The Generalized Laplace Transform: Applications to Adaptive Control*

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- Explore algebraic properties of the transform: convolution, identity element.
- Applications to adaptive control of linear systems (time permitting): controllability and observability for linear systems.

Generalized Laplace Transforms

Definition

For $f : \mathbb{T} \to \mathbb{R}$, the *Laplace transform of* f, denoted by $\mathcal{L}\{f\}$ or $F(z)$, is given by

$$\mathcal{L}\{f\}(z) = F(z) = \int_0^\infty f(t) g^\sigma(t)\, \Delta t, \qquad (1)$$

where $g(t) = e_{\ominus z}(t, 0)$.

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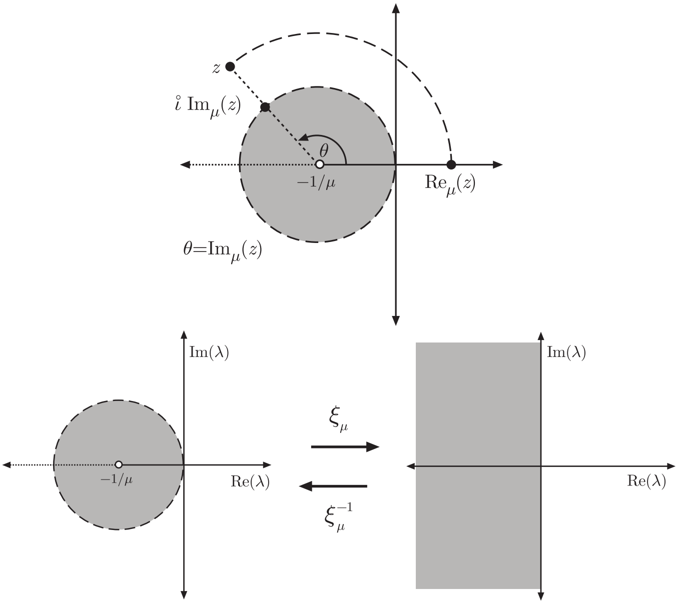
Recall...

- $\ominus z = \frac{-z}{1+\mu(t)z}$,
- $e_\lambda(t, t_0)$ is the unique solution of the dynamic IVP $x^\Delta(t) = \lambda x(t),\ x(t_0) = 1$.

Hilger Complex Plane and Cylinder Transform



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Under these assumptions, if $\lambda \in \mathbb{H}$, where $\mathbb{H}$ denotes the *Hilger circle* given by

$$\mathbb{H} = \mathbb{H}_t = \left\{ z \in \mathbb{C} : \left| z + \frac{1}{\mu(t)} \right| < \frac{1}{\mu(t)} \right\},$$

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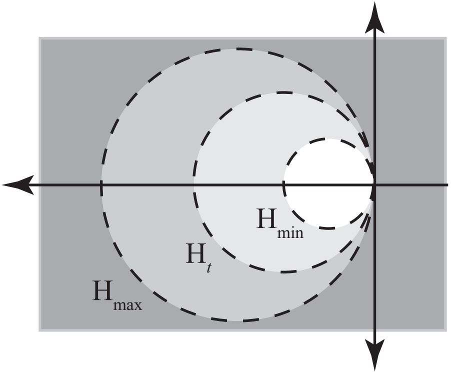
Careful: $\lambda = \ominus z$ is time varying!

To give an appropriate domain for the transform, which of course is tied to the region of convergence of the integral in (1), for any $c > 0$ define the set

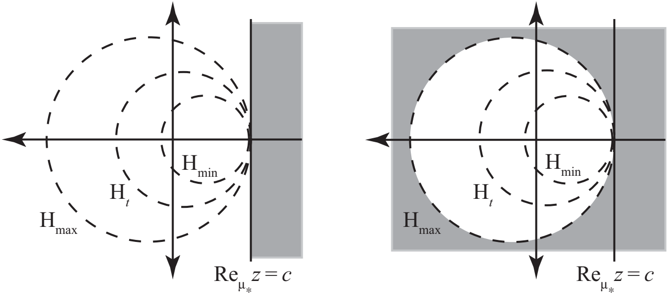
$$\begin{aligned} D &= \{z \in \mathbb{C} : z \in \overline{\mathbb{H}}_{\max}^{\mathsf{C}} \text{ and } z \text{ satisfies } |1 + z\mu_*| > |1 + c\mu_*|\} \\ &= \overline{\mathbb{H}}_{\max}^{\mathsf{C}} \cap \{\operatorname{Re}_{\mu_*}(z) > \operatorname{Re}_{\mu_*}(c)\}, \end{aligned}$$

Note that if $\mu_* = 0$ this set is a right half plane.

Dynamic Hilger circles and R.O.C.s



Time varying Hilger circles. The largest, $\mathbb{H}_{\text{max}}$ has center μ_{min} while the smallest, $\mathbb{H}_{\text{min}}$ has center μ_{max}. In general, the Hilger circle at time t is denoted by $\mathbb{H}_t$ which has center $\mu(t)$. The exterior of each circle is shaded representing the corresponding regions of convergence.



The region of convergence is shaded. On the left, the $\mu_* = 0$ case. On the right, the $\mu_* \neq 0$ case. In the latter, note our proof of the inversion formula is only valid for $\mathrm{Re}\, z > c$, i.e. the right half plane bounded by this abscissa of convergence even though the region of convergence is clearly a superset of this right half plane.

Furthermore, if $z \in D$, then $\ominus z \in \mathbb{H}_{\text{min}} \subset \mathbb{H}_t$ since for all $z \in D$, $\ominus z$ satisfies the inequality

$$\left| \ominus z + \frac{1}{\mu^*} \right| < \frac{1}{\mu^*}.$$

Finally, if $\text{Re}_{\mu_*}(z) > \text{Re}_{\mu_*}(c)$, then $\text{Re}_{\mu}(z) > \text{Re}_{\mu}(c)$ for all $t \in \mathbb{T}$ since the Hilger real part is an increasing function in μ.

Exponential Type I and Type II Functions

Definition

The function $f : \mathbb{T} \to \mathbb{R}$ is said to be of *exponential Type I* if there exists constants $M, c > 0$ such that $|f(t)| \leq Me^{ct}$. Furthermore, f is said to be of *exponential Type II* if there exists constants $M, c > 0$ such that $|f(t)| \leq Me_c(t, 0)$.

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- The time scale exponential function, $e_\alpha(t, t_0)$, is Type II.
- The time scale polynomials $h_k(t, 0)$ are Type I.

Region of Convergence for the Transform

Theorem

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The same estimates used in the proof of the preceding theorem can be used to show that if $f(t)$ is of exponential Type II with constant c and $\mathrm{Re}_\mu(z) > \mathrm{Re}_\mu(c)$, then $\lim_{t\to\infty} e_{\ominus z}(t,0)f(t) = 0$.

Theorem

Let F denote the generalized Laplace transform for $f : \mathbb{T} \to \mathbb{R}$.

- $F(z)$ *is analytic in* $\operatorname{Re}_\mu(z) > \operatorname{Re}_\mu(c)$.
- $F(z)$ *is bounded in* $\operatorname{Re}_\mu(z) > \operatorname{Re}_\mu(c)$.
- $\lim_{|z|\to\infty} F(z) = 0$.

Initial and Final Values Theorem

Theorem

Let $f : \mathbb{T} \to \mathbb{R}$ have generalized Laplace transform $F(z)$. Then

- $f(0) = \lim_{z\to\infty} zF(z)$,
- $\lim_{t\to\infty} f(t) = \lim_{z\to 0} zF(z)$ *when the limits exist.*

Inversion Theorem

Theorem

Suppose that $F(z)$ *is analytic in the region* $\mathrm{Re}_\mu(z) > \mathrm{Re}_\mu(c)$ *and* $\lim_{|z|\to\infty} F(z) \to 0$ *uniformly in this region. Suppose* $F(z)$ *has finitely many regressive poles of finite order* $\{z_1, z_2, \ldots z_n\}$ *and* $\tilde{F}_{\mathbb{R}}(z)$ *is the transform of the function* $\tilde{f}(t)$ *on* $\mathbb{R}$ *that corresponds to the transform* $F(z) = F_{\mathbb{T}}(z)$ *of* $f(t)$ *on* $\mathbb{T}$*. If*

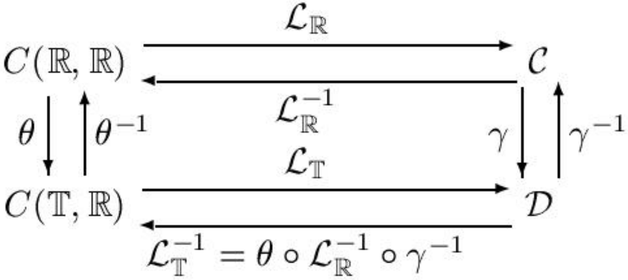
$$\int_{c-i\infty}^{c+i\infty} |\tilde{F}_{\mathbb{R}}(z)| dz < \infty,$$

then

$$f(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i}\, e_z(t,0)F(z),$$

has transform $F(z)$ *for all* z *with* $\mathrm{Re}\,(z) > c$.

Proof. The proof follows from the commutative diagram between the appropriate function spaces.



Let $\mathcal{C}$ be the collection of Laplace transforms over $\mathbb{R}$, and $\mathcal{D}$ the collection of transforms over $\mathbb{T}$, i.e. $\mathcal{C} = \{\tilde{F}_{\mathbb{R}}(z)\}$ and $\mathcal{D} = \{F_{\mathbb{T}}(z)\}$, where $\tilde{F}_{\mathbb{R}}(z) = \tilde{G}(z)e^{-z\tau}$ and $F_{\mathbb{T}}(z) = G(z)e_{\ominus z}(\tau, 0)$ for G and $\tilde{G}$ rational functions in z and for τ an appropriate constant.

Each of θ, γ, θ^{-1}, γ^{-1} maps functions involving the continuous exponential to the time scale exponential and vice versa. For example, γ maps the function $\tilde{F}(z) = \frac{e^{-za}}{z}$ to the function $F(z) = \frac{e_{\ominus z}(a,0)}{z}$, while γ^{-1} maps $F(z)$ back to $\tilde{F}(z)$ in the obvious manner.

If the representation of $F(z)$ is independent of the exponential (that is, $\tau = 0$), then γ and its inverse will act as the identity. For example,

$$\gamma\left(\frac{1}{z^2+1}\right) = \gamma^{-1}\left(\frac{1}{z^2+1}\right) = \frac{1}{z^2+1}.$$

θ will send the continuous exponential function to the time scale exponential function in the following manner: if we write $\tilde{f}(t) \in C(\mathbb{R}, \mathbb{R})$ with $\tilde{f}(t)$ of exponential order as

$$\tilde{f}(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i}\, e^{zt} \tilde{F}_{\mathbb{R}}(z),$$

then

$$\theta(\tilde{f}(t)) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i}\, e_z(t,0) F_{\mathbb{T}}(z).$$

To go from $\tilde{F}_{\mathbb{R}}(z)$ to $F_{\mathbb{T}}(z)$, we simply switch expressions involving the continuous exponential in $\tilde{F}_{\mathbb{R}}(z)$ with the time scale exponential giving $F_{\mathbb{T}}(z)$ as was done for γ and its inverse. θ^{-1} will then act on

$$g(t) = \sum_{i=1}^{n} \text{Res}_{z=z_i}\, e_z(t,0) G_{\mathbb{T}}(z),$$

with $g(t)$ of exponential Type II as

$$\theta^{-1}(g(t)) = \sum_{i=1}^{n} \text{Res}_{z=z_i}\, e^{zt} \tilde{G}_{\mathbb{R}}(z).$$

For a given time scale Laplace transform $F_{\mathbb{T}}(z)$, we begin by mapping to $\tilde{F}_{\mathbb{R}}(z)$ via γ^{-1}. The hypotheses on $F_{\mathbb{T}}(z)$ and $\tilde{F}_{\mathbb{R}}(z)$ are enough to guarantee the inverse of $\tilde{F}_{\mathbb{R}}(z)$ exists for all z with $\mathrm{Re}\,(z) > c$, and is given by

$$\tilde{f}(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i}\, e^{zt}\tilde{F}_{\mathbb{R}}(z).$$

Apply θ to $\tilde{f}(t)$ to retrieve the time scale function

$$f(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i}\, e_z(t,0)F_{\mathbb{T}}(z),$$

whereby $(\gamma \circ \mathcal{L}_{\mathbb{R}} \circ \theta^{-1})(f(t)) = F_{\mathbb{T}}(z)$ as claimed. □

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  - For $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$—yes!
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  - For $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$—yes!
  - If $\mathbb{T}$ is completely discrete, then if we choose any circle in the region of convergence which encloses all of the singularities of $F(z)$, the inversion formula holds.
  - In general? Sorry, just don't know... (Want a problem to work on?)

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- For any transformable function $f(t)$ on $\mathbb{T}$, there is a *shadow* function $\tilde{f}(t)$ defined on $\mathbb{R}$. That is, to determine the appropriate *time scale analogue* of the function $\tilde{f}(t)$ in terms of the transform, we use the diagram to map its Laplace transform on $\mathbb{R}$ to its Laplace transform on $\mathbb{T}$.

Example. Suppose $F(z) = \frac{1}{z^2}$. The hypotheses of the Inversion Theorem are readily verified, so that

$$\mathcal{L}^{-1}\{F\} = f(t) = \mathrm{Res}_{z=0} \frac{e_z(t,0)}{z^2} = t.$$

For $F(z) = \frac{1}{z^3}$, we have

$$\mathcal{L}^{-1}\{F\} = f(t) = \mathrm{Res}_{z=0} \frac{e_z(t,0)}{z^3} = \frac{t^2 - \int_0^t \mu(\tau)\Delta\tau}{2} = h_2(t,0).$$

The last equality is justified since the function

$$f(t) = \frac{t^2 - \int_0^t \mu(\tau)\Delta\tau}{2},$$

is the unique solution to the initial value problem

$$f^{\Delta}(t) = h_1(t,0), \ f(0) = 0.$$

In a similar manner, we can use an induction argument coupled with the Inversion Theorem to show that the inverse of $F(z) = \frac{1}{z^{k+1}}$, for k a positive integer, is $h_k(t, 0)$.

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In fact, it easy to show that the inversion formula gives the claimed inverses for any of the elementary functions that Bohner and Peterson have in their table. These elementary functions become the proper time scale analogues or *shadows* of their continuous counterparts.

A very important inverse...

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Consider the function $F(z) = e_{\ominus z}(\sigma(a), 0)$. The Inversion Theorem cannot be applied. However, if a is right scattered, one can show that the Hilger Delta function which has representation

$$\delta_a^{\mathbb{H}}(t) = \begin{cases} \frac{1}{\mu(a)}, & t = a, \\ 0, & t \neq a, \end{cases}$$

has $F(z)$ as a transform, while if a is right dense, the Dirac delta functional has $F(z)$ as a transform.

Uniqueness of the Inverse

If two functions f and g have the same transform, then are f and g necessarily the same function? As on $\mathbb{R}$, the answer to this question is affirmative if we define our equality in an almost everywhere (a.e.) sense.

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Upshot (after some measure theoretic hoops to jump thru): To show that a property holds almost everywhere on a time scale, it is necessary to show that the property holds for every right scattered point in the time scale, and that the set of right dense points for which the property fails has Lebesgue measure zero.

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Theorem

If the functions $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ have the same Laplace transform, then $f = g$ a.e.

Setting the table for convolution

Definition

The *delay* of the function $x : \mathbb{T} \to \mathbb{R}$ by $\sigma(\tau) \in \mathbb{T}$, denoted by $x(t, \sigma(\tau))$, is given by

$$u_{\sigma(\tau)}(t)x(t, \sigma(\tau)) = \sum_{i=1}^{n} \text{Res}_{z=z_i} X(z)e_z(t, \sigma(\tau)).$$

Here, $u_\xi(t) : \mathbb{T} \to \mathbb{R}$ is the time scale unit step function activated at time $t = \xi \in \mathbb{T}$.

Shifts

Notice that $u_{\sigma(\tau)}(t)x(t,\sigma(\tau))$ has transform $X(z)e^{\sigma}_{\ominus z}(\tau,0)$. Indeed,

$$\begin{aligned} u_{\sigma(\tau)}(t)x(t,\sigma(\tau)) &= \sum_{i=1}^{n} \mathrm{Res}_{z=z_i}\, X(z)e_z(t,\sigma(\tau)) \\ &= \sum_{i=1}^{n} \mathrm{Res}_{z=z_i} \left[X(z)e^{\sigma}_{\ominus z}(\tau,0)\right] e_z(t,0) \\ &= \mathcal{L}^{-1}\{X(z)e^{\sigma}_{\ominus z}(\tau,0)\}. \end{aligned}$$

This allows us to use the term *delay* to describe $x(t,\sigma(\tau))$, since on $\mathbb{T}=\mathbb{R}$, the transformed function $X(z)e^{-z\tau}$ corresponds to the function $u_\tau(t)x(t-\tau)$. With the delay operator defined, we define the convolution of two arbitrary transformable time scale functions.

The Convolution Product

Definition

The convolution of the functions $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$, denoted $f \star g$, is given by

$$(f \star g)(t) = \int_0^t f(\tau) g(t, \sigma(\tau)) \Delta\tau.$$

The Convolution Theorem

Theorem

The transform of a convolution product is the product of the transforms.

Proof. Assuming absolute integrability of all functions involved, then by the delay property and inversion we obtain

$$\begin{aligned}
\mathcal{L}\{f \star g\} &= \int_0^\infty e^\sigma_{\ominus z}(t,0)\left[(f\star g)(t)\right]\Delta t \\
&= \int_0^\infty \left[\int_0^t f(\tau)g(t,\sigma(\tau))\Delta\tau\right] e^\sigma_{\ominus z}(t,0)\,\Delta t \\
&= \int_0^\infty f(\tau)\left[\int_{\sigma(\tau)}^\infty g(t,\sigma(\tau))e^\sigma_{\ominus z}(t,0)\,\Delta t\right]\Delta\tau \\
&= \int_0^\infty f(\tau)\mathcal{L}\{u_{\sigma(\tau)}(t)g(t,\sigma(\tau))\}\Delta\tau \\
&= \int_0^\infty f(\tau)\left[G(z)e^\sigma_{\ominus z}(\tau,0)\right]\Delta\tau \\
&= \int_0^\infty f(\tau)e^\sigma_{\ominus z}(\tau,0)\Delta\tau\cdot G(z) \\
&= F(z)G(z).
\end{aligned}$$

Example. Suppose $g(t, \sigma(\tau)) = 1$. In this instance, we see that for any transformable function $f(t)$, the transform of $h(t) = \int_0^t f(\tau)\Delta\tau$ is given by

$$\mathcal{L}\{h\} = \mathcal{L}\{f \star 1\} = F(z)\mathcal{L}\{1\} = \frac{F(z)}{z},$$

another result obtained by direct calculation in Bohner and Peterson.

The convolution product is both commutative and associative. Indeed, the products $f \star g$ and $g \star f$ have the same transform as do the products $f \star (g \star h)$ and $(f \star g) \star h$, and since the inverse is unique, the functions defined by these products must agree almost everywhere.

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Identity element?

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Identity element?

At first glance, one may think that the identity is vested in the Hilger Delta. Unfortunately, this is not the case. It can be easily shown that any identity for the convolution will of necessity have transform 1 by the convolution theorem. But the the Hilger Delta does not have transform 1 since its transform is just the exponential.

The Generalized Dirac Delta Functional

Let $f, g : \mathbb{T} \to \mathbb{R}$ be given functions with $f(x)$ having unit area. To define the time scale delta functional, we construct the following functional. Let $C_c^\infty(\mathbb{T})$ denote the $C^\infty(\mathbb{T})$ functions with compact support. For $g^\sigma \in C_c^\infty(\mathbb{T})$ and for all $\epsilon > 0$, define the functional $F : C_c^\infty(\mathbb{T}, \mathbb{R}) \times \mathbb{T} \to \mathbb{R}$ by

$$F(g^\sigma, a) = \begin{cases} \int_0^\infty \delta_a^{\mathbb{H}}(x) g^\rho(\sigma(x))\,\Delta x, & a \text{ is right scattered,} \\ \lim_{\epsilon\to 0} \int_0^\infty \frac{1}{\epsilon} f(\frac{x}{\epsilon}) g(\sigma(x))\,\Delta x, & a \text{ is right dense.} \end{cases}$$

The (time scale or generalized) Dirac delta functional is then given by

$$\langle \delta_a(x), g^\sigma \rangle = F(g^\sigma, a).$$

Action of delta functional on $C_c^\infty(\mathbb{T})$?

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In a nutshell...

$$\langle g^\sigma, \delta_a \rangle = g(a),$$

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Also, if $g(t) = e_{\ominus z}(t, 0)$, then $\langle \delta_a, g^\sigma \rangle = e_{\ominus z}(a, 0)$, so that for $a = 0$, the Dirac delta functional $\delta_0(t)$ has Laplace transform of 1, thereby giving us an identity element for the convolution.

However, our definition of the delay operator will only hold for functions. It will be necessary to extend this definition for the delta functional. To maintain consistency with the delta function's action on $g(t) = e^{\sigma}_{\ominus z}(t,0)$, it follows that for any $t \in \mathbb{T}$, the shift of $\delta_a(\tau)$ is given by $\delta_a(t,\sigma(\tau)) = \delta_t(\sigma(\tau))$.

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Just as before, the algebraic properties hold in this setting, e.g.

$$(\delta_0 \star g)(t_0) = (g \star \delta_0)(t_0).$$

When we perform the convolution $\langle g, \delta^\sigma \rangle$, we must give meaning to this symbol and do so by defining $\delta^\sigma(t)$ to be the Kronecker delta when t is right scattered and the Dirac delta if t is right dense. While this *ad hoc* approach does not address convolution with an arbitrary (shifted) distribution on the right, this will suffice (at least for now) since our eye is on solving generalizations of canonical partial dynamic equations which will involve the Dirac delta distribution.

Uniqueness of the inverse of the Dirac delta?

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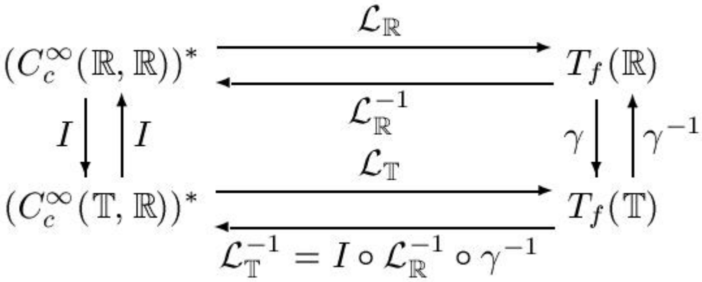
Yes, just need a new commutative diagram betwixt the dual spaces.



Controllability

Definition

The regressive linear state equation

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned} \tag{2}$$

is called *controllable* on $[t_0, t_f]$ if given any initial state x_0 there exists a rd-continuous input signal $u(t)$ such that the corresponding solution of the system satisfies $x(t_f) = x_f$.

Controllability Gramian Condition

Theorem

The regressive linear state equation (2) *is controllable on* $[t_0, t_f]$ *if and only if the* $n \times n$ controllability Gramian matrix

$$W(t_0, t_f) = \int_{t_0}^{t_f} \Phi_A(t_0, \sigma(t)) B(t) B^T(t) \Phi_A^T(t_0, \sigma(t)) \, \Delta t$$

is invertible.

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is invertible.

$\Phi_A(t, t_0)$ is the transition matrix for the (time varying!) problem $X^\Delta = A(t)X(t)$, $X(t_0) = I$.

Controllability Rank Theorem for LTI Systems

Theorem

The regressive linear time invariant system

$$x^{\Delta}(t) = Ax(t) + Bu(t), \;\; x(t_0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

is controllable on $[t_0, t_f]$ *if and only if the* $n \times nm$ *controllability matrix*

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

satisfies

$$\operatorname{rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n.$$

Observability

Definition

The linear state equation

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = x_0,$$
$$y(t) = C(t)x(t)$$

is called *observable* on $[t_0, t_f]$ if any initial state $x(t_0) = x_0$ is uniquely determined by the corresponding response $y(t)$ for $t \in [t_0, t_f)$.

Observability Gramian Condition

Theorem

The regressive linear system given above is observable on $[t_0, t_f]$ *if and only if the* $n \times n$ observability Gramian matrix

$$M(t_0, t_f) = \int_{t_0}^{t_f} \Phi_A^T(t, t_0) C^T(t) C(t) \Phi_A(t, t_0) \, \Delta t$$

is invertible.

Observability Rank Theorem for LTI Systems

Theorem

The autonomous linear regressive system

$$x^{\Delta}(t) = Ax(t), \quad x(t_0) = x_0,$$
$$y(t) = Cx(t)$$

is observable on $[t_0, t_f]$ *if and only if the* $nm \times n$ observability matrix *satisfies*

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$