Control of Linear Time Invariant Systems Part II

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Finite Sum

Theorem

For A a constant $n \times n$ regressive matrix, there exists a collection of n linearly independent functions $\{\gamma_k\}_0^{n-1}$ such that

$$e_A(t, t_0) = \sum_{k=0}^{n-1} \gamma_k(t, t_0) A^k.$$

Infinite Sum

Theorem

For A a constant $n \times n$ regressive matrix, we have that

$$e_A(t, t_0) = \sum_{i=0}^{\infty} A^i h_i(t, t_0).$$

Rank Theorem

Theorem

The regressive linear time invariant system

$$x^{\Delta}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{1}$$
$$y(t) = Cx(t) + Du(t) \tag{2}$$

is controllable on $[t_0, t_f]$ *if and only if the* $n \times nm$ *controllability matrix*

$$\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$$

satisfies

$$\operatorname{rank} \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = n.$$

Proof:

Suppose the system is controllable, but that for the sake of a contradiction, that the rank condition fails. Then there exists an $n \times 1$ vector x_a such that

$$x_a^T A^k B = 0, \quad k = 0, \ldots, n-1.$$

Now, there are two cases to consider: either $x_a^T x_f = 0$ or $x_a^T x_f \neq 0$.
Assume $x_a^T x_f \neq 0$. Then we know that for any t, the solution at time t is given by

$$\begin{aligned} x(t) &= \int_{t_0}^{t} e_A(t, \sigma(s)) B u_{x_0}(s) \, \Delta s \, + e_A(t, t_0) x_0 \\ &= \int_{t_0}^{t} e_A(s, t_0) B u_{x_0}(t, \sigma(s)) \, \Delta s \, + e_A(t, t_0) x_0. \end{aligned}$$

Choose initial state $x_0 = By$, where y is arbitrary. Then we have that

$$\begin{aligned} x_a^T x(t) &= x_a^T \int_{t_0}^{t} e_A(s,t_0) B u_{x_0}(t,\sigma(s)) \;\Delta s + x_a^T e_A(t,t_0) x_0 \\ &= \int_{t_0}^{t} \sum_{k=0}^{n-1} \gamma_k(s,t_0) x_a^T A^k B u_{x_0}(t,\sigma(s)) \;\Delta s \\ &+ \sum_{k=0}^{n-1} \gamma_k(t,t_0) x_a^T A^k B y = 0, \end{aligned}$$

so that

$$x_a^T x(t) = 0 \;\text{ for all }\; t,$$

which is a contradiction since we know that $x_a^T x(t_f) = x_a^T x_f \neq 0$.

Now assume that $x_a^T x_f = 0$. This time, we choose initial state $x_0 = e_A^{-1}(t_f, t_0)x_a$. Similarly to the equation above, we have that

$$\begin{aligned} x_a^T x(t) &= \int_{t_0}^{t} \sum_{k=0}^{n-1} \gamma_k(s, t_0) x_a^T A^k B u_{x_0}(t, \sigma(s)) \; \Delta s \\ &+ x_a^T e_A(t, t_0) e_A^{-1}(t_f, t_0) x_a \\ &= x_a^T e_A(t, t_0) e_A^{-1}(t_f, t_0) x_a. \end{aligned}$$

In particular, at $t = t_f$, we have

$$x_a^T x(t_f) = ||x_a||^2 \neq 0,$$

another contradiction. Thus in either case we arrive at a contradiction, and so controllability implies the rank condition.

Conversely, suppose that the system is not controllable. Then there exists an initial state x_0 such that for all input signals $u(t)$, we have that $x(t_f) \neq x_f$. Thus, it follows that

$$\begin{aligned} x_f \neq x(t_f) &= \int_{t_0}^{t_f} e_A(t_f, \sigma(s)) B u_{x_0}(s) \, \Delta s + e_A(t_f, t_0) x_0 \\ &= \int_{t_0}^{t_f} e_A(s, t_0) B u_{x_0}(t_f, \sigma(s)) \, \Delta s + e_A(t_f, t_0) x_0 \\ &= \int_{t_0}^{t_f} \sum_{k=0}^{n-1} \gamma_k(s, t_0) A^k B u_{x_0}(t_f, \sigma(s)) \, \Delta s \ + e_A(t_f, t_0) x_0, \end{aligned}$$

so that in particular, we have that

$$\sum_{k=0}^{n-1} A^k B \int_{t_0}^{t_f} \gamma_k(s, t_0) u_{x_0}(t_f, \sigma(s)) \, \Delta s \neq x_f - e_A(t_f, t_0) x_0.$$

Notice that the last equation really says that there is no linear combination of the matrices $A^k B$ for $k = 0, 1, \ldots, n-1$ that will satisfy the equation

$$\sum_{k=0}^{n-1} A^k B \alpha_k = x_f - e_A(t_f, t_0) x_0.$$

This statement of course needs verification, which we will not do here, but it does follow from a quite technical linear algebra argument.

Theorem

The regressive linear time invariant state equation

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

is controllable if and only if for every scalar λ the only complex ntimes1 vector p that satisfies

$$p^T A = \lambda p^T, \; p^T B = 0$$

is $p = 0$.

Theorem

The regressive linear state equation

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

is controllable if and only if

$$\mathrm{rank}\begin{bmatrix} zI - A & B \end{bmatrix} = n$$

for every complex scalar z*.*

Proof:

By Theorem 4, the state equation is not controllable if and only if we have the existence of a nonzero complex $n \times 1$ vector p and complex scalar λ such that

$$p^T \begin{bmatrix} \lambda I - A & B \end{bmatrix} = 0, \;\; p \neq 0.$$

But this condition is equivalent to saying that

$$\mathrm{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} < n,$$

so that the claim follows.

Rank Theorem

Theorem

The autonomous linear regressive system

$$\begin{aligned} x^{\Delta}(t) &= Ax(t), \;\; x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

is observable on $[t_0, t_f]$ *if and only if the* $nm \times n$ observability matrix *satisfies*

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

Proof:

Again, we show that the rank condition fails if and only if the observability Gramian is not invertible. Thus, suppose that the rank condition fails. Then, there exists a nonzero $n \times 1$ vector x_a such that

$$CA^k x_a = 0, \;\; k = 0, \ldots, n-1.$$

This implies, using the power series representation of the matrix exponential given earlier, that

$$\begin{aligned} M(t_0, t_f)x_a &= \int_{t_0}^{t_f} e_A^T(t, t_0) C^T C e_A(t, t_0) x_a \, \Delta t \\ &= \int_{t_0}^{t_f} e_A^T(t, t_0) C^T \sum_{k=0}^{n-1} \gamma_k(t, t_0) CA^k x_a \, \Delta t = 0, \end{aligned}$$

so that the Gramian is not invertible.

Conversely, suppose that the Gramian is not invertible. Then there exists nonzero x_a such that $x_a^T M(t_0, t_f)x_a = 0$. As argued before, this then implies that

$$Ce_A(t, t_0)x_a = 0, \quad t \in [t_0, t_f).$$

At $t = t_0$, we obtain $Cx_a = 0$, and differentiating k times and evaluating the result at $t = t_0$ gives

$$CA^k x_a = 0, \quad k = 0, \ldots, n-1.$$

Thus, we have that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_a = 0$$

so that the rank condition fails.

Theorem

The regressive time invariant linear state equation

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

is observable if and only if for every complex scalar λ *the only complex* $n \times 1$ *vector* p *that satisfies*

$$Ap = \lambda p, \;\; Cp = 0$$

is $p = 0$.

Theorem

The regressive time invariant linear state equation

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

is observable if and only if

$$\mathrm{rank} \begin{bmatrix} C \\ zI - A \end{bmatrix} = n$$

for every complex scalar z.

Definition

For any shift $u(t, \sigma(s))$ of the transformable function $u(t)$, the time invariant system

$$\begin{aligned} x^{\Delta} &= Ax + Bu \\ y &= Cx \end{aligned}$$

is said to be *uniformly bounded-input, bounded-output stable* if there exists a finite constant η such that the corresponding zero-state response satisfies

$$\sup_{t \geq 0} ||y(t)|| \leq \eta \sup_{t \geq 0} \sup_{s \geq 0} ||u(t, \sigma(s))||.$$

Theorem

The regressive linear time invariant system

$$\begin{aligned} x^{\Delta} &= Ax + Bu \\ y &= Cx \end{aligned}$$

is bounded-input, bounded-output stable if and only if there exists a finite $\beta > 0$ such that

$$\int_0^{\infty} ||G(t)||\Delta t \leq \beta.$$

Proof:

Suppose we have the existence of the claimed $\beta > 0$. For any time t, we have that

$$y(t) = \int_0^t Ce_A(t, \sigma(s))Bu(s)\Delta s = \int_0^t Ce_A(s, 0)Bu(t, \sigma(s))\Delta s,$$

so that

$$||y(t)|| \leq ||C|| \int_0^\infty ||e_A(s, 0)||\Delta s \; ||B|| \; \sup_{s\geq 0} ||u(t, \sigma(s))||.$$

Therefore, it follows that

$$\sup_{t\geq 0} ||y(t)|| \leq ||C|| \int_0^\infty ||e_A(s, 0)||\Delta s \; ||B|| \; \sup_{t\geq 0} \sup_{s\geq 0} ||u(t, \sigma(s))||.$$

Thus, if we choose $\eta = ||C|| \, \beta \, ||B||$, then the claim follows.

Conversely, suppose that the system is bounded-input bounded-output stable, but for the sake of a contradiction that the integral is unbounded. Then we have that

$$\sup_{t\geq 0} ||y(t)|| \leq \eta \sup_{t\geq 0} \sup_{s\geq 0} ||u(t,\sigma(s))||,$$

and

$$\int_0^\infty ||G(t)||\Delta t > \beta, \text{ for all } \beta > 0.$$

In particular, there exist indices i, j such that

$$\int_0^\infty |G_{ij}(t)|\Delta t > \beta.$$

We now choose $u(t, \sigma(s))$ in the following manner: set $u_k(t, \sigma(s)) = 0$ for all $k \neq j$, and for $u_j(t, \sigma(s))$ choose the function so that

$$u_j(t, \sigma(s)) = \begin{cases} 1, & \text{if } G_{ij}(s) > 0 \\ 0, & \text{if } G_{ij}(s) = 0 \\ -1, & \text{if } G_{ij}(s) < 0, \end{cases}$$

and choose $\beta > \eta > 0$. Notice that

$$\sup_{t \geq 0} \sup_{s \geq 0} ||u(t, \sigma(s)|| \leq 1,$$

so that

$$\sup_{t \geq 0} ||y(t)|| \leq \eta.$$

However, it follows that

$$\begin{aligned}
\sup_{t\geq 0} ||y(t)|| &= \sup_{t\geq 0} || \int_0^t G(s)u(t,\sigma(s))\Delta s|| \\
&= \sup_{t\geq 0} || \int_0^t G_j(s) \cdot u_j(s)\Delta s|| \\
&\geq \sup_{t\geq 0} \int_0^t |G_{ij}(s)|\Delta s \\
&= \int_0^\infty |G_{ij}(s)|\Delta s > \beta > \eta,
\end{aligned}$$

which of course is a contradiction. Thus, the claim follows.