\alpha = \{\vec{x} \mid g(\vec{x}) \geq \alpha\}$$

Let $\vec{x}_1 \in A_f^\alpha$ and $\vec{x}_2 \in A_f^\alpha$. Then there exist $\vec{y}_1$ and $\vec{y}_2$ such that

$$\vec{x}_1 - \vec{y}_1 \in G^\alpha, \quad \vec{x}_2 - \vec{y}_2 \in G^\alpha$$

Now, we examine $\gamma\vec{x}_1 + (1-\gamma)\vec{x}_2$,

$$\gamma\vec{x}_1 + (1-\gamma)\vec{x}_2$$

$$= \gamma(\vec{y}_1 + \vec{x}_1 - \vec{y}_1) + (1-\gamma)(\vec{y}_2 + \vec{x}_2 - \vec{y}_2)$$

$$= [\gamma(\vec{x}_1 - \vec{y}_1) + (1-\gamma)(\vec{x}_2 - \vec{y}_2)]$$

$$+[\gamma\vec{y}_1 + (1-\gamma)\vec{y}_2]$$

Because A and G^α are convex sets, we have

$$\gamma\vec{y}_1 + (1-\gamma)\vec{y}_2 \in A$$

$$\gamma(\vec{x}_1 - \vec{y}_1) + (1-\gamma)(\vec{x}_2 - \vec{y}_2) \in G^\alpha$$

So,

$$\gamma\vec{x}_1 + (1-\gamma)\vec{x}_2 \in A_f^\alpha$$

Therefore our proof is complete.
Q. E. D.

Proof of Theorem 2

Before we prove the Theorem, we show the following Lemma

Lemma 1 *For any $\vec{y} \in A$ and $\vec{x} \notin A_f^\alpha$, then* $\| \vec{x} - \vec{y} \| > R_\alpha$

Proof : Assume that $\| \vec{x} - \vec{y} \| \leq R_\alpha$. Then $\vec{y} \in A$ and $\| \vec{x} - \vec{y} \| \leq R_\alpha$ implies $\vec{x} \in A_f^\alpha$. This contradicts the assumption and our proof is complete. Q.E.D.

Proof of Theorem 2

$$\vec{x}_p^\alpha - \vec{x}_p^1 = \frac{R_\alpha}{\| \vec{x}_0 - \vec{x}_p^1 \|}(\vec{x}_0 - \vec{x}_p^1)$$

Here, $\| \vec{x}_p^\alpha - \vec{x}_p^1 \| = R_\alpha$ and $\vec{x}_p^1 \in A$. Therefore $\vec{x}_p^\alpha \in A_f^\alpha$. Also,

$$\begin{aligned} \vec{x}_0 - \vec{x}_p^\alpha &= (\vec{x}_0 - \vec{x}_p^1) + (\vec{x}_p^1 - \vec{x}_p^\alpha) \\ &= (\vec{x}_0 - \vec{x}_p^1)\left[1 - \frac{R_\alpha}{\| \vec{x}_0 - \vec{x}_p^1 \|}\right] \end{aligned}$$

$$\| \vec{x}_0 - \vec{x}_p^\alpha \| = \| \vec{x}_0 - \vec{x}_p^1 \| - R_\alpha$$

For any $\vec{y} \in A_f^\alpha$, let $\vec{y}_p^1 = \mathcal{P}^1\vec{y}$. Then $\| \vec{y} - \vec{y}_p^1 \| \leq R_\alpha$.

$$\| \vec{y} - \vec{x}_0 \| \geq \| \vec{y}_p^1 - \vec{x}_0 \| - \| \vec{y} - \vec{y}_p^1 \| \geq \| \vec{y}_p^1 - \vec{x}_0 \| - R_\alpha$$

If $\vec{y}_p^1 \neq \vec{x}_p^1$, then

$$\| \vec{y} - \vec{x}_0 \| \geq \| \vec{y}_p^1 - \vec{x}_0 \| - R_\alpha$$

$$> \| \vec{x}_p^1 - \vec{x}_0 \| - R_\alpha = \| \vec{x}_0 - \vec{x}_p^\alpha \|$$

Therefore

$$\| \vec{y} - \vec{x}_0 \| > \| \vec{x}_0 - \vec{x}_p^\alpha \|$$

If $\vec{y}_p^1 = \vec{x}_p^1$, then

$$\| \vec{y} - \vec{x}_0 \| \geq \| \vec{x}_p^1 - \vec{x}_0 \| - \| \vec{y} - \vec{x}_p^1 \|$$

The equality holds for only the case that $\vec{y} = \vec{x}_p^\alpha$.

$$\| \vec{y} - \vec{x}_0 \| > \| \vec{x}_0 - \vec{x}_p^\alpha \|$$

Therefore our proof is complete.
Q. E. D.

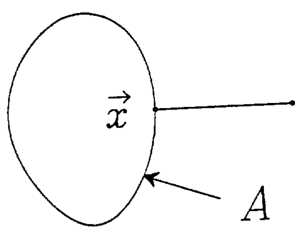


Figure 1: The set, A, is convex. The projection of $\vec{y}$ onto A is the unique element in A closest to $\vec{y}$ in the mean square sense.

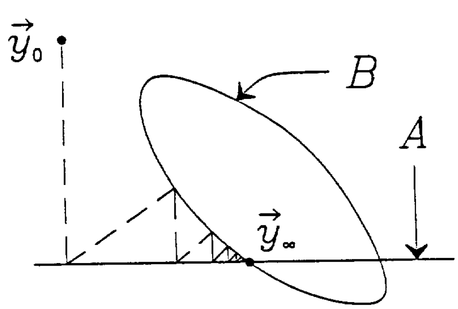


Figure 2: Alternating projection between two or more convex sets with nonempty intersection results in convergence to a fixed point, in that intersection. Here, sets A (a line segment) and B are convex. Initializing the iteration at $\vec{y}_0$, convergence is to $\vec{y}_\infty \in A \cap B$.

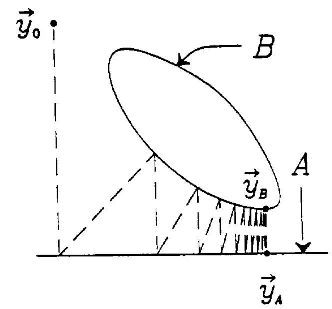


Figure 3: If two convex sets, A and B, do not intersect, POCS converges to a limit cycle, here between the points $\vec{y}_A$ and $\vec{y}_B$. The point $\vec{y}_B \in B$ is the point in B closest to the set A. A similar property is true for $\vec{y}_A$.

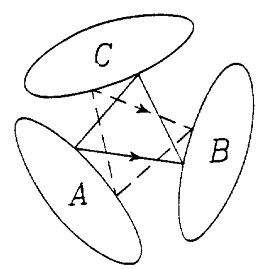


Figure 4: If three or more convex sets do not intersect, the POCS convergences to greedy limit cycles with no particularly useful properties. As illustrated here, the limit cycles can differ for different choices of set ordering.

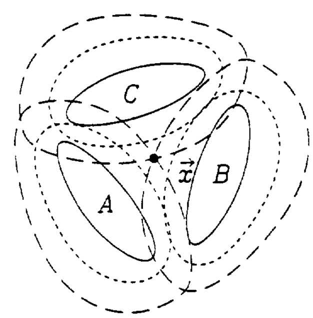
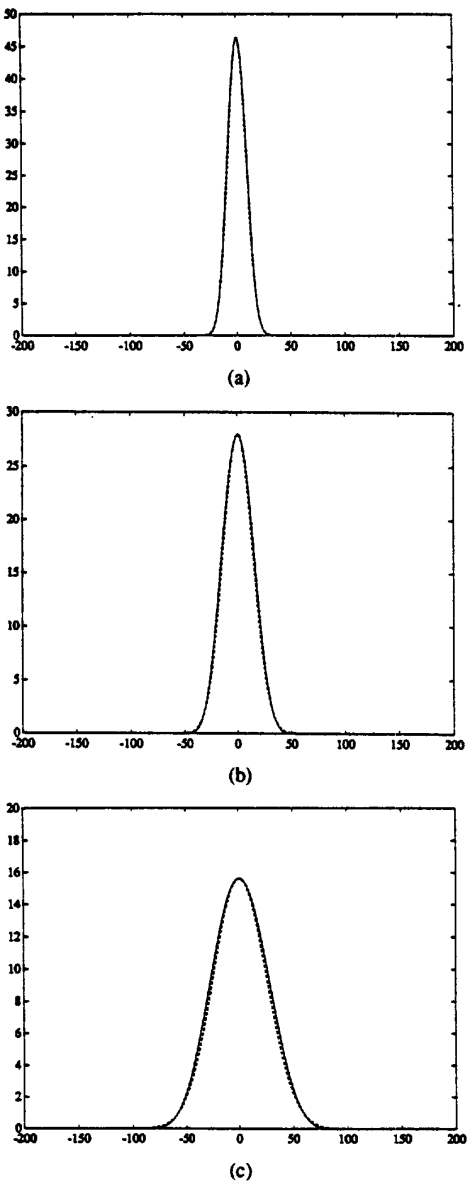


Figure 5: Crisp convex sets can be fuzzified into fuzzy convex sets. The α-cuts of the fuzzifications are convex. As illustrated here, there can exist an α-cut of each of the convex sets such that the resulting intersection is nonempty. Application of POCS to these α-cuts will result to convergence to a point in this intersection. The solution, for large α, is then 'close' to each of the underlying crisp sets.



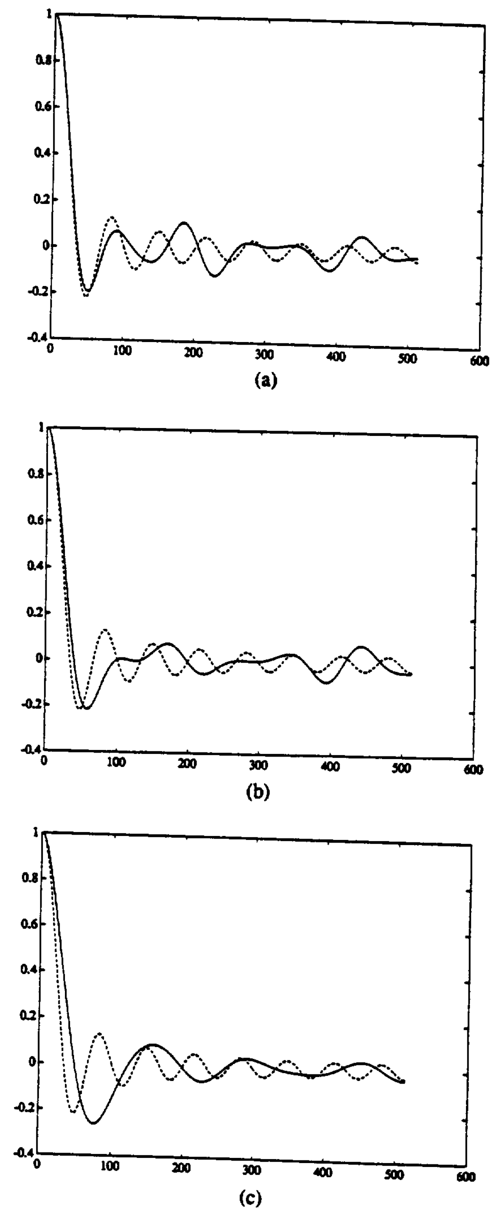


Figure 6: Fuzzy POCS solutions of a signal that is both time limited and bandlimited. The importance of being bandlimited increases from (a) to (c). The result is compared to a Gaussian curve fit (broken line) in each case.

Figure 7: A bandlimited signal is plotted here with a broken line. We attempt to fit a signal with lower bandwidth to the known signal while simultaneously keeping the error within a specified boundary. The fuzzy POCS results are shown, from (a) to (c), as the allowable bandwidth increases and the bound constraint is relaxed.

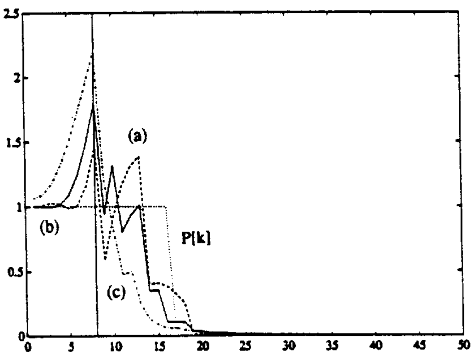


Figure 8: The magnitudes of the discrete Fourier transform of the signals in Figure 7(a) through Figure 7(c).