E&M and Energy Conversion
R.J. Marks II Class Notes
Rose-Hulman Institute of Technology (1971)
Texas Tech University (1976)
ELECTROMAGNETIC WAVES
TUESDAY

VECTOR NOTATION: \( \vec{A} \)

\( \vec{\alpha} \) = UNIT VECTOR \( (\vec{\alpha}_x = \hat{\imath}) \)

\( \vec{R} = (y-y')\vec{\alpha}_y + (z-z')\vec{\alpha}_z \)

(prime point - source point)

(primed - field point)

CYLINDRICAL CO-ORDINATES

\( \vec{\alpha} = \cos \phi \vec{\alpha}_x + \sin \phi \vec{\alpha}_\phi \)

\( \vec{\alpha} = \sin \phi \vec{\alpha}_x + \cos \phi \vec{\alpha}_\phi \)

\( \phi \) measured counter-clockwise

FROM \( x \) AXIS

WORK 1, 2, AND 49 FOR TOMORROW
\[ a_r = \cos (\phi - \phi') \vec{a}_r, \quad a_\theta = \sin (\phi - \phi') \vec{a}_\theta. \]

**More Cylindrical Co-ordinates (Differential Volume)**

\[ dV = r \, dr \, d\phi \, dz \]

**Spherical Co-ordinates**

\[ \vec{R} = r \vec{a}_r - r' \vec{a}_{r'}. \]
SURFACE VECTOR

MAGNITUDE = AREA

DIRECTION: TO AREA (POSITIVE, OUTWARD)

FOR CLOSED SURFACE

THRU

\[ \vec{a}_\phi = \cos \phi \vec{a}_y - \sin \phi \vec{a}_x \]

1.5) \[ \vec{A} = 2\vec{a}_x + \vec{a}_y - 3\vec{a}_z \]

\[ \vec{A} \cdot \vec{a}_\phi = -2 \sin \phi + \cos \phi \]

(if \( \vec{A} = \vec{F} \), then \( \vec{A} \cdot \vec{a}_\phi = 0 \))
\[ ds = r^2 \sin \theta \, d\theta \, d\phi \]

PROJECTIONS ON AXES:

\[ d\vec{s} = r^2 \sin \theta \, d\theta \, d\phi \, \vec{a}_r \]
\[ \sin \theta \, \sin \phi \, d\phi \, \vec{a}_\phi \]
\[ \cos \phi \vec{a}_\theta \]

\[ \oint d\vec{s} = \oint_{\theta=0}^{\pi} \oint_{\phi=0}^{2\pi} r^2 \sin \theta \, d\theta \, d\phi \vec{a}_r \]
\[ + \oint_{\theta=0}^{\pi} \oint_{\phi=0}^{2\pi} \sin^2 \phi \, d\phi \vec{a}_\phi \]
\[ + \oint_{\theta=0}^{\pi} \oint_{\phi=0}^{2\pi} \cos \phi \, d\phi \vec{a}_\theta \]

\[ \oint d\vec{s} = 0 \]

\[ \oint d\vec{s} = \text{AREA} \]
\[ = \int_0^{2\pi} \int_0^{\pi} r^2 \sin \theta \, d\theta \, d\phi = 4\pi r^2 \]
\( \Phi(x, y, z) = \text{constant yields surface} \)

\( \frac{\delta \Phi}{\delta s} \in S = \text{some direction} \)

\[
d\Phi = \frac{\delta \Phi}{\delta s} \cdot ds = \frac{\delta \Phi}{\delta x} dx + \frac{\delta \Phi}{\delta y} dy + \frac{\delta \Phi}{\delta z} dz
\]

\[
\Rightarrow S = \left( \frac{\delta \Phi}{\delta x} \mathbf{a}_x + \frac{\delta \Phi}{\delta y} \mathbf{a}_y + \frac{\delta \Phi}{\delta z} \mathbf{a}_z \right) \cdot (dx \mathbf{a}_x + dy \mathbf{a}_y + dz \mathbf{a}_z) \nabla \Phi \cdot ds
\]

GRADIENT \( \nabla = \mathbf{a}_x \frac{\delta}{\delta x} + \mathbf{a}_y \frac{\delta}{\delta y} + \mathbf{a}_z \frac{\delta}{\delta z} \)

\( \mathbf{a} \cdot \mathbf{b} = 0 \) if \( \mathbf{a} \perp \mathbf{b} \) or \( |\mathbf{a}| = 0 \) or \( |\mathbf{b}| = 0 \)

GRADIENT IS \( \perp \) TO SURFACE \( \Phi = \text{const.} \)

GRADIENT LIES IN GREATEST MAGNITUDE

OF RATE OF CHANGE OF \( \Phi \)
\[ \nabla \cdot \nabla \phi = \nabla^2 \phi = \delta \phi + \phi' + \frac{\phi''}{2} \]

Laplacian

\[ \nabla^2 \left( \frac{1}{r} \right) \]

Source (\( \delta \)) Origin of Force, Co-ord

\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \frac{1}{r} = -\frac{\alpha}{r^2} \]

Thurs.

\[ \nabla \cdot \nabla \left( \frac{1}{r} \right) = \nabla^2 \frac{1}{r} \]

\[ \frac{\partial}{\partial (\theta, \phi)} \nabla^2 \left( \frac{1}{r} \right) = \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \right) = -\frac{\alpha}{r^2} \]

\[ \int \nabla^2 \frac{1}{r^2} \, dV = \int \nabla \cdot \nabla \left( \frac{1}{r} \right) \, dV = \phi \nabla \left( \frac{1}{r} \right) \cdot dS \]

\[ \Rightarrow dS = \rho^2 \sin \theta \, d\theta \, d\phi \, d\alpha \]

\[ \Rightarrow \int \nabla^2 \frac{1}{r^2} \, dV = \int \frac{\partial}{\partial (\theta, \phi)} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \right) \, dS \]

\[ = \int_0^{\pi} \int_0^{2\pi} \sin \theta \, d\theta \, d\phi = -4\pi \]

\[ \Rightarrow \text{Field} = \frac{\text{Linear}}{\text{Area}} = \frac{\alpha}{4\pi} \]

\[ \text{At } R = 0, \text{ Field Intensity} = \frac{\alpha}{4\pi} \]
\[ \phi = \frac{q}{r^3} \cdot dV = - \text{of lines out of surface} \]

\[ \oint \nabla^2 \left( \frac{1}{r} \right) dV = \int \frac{1}{r} \text{IT IF } R=0 \text{ IS NOT IN VOLUME} \]

\[ \int_0^1 \delta(t-\tau) d\tau = \begin{cases} 1 \\
0 \text{ IF } R=0 \text{ IS NOT IN VOLUME} \end{cases} \]

\[ \Rightarrow \nabla^2 \left( \frac{1}{R} \right) = -4\pi \delta(R) \quad \delta(R) \text{ IS A SPAREL DEL FUNCTION} \]

\[ \nabla \cdot \rho \, dV \text{ FOR POINT SOURCE} \quad \rho = \delta(R) \text{ OR } q \delta(R) \]

\[ \text{CAUSE' LAW} \quad \oint \mathbf{F} \cdot d\mathbf{e} = \int \nabla \cdot \mathbf{E} \, dV \]

\[ \text{SUM-3 IN -3 SOURCES= 15 IN} \quad 2 = 2 \]

\[ \text{THE CURL} \]

\[ (\nabla \times \mathbf{E}) \cdot \mathbf{n} = \lim_{A \to 0} \frac{1}{A} \oint \mathbf{F} \cdot d\mathbf{e} = \epsilon_0 \mathbf{n} \cdot \mathbf{E} = \mathbf{E} \times \mathbf{F} \]

\[ \text{THE CURL} (\text{IN}^2) \quad \oint \nabla \times \mathbf{E} \cdot dS = \oint \mathbf{E} \cdot d\mathbf{L} \]
Mon

Test next Tuesday

Review

\[ \nabla \phi \]
\[ \nabla \cdot \vec{E} \]
\[ \nabla \times \vec{E} \]
\[ \int \nabla \cdot \vec{E} \, dV = \oint \vec{E} \cdot d\vec{s} \quad \text{(Gauss' Law)} \]
\[ \int \nabla \times \vec{E} \cdot d\vec{s} = \oint \vec{E} \cdot d\vec{l} \]
\[ \nabla \times \nabla \phi = 0 \quad \nabla \cdot \nabla \times \vec{E} = 0 \]
\[ \nabla \times \vec{E} = \vec{C} \quad \text{=} \quad \text{Current Density (Vortex)} \]
\[ \nabla \cdot \vec{E} = \nabla \phi = \text{Charge Density} \]

For a given \( \vec{E} \) and \( \vec{C} \) there is a unique \( \vec{F} \)

Rotational Field

Rotational (solenoidal) field

\[ \nabla \times \vec{F} = 0 \quad \nabla \cdot \vec{F} = 0 \]

\[ \Rightarrow \vec{F} = \nabla \phi \]

\[ \Rightarrow \vec{F} = \nabla \times \vec{A} \]

\[ \vec{F} = \vec{F}_1 + \vec{F}_2 \]

\[ \nabla \times \vec{F}_1 = 0 \quad \nabla \cdot \vec{F}_1 = \nabla \phi \]

\[ \nabla \cdot \vec{F}_2 = 0 \quad \nabla \times \vec{F}_2 = -\vec{C} \]

\[ \vec{F}_1 = -\nabla \phi \]

\[ \vec{F}_2 = \nabla \times \vec{A} \]

Vector Potential

Scalar Potential

\[ \nabla \cdot \vec{F} = -\nabla \cdot (\nabla \phi) + \nabla \cdot \nabla \times \vec{A} = -\nabla^2 \phi + \nabla \times \nabla \times \vec{A} = \big( \nabla \times \vec{A} \big) - \nabla^2 \phi = \nabla \times \vec{A} = \vec{C} \]

\[ \nabla \times \vec{F} = \nabla \times (-\nabla \phi) + \nabla \cdot \nabla \times \vec{A} = -\nabla^2 \phi + \nabla \times \nabla \times \vec{A} = \big( \nabla \times \vec{A} \big) - \nabla^2 \phi - \vec{C} \]

\[ \text{Choose } \nabla \cdot \vec{A} = 0 \]

\[ \phi = \int \frac{1}{4\pi} \, \nabla^2 \phi \, dV; \quad \vec{A} = \frac{1}{4\pi} \int \frac{\vec{C}}{R} \, dV \]

\[ \nabla \times \phi = 0 \]

\[ \nabla \cdot A = 0 \]
9. Equivalently, (8-1) is a requirement for the linear independence of the three vectors which means the following:

\[ k_1 \hat{a} + k_2 \hat{b} + k_3 \hat{c} = \hat{0} \]

means that \( k_1 = k_2 = k_3 = 0 \) for \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \) to be linearly independent.

10. In tensor or indicial notation the scalar triple product can be denoted as follows:

\[ \hat{a} \cdot (\hat{b} \times \hat{c}) = a_i \epsilon_{ijk} b_j c_k \]

\[ = \epsilon_{ijk} a_i b_j c_k \]

Using the properties of the permutation operator we deduce the following:

\[ \hat{a} \cdot (\hat{b} \times \hat{c}) = \hat{b} \cdot (\hat{c} \times \hat{a}) = \hat{c} \cdot (\hat{a} \times \hat{b}) \]

It can also be shown that:

11. Physically, in absolute value the scalar triple product represents the volume of the parallelepiped formed by \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \). Hence, if \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \) are coplanar it is evident that

\[ \hat{a} \times \hat{b} \times \hat{c} = 0 \]

\[ \hat{a} \times \hat{b} \text{ yields area of side} \]

**Elements of Vector Calculus**

1. Vector calculus applies to vector functions which in turn define vector fields. Hence, the calculus of vectors is applicable to vector fields.

2. Essentially, vector calculus is merely an extension of ordinary calculus and involves "scalar" calculus applied to the scalar components of a vector function. Hence, all the ideas of ordinary or scalar calculus apply to vector calculus.

3. Let \( \mathbf{v}(t) = (v_1, v_2, v_3) \) be a vector function. Then the derivative of \( \mathbf{v}(t) \) is defined by the following:

\[ \mathbf{v}(t) \frac{d}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \]
TUESDAY

\[ E = \lim_{q \to 0} \frac{q}{4\pi \varepsilon_0} \frac{A}{q^2} \]

Coulomb's Law: \[ F = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2} \]  
\[ \phi = \frac{q}{4\pi \varepsilon_0} \]

\[ \Delta \phi = \Delta \phi \cdot \frac{E}{dE} = \int_{R_1}^{R_2} E \cdot d\vec{a} \] (Potential Field)

\[ \phi = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{a} \]

\[ \phi(b) - \phi(a) = \phi = \int_a^b \left( \frac{q}{4\pi \varepsilon_0} \frac{1}{r^2} \right) \frac{d\vec{r}}{\vec{r}} \cdot (d\vec{r} \cdot \vec{a}) \]

\[ \phi = \frac{q}{4\pi \varepsilon_0} \]

\[ dq = \rho \, dV \Rightarrow E = \int \frac{\rho \, dV}{4\pi \varepsilon_0 R^2} \]
\[ \rho = 2 \epsilon_0 E + \rho \]

\[ \phi = -m_{n} \cdot d_{n} / 4 \pi \epsilon_0 R^2 \]

\[ \text{ELECTRONIC POLARIZABILITY} \]

\[ \alpha_p = 9 \pi \epsilon_0 R^3 \]

\[ \text{IONIC POLARIZABILITY} \]

\[ qE_i = k \Delta l \Rightarrow \Delta l = \frac{qE_i}{k} \]

\[ P = \frac{N}{m} \cdot q_i \cdot l_i = \frac{qE_i}{k} \]

\[ \Rightarrow \alpha_i = \frac{q^2}{k} \]

\[ E_i \Rightarrow E + E_0 \]

\[ \text{INTERNAL FIELD CONSTANT} \]
\[ \vec{E} = \nabla \times \vec{B} = \nabla \times (\vec{v} \times \vec{B}) \]

\[ \Rightarrow \vec{P} = \frac{1}{N} N \alpha_0 \vec{E} \times \vec{B} \] \hspace{1cm} \text{ANALOGOUS TO} \hspace{0.5cm} \frac{\alpha}{d} = \frac{A}{d} \]

\[ \dot{\rho} \vec{D} - \nabla \rho \vec{E} = \varepsilon_0 \nabla \times \vec{E} + \nabla \cdot \vec{P} \]

\[ \rho_p = -\nabla \cdot \vec{P} \]

\[ \rho_{ps} = \rho \cdot \alpha_n \]

\[ \rho \rho_0 \]

\[ \text{DO 18, 19, 23, 24} \]

\[ \text{MON} \]

\[ \left( \text{CLOSED BOOK) IDENTITIES GIVEN} \right. \]

\[ \text{KNOW LAWS AND DEFINITIONS} \]

\[ \text{CHAPTERS 1} \] \hspace{0.5cm} \text{UP TO 95, (EXCEPT PGS 60-78)} \]
(23) \[ \phi \] 

\[ E \phi = e \frac{q_e}{\sqrt{m}} \]

\[ \frac{d}{dt} \frac{v}{c^2 m} = \frac{q_e}{2\pi e_0 m} \]

\[ E_r = \frac{q_e e}{2\pi e_0 m} \]

\[ 2\pi e_0 m = r \sqrt{\frac{q_e e}{2\pi e_0 m}} \]

IF \( E_r \) IS ASSUMED CONSTANT, \( R = \frac{3\pi}{4\mu_0} \)

BUT IF NOT:

\[ q_e e = \mu_0 II_0^2 \Rightarrow q_e e = \frac{d^2 R}{c^2 dt} \]

\[ \Rightarrow z = \sqrt{R^2 + \frac{d^2 R}{c^2 dt}} \]

\[ K = z^2 = \frac{q_e e}{2e_0 m} \]

\[ m = \frac{q_e e}{2e_0 m} \]

\[ r\phi' \phi'' \]

\[ \phi = \frac{4\pi e_0 R^3}{m} \]

\[ z = \sqrt{R^2 + \left(\frac{4\pi e_0 R^3}{m}\right)^2} \]

\[ \omega = 2\pi f \]

\[ dF_{Mxy} = \frac{1}{2} \mu_0 \frac{dV}{dr} \]

\[ F_{Mxy} = \int_{\phi_0}^{\phi_1} \int_{r_0}^{r_1} \frac{r'^2 d\phi' dr' dz'}{r'^2 + (z-z_0)^2} \]

\[ r_0 \leq r' \leq r_1 \]

\[ \phi_0 \leq \phi' \leq \phi_1 \]

\[ \frac{1}{2} \left( r'^2 + (z-z_0)^2 \right)^{3/2} \]

\[ \int_{\phi_0}^{\phi_1} \int_{r_0}^{r_1} \left( r'^2 + (z-z_0)^2 \right)^{3/2} \]

\[ \int_{\phi_0}^{\phi_1} \int_{r_0}^{r_1} \left( r'^2 + (z-z_0)^2 \right)^{3/2} \]
\( \nabla \cdot \mathbf{D} = \rho \)

\[
\oint \nabla \cdot \mathbf{D} \, ds = \int \rho \, dv = q
\]

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \]

\[ \nabla \cdot \mathbf{E} = \varepsilon_0 \left[ \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} \right] = \varepsilon_0 \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \]

\[ E_{max} = 30 \text{kV/cm} \]

\[ \phi = 5 \times 10^6 \text{ V} \]

\[
\phi = \int \mathbf{E} \cdot d\mathbf{l} = \int \mathbf{E} \cdot \left( \frac{d\mathbf{r}}{dr} \right) \, dr = 0 \text{ / } \text{curl} \]

\[
\phi = \frac{q}{4 \pi \varepsilon_0 r^2}
\]

\[
E_r = \frac{q}{4 \pi \varepsilon_0 r^2} \quad \text{and} \quad \phi = \frac{q}{4 \pi \varepsilon_0 r^2}
\]

\[
3 \times 10^{-3} \text{ cm} = \frac{q}{10^6} \Rightarrow q = 16.7 \text{ cm}^3
\]
\[ \phi = \frac{m \cdot \mathbf{a}}{4\pi \varepsilon_0 R^2} = \frac{-m \cdot \nabla \left( \frac{1}{r} \right)}{4\pi \varepsilon_0} \]

\[ E_{\text{Axis}} = \frac{a^2}{2 \pi \varepsilon_0} \int_{\frac{1}{2}}^{1} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{e^d_{r'} \, d \phi' \, d z' \, dr' \, d\theta \, \cos \theta}{r'^2 + (Z - Z')^2}^{3/2} \]

\[ E_{\text{Axis}} = \frac{p_{\text{Gz}}}{2 \varepsilon_0} \ln \left( \frac{1 + \left( \frac{Z - Z'}{r} \right)^2}{1 + \left( \frac{Z + Z'}{r} \right)^2} \right) \]

\[ w = \frac{1}{2} \sum q \cdot \phi_i = \frac{1}{2} \int \rho \phi \, d\mathbf{V} = \frac{1}{2} \int (\nabla \cdot \mathbf{D}) \phi \, d\mathbf{V} \]

Due to all other charge:

\[ \nabla \cdot (\mathbf{D} \phi) = \phi \nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla \phi \]

\[ w = \frac{1}{2} \int \left[ (-\mathbf{D} \cdot \nabla \phi) + \mathbf{D} \cdot \nabla \phi \right] \, d\mathbf{V} = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\mathbf{V} + \frac{1}{2} \int \phi \phi \hat{D} \cdot ds \]

\[ w = \frac{1}{2} \sum q i \phi_i = \frac{1}{2} \int \rho \phi \, d\mathbf{V} = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\mathbf{V} \]
\[ \oint \mathbf{D} \cdot d\mathbf{S} = q = D_r \cdot 2\pi r \cdot \ell \]
\[ E_r = \frac{q}{2\pi\varepsilon_0} \Rightarrow W = \frac{\varepsilon_0}{2} \int \frac{V^2}{r^2} \, 2\pi r \, \ell d\phi \, dr \, dz \]
\[ W = \frac{\varepsilon_0}{2} \int_0^{2\pi} \int_0^b \frac{V^2}{r} r^2 \, d\phi \, dr \, dz \]
\[ \Rightarrow \dot{W} = \frac{V^2 \varepsilon_0}{2 \ell_n \beta \alpha} \left( \frac{1}{2} C V^2 \right) \]
\[ \hat{F} = \frac{dW}{dr} = \frac{\varepsilon_0 \pi V^2}{2 \ell_n \beta \alpha} \]
NEXT TUESDAY: OPEN BOOK MULTIPLE CHOICE
OVER MOSTLY HOMEWORK

\[ d(\mathbf{D} \cdot \mathbf{E}) = \mathbf{D} \cdot d\mathbf{E} + \mathbf{E} \cdot d\mathbf{D} \]

\[ d(\mathbf{B} \cdot \mathbf{H}) = \mathbf{B} \cdot d\mathbf{H} + \mathbf{H} \cdot d\mathbf{B} \]

\[ W = -\int \mathbf{F} \cdot d\ell = -\int q \mathbf{E} \cdot d\ell \]

\[ \int \mathbf{F} \cdot d\ell \Rightarrow T = \varepsilon_0 \left[ (\mathbf{A}_n \cdot \mathbf{E}) \mathbf{E} - \frac{\varepsilon^2}{2} \mathbf{A}_n \right] \]

MON

TEST TOMORROW
Coronilin Square:

\[
F = \frac{\Delta \phi_1}{\Delta t_1}
\]

\[
I = \sigma E = -\sigma \frac{\Delta \phi_1}{\Delta t_1}
\]

\[
I_1 = J \Delta \omega, (I_1) = -\sigma \Delta \phi_1 \frac{\Delta \omega_2}{\Delta t_1}
\]

\[
T = -\alpha \frac{\Delta \phi_2}{\Delta t_2}
\]

Flow out: flow in => \[ T_2 = T_1 \]

\[
\nabla \times E = 0 \quad \nabla \times E = 0
\]

\[
J = \sigma E \quad \vec{D} = \vec{E}
\]

\[
\nabla \cdot \frac{\partial \phi}{\partial t} = 0 \quad \nabla \cdot \vec{D} \cdot V = 0
\]

\[
\nabla \cdot J = 0 \quad \nabla \cdot \vec{D} \cdot \n = 0
\]
4. Multiplication of a vector \( \hat{A} \) by a scalar \( k \) is defined to be a vector \( \hat{P} \) given by

\[
\hat{P} = k \hat{A} = (kA_x, kA_y, kA_z)
\]

Physically, \( \hat{P} \) is a vector that has a magnitude that is \( |k| \) times that of \( \hat{A} \) with the same line of action and same sense if \( k > 0 \) and opposite sense if \( k < 0 \).

\[
\hat{A} \quad \hat{P} = k \hat{A} \quad \quad \quad \quad \quad \quad k < 0 \quad \quad \quad \quad \quad \quad |k| > 1 \Rightarrow k < -1
\]

5. The scalar or dot product of two vectors \( \hat{A} \) and \( \hat{B} \) is a scalar and is defined by

\[
\hat{A} \cdot \hat{B} = |\hat{A}| |\hat{B}| \cos \theta = \hat{B} \cdot \hat{A}
\]

Equivalently,

\[
\hat{A} \cdot \hat{B} = A_x B_x + A_y B_y + A_z B_z
\]

Note the following:

a. \( \hat{A} \cdot \hat{B} \) can be +ve, -ve, or even zero (when \( \theta = 90^\circ \))

b. \( \hat{A} \cdot \hat{B} = \hat{B} \cdot \hat{A} \)

6. The vector or cross product of two vectors \( \hat{A} \) and \( \hat{B} \) is defined to be a vector \( \hat{C} \) as follows:

a. \( |\hat{C}| = |\hat{A}| |\hat{B}| \sin \theta \)

b. \( \hat{C} \) is perpendicular to the plane defined by \( \hat{A} \) and \( \hat{B} \).

c. \( \hat{C} \) obeys the "right hand rule".

Equivalently,

\[
\hat{C} = \begin{vmatrix}
\hat{A} & \hat{B} & \hat{C} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix} = \begin{vmatrix}
\hat{A} & A_y & A_z \\
B_x & B_y & B_z \\
A_x & A_y & A_z
\end{vmatrix} = \hat{A} \times \hat{B} = (A_yB_z - A_zB_y) \hat{A} + (A_zB_x - A_xB_z) \hat{B} + (A_xB_y - A_yB_x) \hat{C}
\]

or in triplet notation

\[
\hat{A} \times \hat{B} = (A_x, A_y, A_z) \times (B_x, B_y, B_z) = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)
\]
7. We shall illustrate the use of triplet notation and vector algebra from selected problems in your textbook:

Prob 3 \( \frac{246}{(446)} \) \text{ GIVEN: } \begin{aligned} \hat{a} &= (1, 3, 2) \\ \hat{b} &= (0, -1, 4) \\ \hat{c} &= (3, -4, -1) \end{aligned} \\
\text{a) } (\hat{a} + \hat{b}) \cdot \hat{c} = (1, 2, 6) \cdot (3, -4, -1) \\
&= (3 - 8 - 4) = -9 \\
\text{b) } \hat{a} \cdot \hat{c} + \hat{b} \cdot \hat{c} = (3 - 12 - 2) + (0 + 4 - 4) \\
&= -11

Prob 9 \( \frac{252}{(452)} \) \text{ GIVEN: } \begin{aligned} \hat{a} &= (1, 2, -3) \\ \hat{b} &= (1, 2, 0) \\ \hat{c} &= (-1, 1, 0) \end{aligned} \\
\hat{a} \times (\hat{b} - \hat{c}) = (1, 2, -3) \\
\times (2, 1, 0) \\
&= (2 \times 0 - (-3) \times 1, -3 \times 2 - 1 \times 0, 1 \times 1 - 2 \times 2) \\
&= (3, -6, -3)

8. Some useful properties of vector algebra are the following:

\text{a) } \hat{a} \cdot (\hat{b} + \hat{c}) = \hat{a} \cdot \hat{b} + \hat{a} \cdot \hat{c} \\
\text{b) } (k \cdot m) \hat{a} = k \hat{a} + m \hat{a} \\
\text{c) } (\hat{a} + \hat{b}) \cdot \hat{c} = \hat{a} \cdot \hat{c} + \hat{b} \cdot \hat{c} \\
\text{d) } \hat{a} + \hat{b} = \hat{b} + \hat{a}
\[ \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \]
\[ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \]
\[ \mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} \perp \mathbf{b} \]
\[ \mathbf{a} \times \mathbf{b} = 0 \Rightarrow \mathbf{a} \parallel \mathbf{b} \]
\[ \mathbf{a} \times (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \times \mathbf{b} \pm \mathbf{a} \times \mathbf{c} \]
\[ (\mathbf{a} \pm \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} \pm \mathbf{b} \times \mathbf{c} \]
\[ \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} \]
\[ \mathbf{a} \times \mathbf{a} = 0 \]

Tensor Manipulation of Vectors

1. By means of indicial notation it is possible to manipulate vectors in a highly efficient and convenient manner. For this purpose we need only to consider the following items:
   a. Tensor notation of vectors and scalars
   b. Summation convention
   c. Definition of Kronecker delta
   d. Definition of permutation symbol
   e. The \( \delta_i^i \) -identity

   With these five items it will be possible to represent as well as derive numerous identities involving vectors and scalars.

2. We have previously defined a vector in tensor form or in *indicilal notation* as follows:

\[ \mathbf{a} = A_i \]

where it is important to note that "\( i \)" is a free index and can have values 1, 2, or 3 in the "real world". Hence, \( A_i \) represents a typical vector component and in essence the entire vector \( \mathbf{a} \).

3. A scalar will lack a free index so familiar representations like \( a \)
   \( x \), \( \epsilon \), \( T \), etc., are scalars as well some others involving the summation convention.

4. The summation convention introduced by Albert Einstein merely eliminates the summation symbol \( \sum \) for the case of two dummy subscripts. The convention can be generally represented by the following

\[ (A B)_i (C D)_i = \sum_{i=1}^{n} (A_i C_i) (B_i D_i) \]

where "\( n \)" is usually 3 in the "real world".
5. Examples of the summation convention are the following:
   a) \( \hat{A} \cdot \hat{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = A_i B_i \)
   b) \( a_{i,j} = a_{ii} + a_{i2} + a_{i3} \)
   c) \( a_{i,j} b_{i,k} = a_{i,j} b_{i,k} + a_{2j} b_{2,k} + a_{3j} b_{3,k} \) 
   \( (a_{i,j} = (a_{i1}, a_{i2}, a_{i3}) \)

\[ \begin{align*}
   \hat{A} \times \hat{B} &= A_x B_y \hat{\mathbf{i}} + A_y B_x \hat{\mathbf{j}} + A_z B_z \hat{\mathbf{k}} \\
   &= A_x B_y \delta_{ij} \hat{\mathbf{i}} + A_y B_x \delta_{ij} \hat{\mathbf{j}} + A_z B_z \delta_{ij} \hat{\mathbf{k}}
\end{align*} \]

6. The Kronecker delta is probably familiar and will be defined below:

\[ \delta_{ij} = \begin{cases} 
   1 & i = j \\
   0 & i \neq j
\end{cases} \]

In other words, \( \delta_{11} = \delta_{22} = 1 \)
\( \delta_{12} = \delta_{31} = 0 \)
\( \delta_{13} = \delta_{23} = 0 \)
\( \delta_{23} = \delta_{12} = 0 \)
\( \delta_{31} = \delta_{13} = 0 \)

7. To illustrate the use of the Kronecker delta consider the following examples:
   a) \( \delta_{i,j} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \)
   b) \( \delta_{i,j} A_j = \)
   \[ \begin{align*}
   \delta_{1,1} A_1 &= \delta_{1,1} A_1 + \delta_{1,2} A_2 + \delta_{1,3} A_3 \\
   \delta_{2,2} A_2 &= \delta_{2,2} A_2 + \delta_{2,1} A_1 + \delta_{2,3} A_3 \\
   \delta_{3,3} A_3 &= \delta_{3,3} A_3 + \delta_{3,1} A_1 + \delta_{3,2} A_2
\end{align*} \]

8. The permutation symbol offers a compact means for representing a cross product. It is defined by the following:

\[ \epsilon_{i,j,k} = \begin{cases} 
   1 & i, j, k = 1, 2, 3, \text{ or } 3, 1, 2 \\
   -1 & i, j, k = 2, 1, 3, \text{ or } 3, 2, 1 \\
   0 & \text{ for all other cases}
\end{cases} \]

9. With the permutation symbol the cross product can be represented as follows:

\[ \hat{A} \times \hat{B} = \epsilon_{i,j,k} A_j \hat{B}_k \]

Let us verify this below:
In the usual manner we have in triplet notation,
\[ \hat{A} \times \hat{B} = (A_1, A_2, A_3) \times (B_1, B_2, B_3) = (A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1) \]

For \( i = 1 \) from (9-1) we have
\[ \hat{A} \times \hat{B} \bigg|_{i = 1} = \mathbf{e}_{123} A_2 B_3 + \mathbf{e}_{132} A_3 B_2 \]
\[ = A_2 B_3 - A_3 B_2 \]

For \( i = 2 \) there results
\[ \hat{A} \times \hat{B} \bigg|_{i = 2} = \mathbf{e}_{213} A_1 B_3 + \mathbf{e}_{231} A_3 B_1 \]
\[ = A_3 B_1 - A_1 B_3 \]

For \( i = 3 \) we get
\[ \hat{A} \times \hat{B} \bigg|_{i = 3} = \mathbf{e}_{312} A_1 B_2 + \mathbf{e}_{321} A_2 B_1 \]
\[ = A_1 B_2 - A_2 B_1 \]

Therefore,
\[ \hat{A} \times \hat{B} = \mathbf{e}_{i'j'k} A_{i'} B_{j'} C_k \]

10. Note two inherent properties of the permutation symbol as follows:

(10-1) \[ \mathbf{e}_{i'j'k} \mathbf{e}_{ijl} = \mathbf{e}_{j'k'i} = \mathbf{e}_{kjl} \]

and

(10-2) \[ \mathbf{e}_{i'j'k} = -\mathbf{e}_{kjl} \]

11. The so-called \( \mathbf{e}_x \) identity can be proved as a theorem with considerable effort and is given by the following:

(11-1) \[ \mathbf{e}_{i'j'k} \mathbf{e}_{ijl} = \delta_{j'r} \delta_{k's} - \delta_{j's} \delta_{k'r} \]

The proof follows from the basic definitions of the Kronecker delta and permutation symbol and will not be done here. Note that there must be one common subscript only while the others are different.

12. As an example of the application of the \( \mathbf{e}_x \) identity let us prove (1) on page 258 of your textbook as follows:

(1) \[ \hat{B} \times (\hat{C} \times \hat{D}) = (\hat{B} \cdot \hat{D}) \hat{C} - (\hat{B} \cdot \hat{C}) \hat{D} \]

We proceed as follows, beginning with the left hand side of (1)
\[ \hat{b} \times (\hat{c} \times \hat{d}) = c_{ij} k \cdot b_j \cdot e_k \cdot c_{io} \cdot d_m \]
\[ = c_{ki} \cdot c_{k \cdot e_k \cdot b_j \cdot c_o \cdot d_m} \]
\[ = (\delta_{c \cdot b_j} - \delta_{c \cdot b_j} \cdot e_k \cdot c_o \cdot d_m) \cdot b_j \cdot c_o \cdot d_m \]
\[ = \delta_{c \cdot b_j} \cdot c_o \cdot d_m \cdot b_j - \delta_{c \cdot b_j} \cdot d_m \cdot b_j \cdot c_o \cdot b_j \cdot c_o \cdot d_m \]
\[ = \delta_{c \cdot b_j} \cdot A_j = A_j \cdot b_j \cdot c_o \cdot d_m \]
\[ \cdot \hat{b} \times (\hat{c} \times \hat{d}) = c_{x} \cdot d_{y} \cdot b_j \cdot c_{x} \cdot b_{x} \cdot d_{y} \cdot b_{y} \]
\[ = (\hat{b} \cdot \hat{d}) \cdot \hat{c} = (\hat{b} \cdot \hat{e}) \cdot \hat{d} \]

where we have used freely the associative law for scalar components as well as relationships in items 2, 5, and 7.

**SCALAR AND VECTOR FIELDS**

1. Given a space (3-dimensional as in most engineering situations) if at each point in the space there is defined a scalar quantity or a vector quantity, the situation is called a scalar or a vector field, respectively.

2. In notational form if \( P \) is a point in space and \( f(P) \) or \( \nabla P \) denote the unique associated scalar or vector values, respectively, then \( f(P) \) and \( \nabla P \) denote scalar and vector fields.

3. In 3-dimensional space \( f(P) = f(x, y, z) \) and \( \nabla P = \nabla(x, y, z) \).

4. Fields also exist for "time space" as follows: \( f(t) \) and \( \nabla(t) \)

   We also have fields in spaces that are both geometric and time in nature so that we may have \( f(x, y, z, t) \) and \( \nabla(x, y, z, t) \).

5. A geometric space is said to be spanned by three vectors \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \), if any vector \( \hat{v} \) in the space is given by

   \[ \hat{v} = k_1 \hat{a} + k_2 \hat{b} + k_3 \hat{c} \]

6. The particular vectors \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \) will span a 3-dimensional vector space since any vector in this space can be represented in terms of these particular three.

7. There are an infinite variety of vectors that will span a space, but the essential ingredient is that the vectors \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \) be non-parallel and non-co-planar.

8. A simple test (see page 256, Theorem 1, in your textbook) for 3 vectors to be non-parallel and non-co-planar is the following:

   \[ (8-1) \quad \hat{a} \cdot \hat{b} \times \hat{c} \neq 0 \]

   This is called the scalar triple product.
\[ B \cdot d\mathcal{O} = \mu_0 I = \mu_0 \int_{S} (\mathcal{L} \cdot \hat{a}_z) \cdot (\vec{d}q \, d\mathcal{a}_z) = \mu_0 \int_{S} \vec{d}q \, r^2 \, d\mathcal{a}_z \]

\[ B_{2n\pi} = \frac{\mu_0}{2\pi} \int_{S} \vec{d}q \, d\mathcal{a}_z = \frac{\mu_0}{2\pi} \int_{d\mathcal{a}_z} \vec{d}q \, r^2 \, d\mathcal{a}_z \]

\[ \Rightarrow B = \mu_0 \frac{a^2}{2\pi} \frac{\mathcal{L}}{r^{2n}} \quad \text{for} \quad r > a \]

\[ B_{\phi} = \mu_0 \frac{a^2}{2\pi} \frac{\mathcal{L}}{r^{2n}} \quad \text{for} \quad r > a \]

**Monday**

Class to be re-sectioned - got afternoon section
$\mathbf{5.12} \quad \mathbf{W} = \frac{1}{2} \int \mathbf{\dot{D}} \cdot \mathbf{\dot{E}} \, dV = \frac{\varepsilon}{2} \int \mathbf{\dot{E}} \cdot \mathbf{\dot{E}} \, dV$

\[ \mathbf{\dot{D}} = \varepsilon_0 \mathbf{E} + \mathbf{\dot{P}} \quad \mathbf{\dot{P}} = \chi_\varepsilon \varepsilon_0 \mathbf{E} \quad \text{(special case)} \]

\[ \Rightarrow \mathbf{\dot{D}} = \varepsilon_0 \left( \mathbf{\dot{E}} + \chi_\varepsilon \mathbf{E} \right) \]

\[ \Rightarrow \mathbf{\dot{P}} = \chi_\varepsilon \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E} \]

\[ \mathbf{\dot{B}} = \mu_0 \left( \mathbf{\dot{H}} + \mathbf{\dot{M}} \right) \quad \mathbf{\dot{M}} = \chi_m \mathbf{\dot{H}} \quad \text{(special case)} \]

\[ \Rightarrow \mathbf{\dot{B}} = \mu_0 \left( \mathbf{\dot{H}} + \chi_m \mathbf{\dot{H}} \right) \]

\[ \Rightarrow \mathbf{\dot{B}} = \mu_0 \left( 1 + \chi_m \right) \mathbf{\dot{H}} \]

\[ \Rightarrow \mathbf{\dot{B}} = \mu_0 \mathbf{\dot{H}} = \mu_0 \mathbf{\dot{H}} \]

\[ \nabla \times \mathbf{B} = \mu_0 \left( \nabla \times \mathbf{H} + \nabla \times \mathbf{\dot{M}} \right) = \mu_0 \mathbf{\dot{J}} \]

(Do 3.16) (3.2, 3.4 $\rightarrow$ Hall Effect)
\[
\vec{B} = \frac{\mu_0}{4\pi I} \oint \frac{I \, d^2 \vec{a}}{R^2}
\]

\[
\vec{R} = (x - x') \hat{a}_x + y \hat{a}_y + (z - z') \hat{a}_z = R
\]

\[
I \, d\vec{a} = I_s \, dx' \, dz' \, \hat{a}_z \frac{dz'}{R^2}
\]

\[
\gamma = \frac{\mu_0}{4\pi} \oint \frac{x' \, dx' \, a_x + y' \, dy' + z' \, dz'}{[x(x')^2 + y'(y')^2 + (z - z')^2]^{3/2}}
\]

**PROB. 3-19**

\[
\oint \vec{H} \cdot d\vec{l} = I = 0
\]

\[
H_m \, I_m + H_g \, I_g = 0
\]

\[
B_g = B_m
\]

(\(B \neq \mu \) \text{H \ MUST \ GET \ FROM \ B-H \ CURVE})

In the gap \(B_g = \mu_0 \, H_g \) (\(= B_m \))

\[
\Rightarrow H_m \, I_m + H_g \, I_g = 0 \Rightarrow H_m = \frac{-B_m}{I_m} \frac{B_m}{H_g}
\]

\[
\Rightarrow B_m = -\frac{\mu_0 \, I_m \, H_g}{I_g}
\]

\[
H_m = \frac{-B_m}{\mu_0 \, I_m} \Rightarrow H_m = \frac{-4\pi \times 10^{-7} \times 0.25 \times B_m}{(0.2 \times 10^{-2})} H_m = \frac{-\pi \times 10^{-9} \times H_m}{m}
\]

TWO ANSWERS FOR EACH PART, DEPENDING ON CURRENT DIRECTION

\[
H_m = -40 \Rightarrow B_m = 4\pi \times 10^{-7} \times B_m
\]
\[ \vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{a} \times \vec{A}_r}{R^2} \]

\[ \vec{R} = (x - x')\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z \]

\[ I d\vec{a} = I_S \, dx' \, dz' \hat{a}_z \]

\[ \Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \int \frac{I_S \, dx' \, dz' \hat{a}_z}{\sqrt{(x-x')^2 + y^2 + (z-z')^2}} \]

\[ \text{Prob. 3-19} \]

\[ \oint H \cdot d\vec{l} = I = 0 \]

\[ H_m l_m + H_g l_g = 0 \]

\[ B_g = B_m \]

\[ (B \neq \mu H \text{ must get from } B-H \text{ curve}) \]

\[ B_g = \mu_0 H_g \]  \( (= B_m) \)

\[ \Rightarrow H_m l_m + H_g l_g = 0 = H_m l_m + \frac{B_m}{\mu_0} l_g \]

\[ \Rightarrow B_m = -\frac{H_m}{\mu_0} l_m \]

\[ H_m = \frac{2\pi \times 10^{-3}}{4\pi \times 6.3 \, \text{mm}^3} \]  \( H_m = 1.1 \times 10^{-4} \, H_m \)

\[ \text{Two answers for each part, depending on current direction} \]

\[ H_m = -40 \Rightarrow B_m = 4\pi \times 6.3 \, \text{mm}^3 \frac{wB_B}{m} \]
3-20) IN MAGNETIC FIELD

\[ W = \int B \cdot \mathbf{H} \, dV \]

\[ \beta_m A_m = \beta_H A_H \Rightarrow A_m = \frac{\beta_H A_H}{\beta_m} \]

\[ A_m = \frac{A_m}{H_m} \]

\[ A_m = \frac{-\frac{2}{3} \mu_0 H_0 B_H \beta_H A_H}{2B_H^2 + B_m^2} \]

\[ \beta_H = \frac{\mu_0 H_0}{2B_H^2 + B_m^2} \]

FOR MINIMUM \( \beta_H \)

\[ \beta_H = \frac{\mu_0 H_0 B_m A_m}{2} \]

CHARGES AND CURRENTS (OR NOT AND) DIPOLE MOMENTS

\[ J_p = \frac{\mathbf{B}}{\varepsilon_0 \tau} \]

\[ \nabla \cdot J_p = \frac{\mathbf{B}}{\varepsilon_0 \tau} \nabla \cdot \mathbf{B} = -\frac{\mathbf{B}}{\varepsilon_0 \tau} \]

\[ \Rightarrow \nabla \cdot J_p + \frac{\mathbf{B}}{\varepsilon_0 \tau} = 0 \]

ELECTRONS

MAGNETIC POLES (OR CHARGES)

\[ \text{ALL THREE ARE EQUIVALENT} \]
TUES.

TEST ON 5, 6, 7, 8.

\[ \mathbf{\nabla} \times \mathbf{\Phi} = \mathbf{j} + \frac{\dot{\mathbf{q}}}{\varepsilon} \]

\[ \mathbf{\Phi} \cdot d\mathbf{\ell} = \frac{\mathcal{E}}{\varepsilon} \int_{\partial V} \mathbf{\Phi} \cdot d\mathbf{S} \]

COMPUTE \( \mathbf{\Phi}(t) \) @ \( A \)

\[ D = \varepsilon_0 \varepsilon \]

\[ L = \frac{q_i}{I} = \frac{N\phi_i}{I} \]

\[ J = \frac{1}{\pi a^2} \int \frac{d\phi}{(a^2 + \phi^2)^{3/2}} \]

\[ W = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{\Phi} \, dv = \frac{1}{2} \frac{I^2}{2} \left( 1 + \frac{q_0}{q_1} \cos \frac{\pi}{q_1} \right) \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]
\[ \nabla \cdot \mathbf{D} = \rho \]
\[ \nabla \cdot \mathbf{B} = 0 \]

For non-time varying:

\[ \phi = \frac{-1}{4\pi\epsilon_0} \int \mathbf{A} \cdot d\mathbf{A} \quad \text{and} \quad \mathbf{A} = \frac{1}{4\pi} \int \nabla \times \mathbf{E} \cdot d\mathbf{A} \]

\[ \nabla \times \nabla \times \mathbf{E} = -\frac{\delta}{\epsilon_0} (\nabla \times \mathbf{B}) = -\frac{\delta}{\epsilon_0} \left( \mu_0 \mathbf{J} + \mu_0 \frac{\delta \mathbf{D}}{\delta t} \right) \]
\[ \mathbf{B} = \mu_0 \mathbf{H} \quad \mathbf{D} = \epsilon_0 \mathbf{E} \]
\[ \Rightarrow \nabla \times \nabla \times \mathbf{E} + \mu_0 \mathbf{E} = \frac{\delta^2 \mathbf{E}}{\delta t^2} = -\mu_0 \frac{\delta \mathbf{D}}{\delta t} \]

\[ \Rightarrow \nabla \cdot \left( \nabla \times \nabla \times \mathbf{E} \right) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\delta^2 \mathbf{D}}{\delta t^2} \]

\[ \Rightarrow \nabla^2 \mathbf{E} = -\mu_0 \frac{\delta \mathbf{D}}{\delta t} \quad \text{(wave eq)} \]

\[ \nabla^2 \mathbf{E} = -\mu_0 \frac{\delta \mathbf{D}}{\delta t} - \mu_0 \frac{\delta^2 \mathbf{D}}{\delta t^2} = 0 \]

\[ \frac{\partial^2 \mathbf{E}}{\partial x^2} - (\mathbf{LR} \cdot \mathbf{GC}) \frac{\delta^2 \mathbf{E}}{\delta t^2} - \mathbf{C} \frac{\delta \mathbf{E}}{\delta t} = 0 \]
9.7) \( \vec{E} = j E_0 \sin k_0 z \hat{a}_z \)

\[ \vec{H} = \frac{1}{\eta} E_0 \cos k_0 z \hat{a}_y \]

\[ \eta = \sqrt{\mu_0 / \varepsilon_0} \]

\[ \text{LET} \quad E_{x_1} = E_1 e^{j(\omega t - k z)} \]

\[ E_{x_2} = E_1 e^{j(\omega t - k z)} \]

\[ E_{x_1} = E_1 e^{-j k z} \]

\[ E_{x_2} = E_1 e^{j k z} \]

\[ E_{\text{tot}} = E_1 e^{-j k z} + E_2 e^{j k z} \]

\[ E_{\text{tot}} = E_1 + E_2 \]

\[ E_{\text{tot}} = E_1 \]

\[ E_1 = -E_2 \]

\[ E_{\text{tot}} = E_1 (e^{-j k z} - e^{j k z}) \]

\[ E_{\text{tot}} = -j \omega E_0 \sin k z \]

\[ \text{BACK TO 9.7} \]

\[ S = E_x \vec{H} = \text{power flow} / m^2 = \frac{\text{watts}}{m^2} \]

\[ E(x, y, z, t) = \Re (e^{j \omega t} E) \]

\[ H(x, y, z, t) = \frac{E_0}{\mu_0} \cos k z \cos \omega t \hat{a}_y \]

\[ \vec{S} = -\frac{E_0^2}{2} \sin k z \cos k z \sin \omega t \cos \omega t \hat{a}_x \]

\[ \text{SAVE} = \int_0^T \frac{E_0^2}{2} \sin k z \cos k z \sin \omega t \cos \omega t \, dt \]

\[ T = \frac{2\pi}{\omega} \]
\[ P_{av} = \frac{1}{T} \int_0^T v_i \, i \, dt = \frac{1}{2} \Re \left( v_i I^* \right) \]

Similarly, \( S_{ave} = \frac{1}{2} \Re \left( E \times H^* \right) = \frac{1}{2} \Re \left[ \left( j \, E_0 \sin k_a x \right) \times \left( E_0 - \omega_0 k_s z \alpha \gamma \right) \right] = 0 \]

Mon

Des

Work 10.5, 10.6

Set wave eq - restrict it, for sinusoidal
- Most general solution
For TEM mode:
\[ \nabla^2 E + k^2 E = \nabla^2 E + k^2 E = 0 \]
\[ \text{Also: } \nabla^2 \mathbf{E} = 0 \]
\[ E(x, y) = e^{i(kx - \omega t)} \]

\[ \text{MODE: } \oint \text{OD} = 0 \]
\[ E_r = \frac{k}{2\pi} \ln \left( \frac{r}{a} \right) \]
\[ v = \frac{a}{2\pi} \ln \left( \frac{r}{a} \right) \]
\[ L = r \frac{v}{k} = \frac{r}{2\pi} \ln \left( \frac{r}{a} \right) \frac{a}{2\pi} \]
\[ \nabla^2 E = \frac{\partial^2}{\partial x^2} E + \frac{\partial^2}{\partial z^2} E + \frac{\partial^2}{\partial y^2} E = 0 \]

Also,
\[ \nabla \cdot \mathbf{E} = 0 \]

\[ (E_x, y) = e^{ix-ct} \]
\[ V = \int E \, dA = \int_{0}^{b} \frac{Q(x+dA)}{2\varepsilon_{0}} \, dx = \frac{3dQ}{2\varepsilon_{0}} \]

\[ C = \frac{Q}{V} = \frac{4\varepsilon_{0}A}{3d} \]

\[ \bar{p} = (\varepsilon_{r}-1)\varepsilon_{0} \bar{E} = \left(\frac{2d_{p}}{x+dA}-1\right)\varepsilon_{0} \frac{Q(x+dA)}{2\varepsilon_{0}dA} (-\bar{a}_{x}) = -\frac{Q}{2dA} (d-x)\bar{a}_{x} \]

\[ p_{p} = -V_{p} \bar{D} = -\frac{Q}{2dA} \]

\[ \left[ p_{p} = \bar{p}_{p} \bar{a}_{p} \right]_{x=d} = 0 \]

2. \[ \oint \vec{B} \cdot d\vec{l} = \mu_{0}NI \]

\[ \Phi = \int B \cdot d\zeta = \int_{a}^{b} \frac{UNI}{2\mu_{0} \ln \frac{b}{a}} (t \, dt) = \frac{UNI}{2\mu_{0}} \ln \frac{b}{a} \]

\[ L = \frac{NI}{I} = \frac{UNI^{2}}{2\mu_{0}} \ln \frac{b}{a} \]

3. \[ \vec{M} \cdot \bar{a}_{x} = I \]

\[ H_{m} \times A + H_{g} \times g = NI \]

\[ B_{y} = H_{y}H_{g} = B_{m} \]

\[ H_{m} \times A + \frac{B_{m}}{H_{g}} A_{y} = NI \]

\[ B_{m} = \mu_{0} \left( -\frac{L_{m}}{I} H_{m} + \frac{NI}{I} \right) = \mu_{0} \left( -7500H_{m} + 3.5 \times 10^{6} \right) \]

\[ \text{AT} E_{y} = 0 \]

\[ I_{m} = \frac{3.5 \times 10^{3}}{7.5 \times 10^{3}} = 0.467 \frac{a}{m} \text{ (HORIZONTAL INTERCEPT)} \]

\[ H_{m} = 400 \]

\[ B_{m} = \mu_{0} \left( -7.5(0.4) + 3.5 \right) 10^{6} = 0.628 \text{ Wb/m}^{2} \]

\[ \text{FROM: PLOT} \]

\[ B_{m} = 1.125 \frac{wB}{m^{2}} = B_{g} \]

\[ N \times 0 \]

\[ B_{m} = \mu_{0} \left( -\frac{L_{m}}{I} H_{m} + \frac{NI}{I} \right) = \mu_{0} \left( H_{m} + M \right) \]

\[ M = \frac{NI}{I} \left( 1 + \frac{L_{m}}{I} \right) H_{m} = 3.5 \times 10^{6} - 7501H_{m} = 3.5 \times 10^{6} \left( 1 - 0.75 \right) = 8.75 \times 10^{5} \text{ a-m} \]
\[ E - V + \mathbf{B} = \omega r \mathbf{B}_0 \mathbf{a}_r \]
\[ \int \mathbf{E} \cdot d\mathbf{l} = \int_0^a \omega r B_0 a_r \, dr = \omega B_0 \frac{a^2}{2} \quad V = -\omega B_0 \frac{a^2}{2} \]

6. \[ \rho_m = -\nabla \cdot \mathbf{M} = -\frac{\partial M_z}{\partial z} = -2 \pi M_0 (a-r) \]
\[ \mathbf{P}_m = \mathbf{M} \cdot \mathbf{a}_r = M_0 (a-r) \frac{L^2}{a} \text{ at } z = \pm \frac{L}{2} \]
\[ \mathbf{J}_m = \nabla \times \mathbf{M} = -\mathbf{a}_\phi \frac{\partial B_z}{\partial r} = \mathbf{a}_\phi M_0 a \frac{q e}{r} \]
\[ \mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n = 0 \text{ at } r = a. \]

6. \[ V = 0 \quad (E \text{ inside a perfect conductor} = 0 \text{ always!}) \]
In magnetic circuit problems where the length $l$ and the cross-sectional area $A$ of the flux paths are well defined, it is convenient to define the reluctance of the magnetic path as

$$\rho = \frac{l}{A} \text{ amp-turns/weber} \ (3.61)$$

The reluctance $\rho$ of a flux path, the mmf $N\mu$ which causes the flux, and the flux $\phi$ are related by

$$\phi = \frac{N\mu}{\rho} \text{ webers} \ (3.62)$$

The induced emf caused by a changing magnetic flux is given by

$$\oint E \cdot ds = -\frac{d\phi}{dt} \text{ volts} \ (3.63)$$

If a conductor of length $l$ is moving with a velocity $v$ perpendicularly to a magnetic field of flux density $B$, the voltage induced between the ends is

$$v = Blu \text{ volts} \ (3.64)$$

The energy density of a magnetic field is

$$U = \frac{1}{2}\mu B^2 \text{ joules/meter}^2 \ (3.65)$$

For an electromagnetic wave, the velocity of propagation is given by

$$u = \frac{1}{\sqrt{\mu\varepsilon}} \text{ meters/sec} \ (3.66)$$

The ratio of the electric field intensity to the magnetic field intensity in a plane electromagnetic wave is

$$\frac{E}{H} = \frac{\mu}{\varepsilon} \text{ ohms} \ (3.67)$$

**PROBLEMS**

3.1. An electron is traveling parallel to a straight, current-carrying conductor. If the speed of the electron is 10$^4$ meters/sec and the current in the conductor is 50 amp, what force is exerted on the electron if it is 10 cm from the wire? (The charge of an electron is $e_0 = -1.602 \times 10^{-19}$ coulombs.)

3.2. Consider an $xyz$ coordinate system. The electric field is known to be zero in this test. An electron is projected in the direction of the $y$ axis with a speed of 2.00 $\times 10^9$ meters/sec. If it is projected in the direction of the $z$ axis, the side-wise acceleration is found to be zero. If it is projected in the direction of the positive $y$ axis, the acceleration is found to be 1.78 $\times 10^6$ meters/sec$^2$ in the positive $z$ direction. What are the direction and magnitude of the magnetic flux density $B$ at the origin at this moment?

3.3. Refer to Fig. P3.3. A nonmagnetic insulating disk is inclined with its axis at an angle $\theta$ to a magnetic field $B$ and is rotating at an angular speed of $\omega$ rad/sec. A point charge $q$ is located on the periphery of the disk. Find the instantaneous torque when the charge is in positions 1, 2, 3, and 4. The radius of the disk is $r$. Note that the torque tends to turn the axis of the disk into line with $B$.

3.4. This problem concerns the Hall effect. Consider a solid body such as that shown in Fig. P3.4. The substance carries a current density $J_x$ and is subjected to a magnetic field $B_z$. The sketch shows the effect if the current is one of negative charges (i.e., electrons). The magnetic field deflects the moving charges downward, thus charging the lower face negatively and leaving the upper face with a positive charge. This produces an electric field in the negative $y$ direction that, in equilibrium, annuls the deflecting effect of the magnetic field on the charge carriers.

(a) Show that, if the current is that of positive charge carriers (i.e., holes), the electric field is opposite, i.e., in the positive $y$ direction.

(b) The Hall coefficient is defined by $R_H = e_0/J_xB_z$. Assume that all the charge carriers are alike and that each has a charge $q$ (positive for holes, negative for electrons). The number of charge carriers per unit volume is $N$. Express $R_H$ in terms of $q$ and $N$.

(c) For copper at room temperature, $R_H = -5.5 \times 10^{-11}$ volt-meter$^2$/weber-amp. The electronic charge is $-1.602 \times 10^{-19}$ coulomb. Compute the number of free electrons per unit volume. Compare this with the number of atoms per unit volume as computed by Avogadro's number, which is $6.02 \times 10^{23}$ atoms per gram atom; i.e., the number of atoms per gram of a substance is $6.02 \times 10^{23}/A$, where $A$ is the atomic weight of the substance. The atomic weight of copper is 63.6, and its density is 8.99 grams/cm$^3$.

(d) The Hall coefficient for zinc is positive. What does this fact suggest concerning the charge carriers in zinc?
magnetic field schema. If the electrons enter the region of the magnetic field with a velocity of $22.9 \times 10^4$ meters/sec, determine the magnetic flux density $B$ that is required to make the beam emerge at an angle of $10^\circ$.

3.6. Consider two parallel wires of diameter much smaller than $d$, the distance between their centers. Suppose that the current carried by the wires is $5000$ amp in opposite directions. If the spacing between them is $0.5$ meter, calculate the force in pounds between two 10-meter-length sections of the wires. (One pound is equivalent to 4.448 newtons.) Is the force one of repulsion or attraction?

3.7. In the d’Arsonval meter of Fig. P3.7, the poles of the permanent magnet are shaped around the cylindrical iron core so that there is a uniform, radial magnetic field $B$ in the air gap. The rectangular coil has $N$ turns and a radius $a$. The magnetic field acts over a height $h$. The coil carries a current of $i$ amp.

(a) Determine an expression for the torque tending to turn the coil.
(b) The coil turns against a spring that resists rotation with a torque $j = k \theta$, where $k$ is the torsional spring constant in newton-meters/radian and $\theta$ is the angular deflection in radians. Write the expression for the equilibrium value of $\theta$.
(c) A uniform magnetic field with a density of $B$ webers/meter$^2$ emerges through a small plane area of $A$ meters$^2$. The angle between $B$ and the normal to the area is $\theta$. Write the expression for the magnetic flux $\phi$ that emerges from the surface.

3.8. A rectangular turn of wire has dimensions $h$ and $w$, as shown in Fig. P3.10, and carries a current $i$. The coil is placed in a uniform magnetic field so that the vector $B$ makes an angle $\theta$ with the normal to the plane of the rectangle.

(a) Determine the expression for the torque that tends to turn the rectangle at right angles to $B$.
(b) Express the result of part $a$ in terms of the magnetic moment of the current-carrying rectangle.
(c) A small bit of ferromagnetic material of volume $V$ has a uniform magnetic polarization $M$. Use the results of part $b$, and discuss the torque that the ferromagnetic body will experience when placed in a magnetic field $B$.

3.9. Figure P3.9 shows two views of a rectangular area of width $w$ and height $h$. Through this surface emerges a magnetic flux that has a density $B = 1/\sqrt{1 + y^2}$. The angle between the vector $B$ and the normal to the $yz$ plane is $\theta = \cot^{-1} y$. Determine the expression for the magnetic flux $\phi$ that emerges through the surface.

3.10. A rectangular turn of wire has dimensions $h$ and $w$, as shown in Fig. P3.10, and carries a current $i$. The coil is placed in a uniform magnetic field so that the vector $B$ makes an angle $\theta$ with the normal to the plane of the rectangle.

(a) Determine the expression for the torque that tends to turn the rectangle at right angles to $B$.
(b) Express the result of part $a$ in terms of the magnetic moment of the current-carrying rectangle.
(c) A small bit of ferromagnetic material of volume $V$ has a uniform magnetic polarization $M$. Use the results of part $b$, and discuss the torque that the ferromagnetic body will experience when placed in a magnetic field $B$.

3.11. A typical sample of iron is fairly well saturated magnetically at $B = 1.6$ webers/meter$^2$. Take $H = 5000$ amp/meter$^2$ at this value of $B$. Compute the magnetic moment per atom. See Prob. 3.4 for Avogadro’s number. The atomic weight of iron is 55.85, and its density is 7.87 grams/cm$^3$.

3.12. The flux density in a certain substance is $3.10$ webers/meter$^2$. Calculate the magnetic field intensity $H$ and the magnetic polarization $M$ if the material has a relative permeability of $\mu = 1.00002$; (b) $4.0$; (c) $10,000$.

3.13. Find the exact expression for the total magnetic flux threading the toroidal core shown in Fig. P3.13.

3.14. At a particular instant of time, two isolated electrons are travelling in the plane of the paper with velocities whose magnitudes are $3 \times 10^7$ meters/sec for each one and whose directions are shown in Fig. P3.14. Compute all of the forces exerted on the two particles at that instant, including gravitational effects between each other and between each one and the earth. The electrons are near the earth’s surface.

3.15. Using Gauss’s law and the mmf law, show that, if there are no electrical forces acting to deflect a uniform cylindrical beam of charged particles moving with a maximum velocity, then this velocity must be the velocity of light.
3.16. Figure P3.16 shows a d-c electromagnet in which we wish to obtain a flux density of 1 weber/meter^2 in the air gap. Let \( g = 0.25 \text{ in.} \) Assume that the steel core has infinite permeability.

**Fig. P3.16.** For the design of a d-c electromagnet.

(a) How many ampere-turns are required on the coil?

(b) The “window” has dimensions \( w \) and \( h \). Assume that half the area of the window is filled with copper, the remaining area being insulation and air spaces. A conservative value of current density in the copper is 1000 amp/in.\(^2\). What is the required area of the window?

(c) From the results of part (a) select values for the dimensions \( w \) and \( h \), and estimate the “mean length of turn” for the coil, i.e., the length of one turn of a conductor lying at the middle of the coil. Compute the electric field intensity in the copper and the voltage required for one turn. \( g = 0.020 \times 10^{-3} \text{ ohm-meter} \) at the reasonable operating temperature of 70°C. The coil is to use 50 volts d-c. How many turns are required? What is the cross-sectional area of the copper of one wire? What will be the current in the coil?

(d) Make a detailed check of the coil design as follows: From the table of wire gauges in Appendix C, select the standard wire nearest to the one computed in part (c). Add 0.003 in. to the diameter for enamel insulation. Assume that the width of the winding is the dimension \( h \) minus 0.5 in., and compute the turns per layer and the number of layers required. Compute the “build” of the coil in the direction of the dimension \( w \), assuming 0.006 in. pressboard under the coil, 0.005 in. paper between each layer, and a wrapper 0.020 in. thick. The calculated build should lie between 0.75\( w \) and 0.9\( w \). The latter limit allows for 10 per cent “bulge” in winding. If the first wire size that you choose does not work out properly, choose another and try again.

3.17. Refer to Fig. P3.16. Let \( w = 3 \text{ in.}, h = 6 \text{ in.}, g = 0.25 \text{ in.} \), and assume that the core has a constant permeability of 4000.

(a) Compute the (approximate) mean length of magnetic path in the core.

(b) Compute the reluctance of each of the two parts of the magnetic circuit, neglecting leakage flux and fringing at the gap.

(c) The coil has 2000 turns of wire. What current is required to produce a magnetic flux of 5 \( \times 10^{-2} \) weber across the gap?

3.18. Refer to Fig. P3.16. Let \( w = 3 \text{ in.}, h = 6 \text{ in.}, g = 0.10 \text{ in.} \). The coil has 2000 turns of wire. The \( B-H \) relationship for the steel of the core is given in the table below:

<table>
<thead>
<tr>
<th>( B ), webers/meter^2</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H ), amp/meter</td>
<td>0</td>
<td>78</td>
<td>219</td>
<td>414</td>
<td>1020</td>
<td>5580</td>
<td>9579</td>
</tr>
</tbody>
</table>

3.19. A permanent magnet, such as the one shown in Fig. P3.21, has a magnetic path 20 cm long in the magnetic material. The \( B-H \) curve of the material is given in Fig. P3.19.

**Fig. P3.19.** \( B-H \) curve for an Alnico steel.

Neglect leakage flux and fringing at the gap, and compute the magnetic flux density in the core. Compute the (approximate) mean length of the magnetic path in the core.

(a) 0.1 cm

(b) 1 cm

(c) 2 cm

3.20. With the permanent magnet of Fig. P3.20 we wish to produce a flux density of \( B_w \) webers/meter^2 in the gap of area \( A_w \) and length \( l_w \), and we wish to do this with a permanent magnet of minimum volume. The permanent magnet has a cross-sectional area \( A_m \) and a length \( l_m \) that are to be selected so that the volume \( A_ml_m \) is a minimum. The iron pole pieces may be considered to have infinite permeability. Neglect leakage flux and fringing at the gap.

(a) Show that \( A_ml_m = B_w A_m l_m/\mu_B H_m \), where \( B_w \) is the flux density in the permanent magnet and \( H_m \) is the magnetic field intensity in the permanent magnet. This relation shows that the criterion for minimum \( A_ml_m \) is that the product \(-B_w H_m \) be a maximum.

(b) Refer to the \( B_w-H_m \) curve of Fig. P3.19. Plot the product \(-B_w H_m \) versus \( H_m \) for the portion of the curve in the second quadrant. For this material, what value of \( B_w \) and \( H_m \) satisfy the criterion of part (a)?
(c) Given \( A_2 = 10 \text{ cm}^2 \), \( l_2 = 0.25 \text{ cm} \), and \( B_2 = 1.3 \) webers/meter\(^2\). Determine \( A_m \) and \( l_m \), assuming the material of Fig. P3.19.

(d) If the area determined in part (c) should be increased by 50 per cent to take care of leakage flux, determine the proper diameter for the cylindrical magnet.

3.21. A sectional view of a loudspeaker magnet is shown in Fig. P3.21. The diameter of the magnetic pole is 5 cm. The thickness of the rectangular steel structure is 1.0 cm. We wish to have a flux density of 1.3 webers/meter\(^2\) in the gap. We may assume that the magnetic structure of loudspeaker magnet. 

3.22. When the length of an air gap is appreciable compared with its breadth and depth, there will be an appreciable fringing flux at the edges of the gap that should be included in the total flux. The additional flux can be taken into account approximately by imagining that the breadth of the air gap is increased by the amount indicated in Fig. P3.22, where the dotted lines are semicircles. Given a gap of 1 cm length, compute the ratio of effective area of the air gap to the area of one pole face if:

(a) The pole pieces are circular with a diameter of 5 cm.
(b) The pole pieces are rectangular with dimensions 5 by 100 cm.

3.23. (a) For the transformer shown in Fig. P3.23, calculate the reluctance \( r \) of flux set up by current flowing in the \( N_1 \) turns. Assume a constant permeability \( \mu = 2000 \mu_0 \).

(b) If the current through the primary winding increases linearly with time according to the law \( i = 0.1t \) amp, where \( t \) is measured in seconds, calculate how large the product \( N_1 i \) must be for a constant voltage of 1 volt to be induced in the secondary winding.

3.24. The ring shown in Fig. P3.24 has a constant permeability \( \mu \). The mean length of the magnetic path is \( l \), and the cross-sectional area of the magnetic path is \( A \). The wire that goes through the opening of the ring carries a current \( i \) that is a function of time.

(a) Determine the expression for the magnetic flux \( \phi \) in the ring as a function of \( t \).
(b) Determine the expression for the voltage \( v \) induced in the \( N \)-turn coil.
(c) Let \( i = I_0 \sin \omega t \), where \( t \) is time in seconds. Determine the expression for \( v \) as a function of time. Sketch both \( v \) and \( i \) versus \( t \).

3.25. Refer to Fig. P3.25. A voltage \( v_1 = V_0 \sin \omega t \), where \( t \) is time, is applied to a coil of \( N_1 \) turns which encircles a high permeability toroid. Assuming that \( \mu \) is constant, and neglecting leakage flux, calculate:

(a) The expression for the flux threading the coil as a function of \( t \).
(b) The expression for current \( i \) as a function of time.
(c) The expression for \( v_2 \) as a function of time.

Sketch \( v_1 \), \( v_2 \), \( \phi \), and \( i \) versus time.

3.26. The sketches of Fig. P3.26 show an instrument used in measuring the vibration of machines. The magnetic flux density in the gap is \( B = 1.0 \) weber/meter\(^2\).

(b) The coil has 20 turns and a diameter of 3 cm. If the motion of the coil is \( x = 10^{-2} \) meter, what is the voltage induced in the coil?
3.27. The phonograph pickup of Fig. P3.27 has an \( N \)-turn coil of area \( A \). The permanent magnet supplies a uniform magnetic field of density \( B \). The side-to-side motion of the stylus rotates the coil.

![Fig. P3.27. Moving-coil phonograph pickup.](image)

(a) Determine the expression for the voltage induced in the coil in terms of \( \theta \). What does this expression reduce to for small \( \theta \)? Write this approximate expression in terms of \( x \) (\( x \ll l \)).

(b) Let \( N = 10 \), \( A = 0.10 \) cm\(^2\), \( B = 0.3 \) weber/meter\(^2\), \( l = 0.7 \) cm, and \( x = 10^{-3} \) sin \( 2\pi f t \) cm, where \( f \) is the frequency of the sinusoidal signal in cycles per second and \( t \) is the time in seconds. Determine the induced voltage as a function of time for \( f = 100 \) cycles per second (cps), 1000 cps, 10,000 cps.

(c) For the data in part \( b \), what is the induced emf in millivolts for an instantaneous stylus velocity of 10 cm/sec?

![Fig. P3.28. Voltage induced by changing current.](image)

3.28. A current of \( 10 \cos 3770t \) amp flows through a thin wire as shown in Fig. P3.28. Calculate as a function of time the voltage which is induced between the terminals of the rectangular coil. Sketch \( v \) versus \( t \).

3.30. The betatron sketched in Fig. P3.30 has cylindrical pole pieces. A magnetic flux density of \( B \) webers/meter\(^2\) is changing at the rate \( dB/dt \). Neglect fringing flux. It is observed that a charge placed in the air gap experiences a force even when it is not moving. What are the magnitude and direction of the induced electric field within the gap at a radius \( r \)?

![Fig. P3.30. Betatron.](image)

3.31. The top sketch in Fig. P3.31 shows two resistors connected in a simple circuit. A magnetic flux within the loop is changing at the rate of 1 weber/sec. A high-impedance d-c voltmeter is connected successively in the three positions shown. State the reading of the voltmeter in each of the three positions, and give the reasons for your answers.

![Fig. P3.31. Induced emf.](image)
3.32. This problem is similar to Prob. 3.31, except that the lower conductor is looped around the flux as shown in Fig. P3.32. Four high-impedance d-c voltmeters are arranged as shown. State the readings of the meters, and give the reasons for your answers.

![Diagram of a looped conductor with voltmeters](image)

**Fig. P3.32. What do the voltmeters read?**

3.33. (a) A circular thin sheet of nichrome of resistivity $\rho$ is suspended perpendicular to a magnetic field which is changing at the rate of $dB/dt$ webers/meter$^2$-sec. Sketch the direction of the induced electric field at various points in the sheet and the paths of the resulting currents (often termed "eddy currents"). Calculate the magnitude of the total induced current. Assume that the resistivity of the sheet is large, so that the induced currents are correspondingly small and do not affect the value of $B$ appreciably. (See Fig. P3.33.)

![Diagram of a circular thin sheet with eddy currents](image)

**Fig. P3.33. For the calculation of eddy currents.**

(b) If the sheet is free to turn, what will be its stable equilibrium position?

(c) What will be the stable equilibrium if $B$ is in the direction shown but is decreasing with respect to time?

3.34. Given a toroid with a mean length of magnetic path $l$, a cross-sectional area $A$ normal to the magnetic flux, and a permeability $\mu$. The toroid is wound with $N$ turns of wire that carry a current of $i$ amp. Obtain an expression for the energy in the magnetic field of the toroid.

3.35. An electromagnet is shown in Fig. P3.16. The length of the air gap is one one-hundredth the length of the flux path in the iron. The relative permeability of the iron is 5000. Calculate the ratio of the energy stored in the air gap compared with that in the iron.
"I shall try to correct errors where shown to be errors, and I shall adopt new views as fast as they shall appear to be true views."

...Abraham Lincoln

"The lady bearer of this says she has two sons who want to work. Set them at it if possible. Wanting to work is so rare a want that it should be encouraged."

...Abraham Lincoln
3. The expression for $\vec{a}_\theta$, the unit vector in spherical coordinates, in terms of the rectangular unit vectors is:
   
   (a) $-\sin \theta \vec{a}_x + \cos \theta \vec{a}_y$
   
   (b) $\sin \theta (-\sin \theta \vec{a}_x + \cos \theta \vec{a}_y)$

   (c) $\sin \theta \vec{a}_x - \cos \theta \vec{a}_y$

   (d) $\cos \theta (\sin \theta \vec{a}_x - \cos \theta \vec{a}_y)$

4. The component of $\vec{A}$ in the direction of $\vec{a}_r$ (in cylindrical coordinates) is:
   
   (a) $r - r' \sin(\phi - \phi')$
   
   (c) $r \sin(\phi - \phi') - r' \cos(\phi - \phi')$

   (b) $r' \sin(\phi - \phi')$

   (d) $r - r' \cos(\phi - \phi')$

5. $\int \, A \, dS$ over the side of the prism in the $x-z$ plane is:
   
   (a) $6(\vec{a}_x + \vec{a}_z)$
   
   (c) $6 \vec{a}_y$

   (b) $\frac{6}{2}(\vec{a}_x + \vec{a}_z)$

6. $\int \, A \, dS$ over the top of the cube is:
   
   (a) $1/24$
   
   (c) $2/3$

   (b) $-1/48$

   (d) $1/96$

7. $\int \, A \, dS$ along the portion of the contour described by the line $y=2$ is:
   
   (a) $2$

   (b) $-2$

   (c) $-16/3$

   (d) $4$
12. 17. 2.25 The vectors which may be derived from the gradient of a scalar function are:
(a) \( \vec{\nabla} \), \( \vec{\nabla} \) (b) \( \vec{\nabla}, \vec{\nabla} \) (c) \( \vec{\nabla}, \vec{\nabla} \) (d) \( \vec{\nabla} \)

12. 17. 2.25 The vectors which may be derived from the result of another vector are:
(a) \( \vec{\nabla} \times \vec{\nabla} \) (b) \( \vec{\nabla}, \vec{\nabla} \) (c) \( \vec{\nabla}, \vec{\nabla} \) (d) \( \vec{\nabla} \)

4. 7. The answer to the problem is approximately 5.09 m.
(a) 2 (b) 0.5 (c) 1 (d) 3

6. 21. The electric field intensity in the gap junction is:
(a) \( 4.25 \times 10^2 \) V/m (b) 200 V/m.
(b) 3425 V/m (c) 240 kV/m.

6. 22. The electric field intensity with \( r \) is:
(a) \( 1/r^2 \) (b) \( 1/r \) (c) \( 1/r^{3/2} \) (d) \( 1/r^3 \)

6. 23. The radial force on the electron varies with \( r \) as:
(a) \( 1/r^3 \) (b) \( 1/r^2 \) (c) \( 1/r^{3/2} \) (d) \( 1/r^{1/2} \)

6. 24. 3.1 The surface polarization charge density on the bottom is:
(a) \( \sigma(\vec{\nabla} \cdot \vec{\nabla} + \lambda) \) (b) \( \tau(\vec{\nabla} \cdot \vec{\nabla} + \lambda) \)
(b) \( \sigma(\vec{\nabla} \cdot \vec{\nabla} + \lambda) \) (d) \( \tau(\vec{\nabla} \cdot \vec{\nabla} + \lambda) \).

6. 25. 3.2 If one evaluates the electric field directly (i.e., not using the potential \( \phi \)), it is necessary to evaluate an integral of the form:
(a) \( \int \frac{1}{r^2} \frac{d\phi}{dx} \) (b) \( \int \frac{d\phi}{r^2} \)
(b) \( \int \frac{d\phi}{r^2} \) (d) \( \int \frac{d\phi}{(r^2 + \lambda \phi)} \)

6. 27. 3.5 The surface polarization charge on the right surface can be written as:
(a) \( \sigma \cos \theta / 2 \) (b) \( \sigma \cos \theta / 2 \)
(b) \( \sigma \cos \theta / 2 \) (d) \( \sigma \sin \theta / 2 \)

6. 27. 3.5 In part (b) of the problem, it is necessary to differentiate the electric field with respect to \( t \) with maintained voltage.
(a) \( u \) (b) \( v \) (c) \( -t \)
(b) \( v(t) \) (d) \( (t^2) \)
30. 3.9 The polarization surface charge density on the surface between the dielectric and the oil (i.e., \( r = r_0 \)) will be:

(a) positive  
(b) negative  
(c) positive over half of the cylinder.  
(d) zero

CHECK YOUR CARD:

Does your last mark appear in column #19?

Is your student number marked?

Is your name written on the back of the card?

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<td>Jan. 31 - Feb. 4</td>
<td>Electromagnetic Fields</td>
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<td>8</td>
<td>Feb. 7 - 11</td>
<td>Waves, &amp; Antennas</td>
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<td>Feb. 28 -</td>
<td>Final Exam</td>
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I) VECTOR ANALYSIS

\[ \nabla = \frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z \]

\[ \text{GRAD } \phi = \nabla \phi \]

\[ \text{DIV } \vec{F} = \nabla \cdot \vec{F} \]

\[ \text{CURL } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \]

\[ \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

If \( \nabla \cdot \vec{F} = 0 \), \( \vec{F} \) is SOLENOIDAL

If \( \nabla \times \vec{F} = 0 \), \( \vec{F} \) is IRROTATIONAL

**GAUSS' LAW:** \[ \int_V \nabla \cdot \vec{F} \, dV = \oint_S \vec{F} \cdot d\vec{S} \]

**STOKES' THEOREM:** \[ \oint_C \vec{F} \cdot d\vec{L} = \int_S \nabla \times \vec{F} \cdot d\vec{S} \]

\[ R = R' - R' = (x - x') \hat{a}_x + (y - y') \hat{a}_y + (z - z') \hat{a}_z \]

**MOISEM OF A SOURCE SYSTEM:** \[ m = \frac{1}{2} \int_V \rho \, dV \]

**HELMHOLTZ' THEOREM:** \( \nabla \times \vec{F} = 0 \) AND \( \nabla \cdot \vec{F} = 0 \)

\[ \vec{F} = -\nabla \phi \]

\[ \oint_S \vec{F} \cdot d\vec{S} \]

**RECTANGULAR**

**CYLINDRICAL**
II) ELECTROSTATICS

A) COULOMB'S LAW: 

\[ \mathbf{F}_{12} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2} \hat{\mathbf{r}}_r \]

\[ C_0 = \frac{36II \times 10^{-9}}{M} \]

B) ELECTRIC FIELD

\[ \mathbf{E} = \frac{\rho}{4\pi \varepsilon_0} \frac{\mathbf{r}}{R_i^2} \]

DEFINITION: 

\[ \rho = \frac{\partial \mathbf{E}}{\partial \mathbf{V}} \]

\[ \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{\infty} \frac{q_i}{R_i^2} \hat{\mathbf{r}}_{R_i} \]

DEFINITION: 

\[ Q = \int_V \rho \, dV \]

\[ \Rightarrow \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \int_V \frac{\rho(x',y',z')}{R^2} \hat{\mathbf{r}}_r \, dV' \] (FOR VOLUME)

\[ = \frac{1}{4\pi \varepsilon_0} \int_S \frac{\rho_S}{R^2} \hat{\mathbf{r}}_n \, dS \]

FOR AN INFINITELY CHARGED PLANE

\[ dE_z = \frac{\rho_S 2\pi R \, d\phi}{4\pi \varepsilon_0 R^2} \cos \phi \]

\[ E_z = 2\varepsilon_0 \int_0^{2\pi} \sin \phi \, d\phi = \frac{\rho_S}{2\varepsilon_0} \hat{\mathbf{n}} \]
c) GAUSS' FLUX THEM
\[ E \cdot dS = \frac{q}{4\pi\varepsilon_0 r^2} \cdot dS \]
\[ d\Phi = \frac{q}{4\pi\varepsilon_0 r^2} \cdot dS \quad \theta < \omega < \pi \]
\[ \Phi E \cdot dS = \frac{q}{\varepsilon_0} \quad \text{for} \quad \theta \]
\[ \Phi E \cdot dS = \frac{q}{2\pi\varepsilon_0} = \frac{Q}{\varepsilon_0}. \]

FIELD FROM INFINITE LINE CHARGE:

GAUSS' LAW

\[ \oint S E \cdot dS = \oint S E_0 \cdot dS = \int S_1 E_0 \cdot dS - \int S_2 E \cdot dS \]
\[ 2\pi h E_r = \frac{q}{\varepsilon_0} \Rightarrow E_r = \frac{q}{2\pi\varepsilon_0 h} \]

FROM A CHARGED SPHERE
\[ \Phi E \cdot dS = \frac{q}{\varepsilon_0} \quad \forall \quad \theta \]
\[ = \pi \quad \forall \quad \theta < \theta \]
\[ E_r = \frac{q}{4\pi\varepsilon_0 r^2} \quad \forall \quad \theta > \theta \]
\[ = 0 \quad \forall \quad \theta < \theta \]

D) ELECTROSTATIC POTENTIAL

\[ \text{WORK} = \phi (P_2) - \phi (P_1) \]
\[ \vec{E} = -\nabla \phi \quad \phi = \frac{q}{4\pi\varepsilon_0 R^2} \quad \text{or} \quad \phi = \int_V \frac{\rho dV}{4\pi\varepsilon_0 r^2} \]
\[ \nabla \times \vec{E} = 0 \]

POTENTIAL ON AXIS OF A CHARGED DISK

\[ d\Phi = \frac{\rho_2^2}{4\pi\varepsilon_0 (\omega^2 + z^2)^{3/2}} \]
\[ \Phi_{\text{AXIS}} = \frac{\rho_2^2}{2\varepsilon_0} \int_0^\infty \frac{1}{(\omega^2 + z^2)^{3/2}} \, dz \]
\[ = \frac{\rho_2^2}{2\varepsilon_0} \left[ (\omega^2 + z^2)^{1/2} - |z| \right] \quad \text{Z}>0 \]
\[ E_{\text{AXIS}} = \frac{\partial \Phi}{\partial z} = \frac{\rho_2^2}{2\varepsilon_0} \left[ 1 - z \left( \frac{\omega^2 + z^2}{2} \right)^{1/2} \right] \quad \text{Z}>0 \]
\[ = \frac{\rho_2^2}{2\varepsilon_0} \left[ 1 + z \left( \frac{\omega^2 + z^2}{2} \right)^{-1/2} \right] \quad \text{Z}<0 \]
POTENTIAL FROM A LINE CHARGE IS INFINITE

E) CONDUCTING BOUNDARIES

\[ E_r = \hat{n} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

FIELD TWIXT 2 COAXIAL CYLINDERS

\[ E_r = \frac{\rho_r}{2\pi\varepsilon_0} \]

\[ \phi(a) - \phi(b) = V = \int_a^b E_r dr = \left(\frac{\rho}{2\pi\varepsilon_0} \ln \frac{b}{a}\right) \frac{\rho_a}{2\pi\varepsilon_0} \]

\[ V = \frac{\rho_a}{2\pi\varepsilon_0} \ln \frac{b}{a} \]

\[ \rho_a = 2\pi\varepsilon_0 \sqrt{\varepsilon_0 \mu_0} b \]

\[ E_t = \frac{V}{2\pi\varepsilon_0 (b-a)} \quad a < r < b \]

F) POISSON'S EQUATION

\[ \int_v \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0} \int_v \rho dV \]

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

\[ \nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \quad \text{FOR NO SOURCES,} \quad \nabla^2 \phi = 0 \]
III) ELECTROSTATIC FIELDS IN MATERIAL BODIES, ENERGY, AND FORCES

A) POLARIZABILITY

For spherical atom model: \( \rho = N \rho_0 = N \frac{4 \pi n \varepsilon_0 R^3}{3} \) \( \varepsilon_0 \) (ELECTRIC)

Polarizability: \( \alpha_e = 4 \pi n R^3 \varepsilon_0 \) (ELECTRIC)

\[ \alpha_i = \frac{p^2}{3kT} \] (IONIC)

\[ \rho = N (\alpha_e + \alpha_i + \frac{p^2}{3kT}) \varepsilon_0 \]

\[ \rho = N \left( \frac{\alpha_e}{\varepsilon_0} \right) \varepsilon_0 \]

\( \rho = \varepsilon_0 \chi_e \varepsilon_0 \) ELECTRIC SUSCEPTIBILITY

B) FLUX DENSITY

\( \rho = \varepsilon_0 \varepsilon E + \rho = \varepsilon E \)

\( \nabla \cdot \mathbf{D} = \rho \)

\( \rho = \varepsilon_0 \varepsilon_0 E = (\varepsilon - \varepsilon_0) \varepsilon_0 \)

\( \oint S \cdot dS = \int P \cdot dV = 0 \)

\( \mathbf{n} \cdot \mathbf{D}_1 = \mathbf{n} \cdot \mathbf{D}_2 \)

\( \frac{\mathbf{n} \cdot \mathbf{E}_1}{\mathbf{n} \cdot \mathbf{E}_2} = \frac{\varepsilon_2}{\varepsilon_1} \)

\( \mathbf{D}_1 = \frac{\varepsilon_1}{\varepsilon_2} \mathbf{D}_2 \)

\( \varepsilon \mathbf{E} = \mathbf{K} \mathbf{E} \)

\( \varepsilon \) DIELECTRIC
\( \text{D) STATIONARY CURRENTS} \)

\( \text{OHM'S LAW: } \mathbf{J} = \sigma \mathbf{E} \)

\( \mathbf{I} = \int_S \mathbf{J} \cdot d\mathbf{S} \); CURRENT THRU A SURFACE

\( \text{FOR A CONSERVATIVE FIELD: } V = \phi_1 - \phi_2 = \mathbf{E} \cdot \mathbf{L} \)

\( \text{B) NONCONSERVATIVE FIELDS (EMF)} \)

\( \mathbf{J} = \sigma (\mathbf{E} + \mathbf{E}') \); \( \mathbf{E}' \) IS A CONSERVATIVE FIELD

\( \mathbf{R} = \int_S \frac{d\mathbf{r}}{\mathbf{A}} \); FOR A UNIFORM CYLINDRICAL CONDUCTOR

\( \text{C) CONSERVATION OF CHARGE} \)

\[ - \int_S \mathbf{J} \cdot d\mathbf{S} = - \int_V \nabla \cdot \mathbf{J} \, dV \]  

(\text{GAUSS' THEM})

\[ \Rightarrow \int_V \left( \nabla \cdot \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \right) \, dV = 0 \]  

(\text{CONSERVATION OF CHARGE})

\[ \text{OR } \nabla \cdot \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} = 0 \]  

(\text{DIFFERENTIAL FORM})

\( \text{FOR STATIONARY FIELD, } \frac{\partial \mathbf{E}}{\partial t} = 0 \Rightarrow \nabla \cdot \mathbf{J} = 0 \)

\( \text{WHEN CURRENTS FORM CLOSED LOOPS} \)

\( \text{D) RELAXATION TIME} \)

\( \mathbf{V} = \frac{\mathbf{E}}{\sigma} \)

\( \text{E) RESISTANCE OF ARBITRARY SHAPED CONDUCTORS} \)

\[ \int_c \mathbf{E} \cdot d\mathbf{S} = \Phi_c \]

\[ \int_A \mathbf{J} \cdot d\mathbf{S} = 0 - \int_A \mathbf{E} \cdot d\mathbf{S} = I \]

\[ \Rightarrow \mathbf{R} = \frac{\Phi_1 - \Phi_2}{I} = \frac{\int_c \mathbf{E} \cdot d\mathbf{S}}{\int_A \mathbf{E} \cdot d\mathbf{S}} \]

\( \mathbf{I} \cdot \mathbf{R} = \int_V \mathbf{J} \cdot \mathbf{E} \, dV = \int_V \sigma \mathbf{E} \cdot \mathbf{E} \, dV = \frac{1}{\sigma} \int_V \mathbf{J} \cdot \mathbf{J} \, dV \)

\( \text{F) DUALITY TWIXT } \mathbf{J} \text{ AND } \mathbf{D} \)

\( \mathbf{D} = \text{DISPLACEMENT FLUX DENSITY} \)

\( \text{FOR LINEAR ISOTROPIC MATERIALS} \)

\( \text{CONDUCTOR} \quad \text{DIELECTRIC} \)

\( \nabla \times \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = 0 \)

\( \mathbf{J} = \sigma \mathbf{E} \quad \mathbf{D} = \epsilon \mathbf{E} \)

\( \nabla \cdot \mathbf{J} = 0 \quad \nabla \cdot \mathbf{D} = 0 \)

\( \text{IF } \epsilon \neq \sigma \text{ CONSTANT } \nabla \times \mathbf{J} = 0 \quad \nabla \times \mathbf{D} = 0 \)

\( \mathbf{J} \leftrightarrow \mathbf{D} \quad \sigma \leftrightarrow \epsilon \)
\[ C = \frac{\int_{C} E \cdot ds}{\int_{C} E \cdot dl} \]

\[ RC = \frac{q}{\sigma} ; \text{ IF FRINGING EFFECTS ARE NEGLECTED} \]

6) JOUTE'S LAW;

\[ P = \frac{dW}{dt} = I^2 R \]

7) CONVECTION CURRENT

\[ \mathbf{J} = \rho \mathbf{V} \quad (\mathbf{V} \text{ IS VELOCITY}) \]

8) FLUX PLOTTING:

1) TAKE ADVANTAGE OF SYMMETRY

2) DRAW IN BOUNDARIES SEPARATING CONDUCTING AND THOSE WITH 0 CONDUCTIVITY

3) STARTING WITH KNOWN POTENTIALS AND FLOW LINES, SKETCH FIELD, MAINTAINING ORTHONALITY

4) REFINE TO CURVILINEAR SQUARES

\[ R = \frac{N_p}{\sigma N_f} \]

\[ C = \frac{C_{N_f}}{N_p} \]
VI) STATIC MAGNETIC FIELD IN A VACUUM

A) AMPERE'S LAW OF FORCE
\[ \mu_0 = \text{FARADAYS/METER} (\text{PERMEABILITY}) \]

B) MAGNETIC FIELD (B = \text{WEBER/M}^2)
\[ \mathbf{E} = q \mathbf{V} \times \mathbf{B} \] (LORENZ FORCE)
\[ \mathbf{B} = \nabla \times \mathbf{A} \Rightarrow \mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}}{r^2} \, dV \]

C) MAGNETIC DIPOLE
FOR ARBITRARILY SHAPED LOOP:
\[ \mathbf{M} = \frac{1}{2} \mathbf{\Phi} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} \int_V \mathbf{F} \times \mathbf{J} \, dV \]
\[ \mathbf{T} = \mathbf{E} \times \mathbf{B} \quad \text{or} \quad \mathbf{T} = \frac{1}{2} \int \mathbf{E} \times \mathbf{B} \, d\mathbf{S} \]

D) MAGNETIC FLUX AND \( \nabla \cdot \mathbf{B} \)
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \text{FLUX PASSING THRU A SURFACE} \]
\[ = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} \]
\[ = \Phi_c \mathbf{A} \cdot d\mathbf{S} \]

E) AMPERE'S CIRCUITUAL LAW
\[ \oint_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int_S \mu_0 \mathbf{J} \cdot d\mathbf{S} = \Phi_c \mathbf{B} \cdot d\mathbf{S} \]
\[ \nabla \cdot \mathbf{E} = \rho / \epsilon_0 \]
\[ \nabla \cdot \mathbf{D} = \rho \]
\[ \nabla \times \mathbf{D} = \nabla \times \mathbf{P} \]
Magnetic Field in Material Bodies

1) Magnetic dipoles align in accordance with torque.
2) If atom has no moment, external field distorts, creating magnetic dipole:

\[ \vec{M} = n \vec{M}_0 \left( \frac{\Delta M}{\vec{M}_0} \right) \]

B) Equivalent volume and surface polarization currents:

\[ \vec{A} = \frac{\mu_0}{4\pi} \nabla \times \frac{\vec{M}}{R} \]

\[ \vec{J}_m = \text{polarization current}, \quad \vec{J}_{ms} = \text{surface pol. current} \]

\[ \begin{align*}
J_m &= \nabla \times \vec{M} \\
J_{ms} &= \vec{M} \times \hat{n}
\end{align*} \]

\[ m = \vec{M} \, dV = I \, d\vec{s} \]

\[ \Rightarrow I_{x,y,z} = M_{xyz} \, dx \, dy \, dz \]

\[ J_{x,y,z} = -\frac{dM_{xyz}}{dx, dy, dz} \]

C) Magnetic Field Intensity (H)

\[ \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \]

\[ \Delta \times \vec{H} = \vec{J} \]

\[ \vec{M} = \chi_m \vec{H} \]

\[ \vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \mu H \]

D) \( \vec{B} \cdot \vec{H} \) curve

\[ \int_S \nabla \times \vec{H} \cdot d\vec{s} = \oint_c \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s} \]

E) Boundary Conditions for \( \vec{B} \) and \( \vec{H} \)

F) Scalar potential for \( \vec{H} \)

(\( \hat{P} \) = dipole moment/volume)

For a permanent magnet:

\[ \nabla \times \vec{H} = \vec{0}; \quad \nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \]

\[ \vec{B} = \mu_0 (\vec{H} + \vec{M}) \quad \text{(inside body)} \]

\[ \oint_c \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s} = NI \]
VIII) QUASI-STATIONARY MAGNETIC FIELD

A) FARADAY'S LAW

\[ \psi = \int_S \mathbf{B} \cdot d\mathbf{S} \]

\[ \oint_c \mathbf{E} \cdot d\mathbf{c} = -\frac{\Delta}{\Delta t} \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = -\frac{\sigma}{\varepsilon} \mathbf{A} \cdot d\mathbf{L} \]

\[ \nabla \times \mathbf{E} = \frac{\mathbf{B}}{\varepsilon} \]

\[ F = q \mathbf{V} \times \mathbf{B} \Rightarrow \mathbf{E} = \frac{\mathbf{V}}{\varepsilon} = \mathbf{V} \times \mathbf{B} \]

\[ V_{\text{induced}} = -\int_S \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{S} + \oint_c \mathbf{V} \times \mathbf{B} \cdot d\mathbf{L} \]

FOR MOTIONAL EMF

\[ \oint_c \mathbf{E} \cdot d\mathbf{c} = -\frac{\sigma}{\varepsilon} \int_S \mathbf{B} \cdot d\mathbf{S} \]

\[ \psi_{21} = \]
WHAT DISTANCE $\Delta x$ MUST A STRING BE STRETCHED ON THE N$^{th}$ FRET TO RAISE IT ONE CHROMATIC STEP?

THE STRING OBEYS HOOPE'S LAW, AND IS INITIALLY UNDER TENSION $T_0$. ASSUME STRETCHED AND UNESTRETCHED LINEAR MASS DENSITIES EQUIVALENT.

**ANSWER:**

**UNSTRETCHED**

$$f_n^2 = \frac{T_0}{m \cdot l_n} = \frac{T_0}{m \cdot l_n} \quad \text{and} \quad m = \text{mass of string}$$

**STRETCHED**

$$\Delta x$$

$$f_s^2 = \frac{T_0 + k \cdot l_0 \cdot R^{-n} \cdot \csc \phi}{m \cdot l_0 \cdot R^{-n} \cdot \csc \phi}$$

$$T_s = T_0 + k \cdot l_0 \cdot R^{-n} \cdot (\csc \phi - 1)$$

$$\mu_s = \mu_n = \frac{L_0}{L_n}$$

$$\Rightarrow \quad f_s^2 = \frac{\mu_s \cdot l_0 \cdot R^{-n} \cdot \csc \phi}{m \cdot l_0 \cdot R^{-n} \cdot \csc \phi} = \frac{[T_0 + k \cdot l_0 \cdot R^{-n} \cdot (\csc \phi - 1)]}{m \cdot l_0 \cdot \csc^2 \phi}$$

**NOW:**

$$f_n^2 = \frac{R^2}{R^2} = \frac{[T_0 + k \cdot l_0 \cdot R^{-n} \cdot (\csc \phi - 1)]}{T_0 \cdot \csc^2 \phi}$$

$$= \sin^2 \phi + k \cdot l_0 \cdot R^{-n} \cdot \csc^2 \phi - R^2 = 0$$

$$\Rightarrow \quad \sin^2 \phi + k \cdot l_0 \cdot R^{-n} \cdot \csc^2 \phi - R^2 = 0$$

$$\Rightarrow \quad \sin \phi = \frac{l_0 \cdot R^{-n} \cdot \csc \phi}{(\Delta x_n^2 + l_0^2 \cdot R^{-2n})^{1/2}}$$

$$\Rightarrow \quad (1 - \frac{k \cdot l_0}{T_0} \cdot \frac{l_0^2 \cdot R^{-2n}}{\Delta x_n^2 + l_0^2 \cdot R^{-2n}}) + \frac{k \cdot l_0^2 \cdot R^{-n}}{T_0^2 (\Delta x_n^2 + l_0^2 \cdot R^{-2n})^{1/2}} = R^2 = 0$$

$$\Rightarrow \quad T_0 \cdot R^{2n} (\Delta x_n^2 + l_0^2 \cdot R^{-2n})^{1/2} = 0$$
LET $\psi_n = (\Delta x_n^2 + l_o^2 R^{-2n})^{\frac{1}{2}}$

$\Rightarrow (T_o - k l_o)(l_o^2 R^{-2n}) + k l_o^2 R^{-n} \psi_n = T_o R^{+2} \psi^2 = 0$

BY QUADRATIC THEOREM:

$$\psi_n = \frac{-k l_o^2 R^{-n} \pm \left[ k^2 l_o^4 R^{-2n} + 4 T_o R^{+2} (T_o - k l_o) (l_o^2 R^{-2n}) \right]^{\frac{1}{2}}}{2 l_o^2 T_o R^{+2}}$$

$\Rightarrow \Delta x_n = \left( \Delta x_n^2 + l_o^2 R^{-2n} \right)^{\frac{1}{2}}$

$$\Rightarrow \Delta x_n = \frac{-k l_o^2 R^{-n} \pm \left[ k^2 l_o^4 R^{-2n} + 4 T_o R^{+2} (T_o - k l_o) (l_o^2 R^{-2n}) \right]^{\frac{1}{2}}}{4 T_o R^{+2}}$$

FOR $l_o = l_n$ (i.e., $n = 0$)

$$\Delta x = \frac{-k l_o^2 R^{-n} \pm \left[ k^2 l_o^4 R^{-2n} + 4 T_o R^{+2} (T_o - k l_o) (l_o^2 R^{-2n}) \right]^{\frac{1}{2}}}{4 T_o R^{+2}} - l_o^2$$
Find \( \Delta x_n \) pitch of string increases one chromatic step, assume Hooke's law. Let \( \phi = \tan^{-1} \frac{\Delta x_n}{\Delta y_n} \)

\[ m = \text{string's mass} \]

\[ \theta = \tan^{-1} \frac{R(1 - R^{-n})}{\Delta x_n} \]

\[ \mu_n = \frac{m}{l_0 R^{-n}} \Rightarrow f_n^2 = \frac{T_0}{m l_0 R^{-2n}} = \frac{T_0}{m l_0} R^{2n} \]

\[ \mu_0 = \frac{m}{l_0 R^{-n} (\csc \phi + R^{-n} \csc \phi)} \]

\[ T_0 = T_0 + k l_0 R^{-n} (\csc \phi - 1) \]

\[ l_0 = l_0 R^{-n} \csc \phi \]

\[ \Rightarrow f_0^2 = \frac{[T_0 + k l_0 R^{-n} (\csc \phi - 1)] [1 - R^{2n} \csc \phi + R^{-n} \csc \phi]}{\csc \phi} \]

\[ f_0^2 = R^2 = \frac{[T_0 + k l_0 R^{-n} (\csc \phi - 1)] [1 - R^{2n} \csc \phi - R^{-n} \csc \phi]}{\csc \phi} \]
1. Write the cylindrical coordinate unit vectors, $\hat{a}_r, \hat{a}_\theta, \hat{a}_z$, in terms of $\hat{a}_x, \hat{a}_y, \hat{a}_z$.

2. Convert your expressions in (1) so that only the rectangular variables $x, y, z$ are used (i.e. no "r" or "\( \theta \)").

3. Repeat (1) & (2) for the spherical coordinate unit vectors $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$.

4. Write the complete expression for

$$\vec{R} = (x-x')\hat{a}_x + (y-y')\hat{a}_y + (z-z')\hat{a}_z$$

in terms of (a) cylindrical coordinates and
(b) spherical coordinates.
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<th>Subject</th>
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<td>( \sqrt{17} )</td>
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18. $1.01 \times 10^6$ tons

19.

20. $-\frac{1}{2} \alpha r$ (cylindrical coordinates)

21. $9.3 \times 10^6$ m/sec

22. 16.76 cm

23. 0.34 mm

25. 

$$E_{\text{axis}} = \frac{P}{S} \ln \left( \frac{1 + \sqrt{1 + \left(\frac{a}{\sqrt{3}} - \frac{v}{2}\right)^2}}{1 + \sqrt{1 + \left(\frac{a}{\sqrt{3} + \frac{v}{2}}\right)^2}} \right)$$

26. 

$\Delta l_1 = 0.146 \times 10^{-13} R_{\text{He}}$  \hspace{1cm} $\Delta l_2 = 0.234 \times 10^{-13} R_{\text{He}}$

27. $\theta_1 = 14^\circ$

28. 

$c = \frac{2w e}{\ln \frac{b}{a}}$  \hspace{1cm} $\ln \frac{b}{a} = e$ (base of natural log)

29.

30. $r_0 = \frac{4}{3}$ cm  \hspace{1cm} $\phi = 358$ K

31. $F = -4 \times 10^{-5}$ newtons

32.
1-5) \( \overline{A} = 2 \overline{a}_x + \overline{a}_y - 3 \overline{a}_z \)

a) **FOR CYLINDRICAL CO-ORDINATES**

\[
\begin{align*}
\overline{a}_r &= \cos \phi \overline{a}_x + \sin \phi \overline{a}_y \\
\overline{a}_\phi &= -\sin \phi \overline{a}_x + \cos \phi \overline{a}_y \\
\overline{A}_r &= \overline{A} \cdot \overline{a}_r = (2 \overline{a}_x + \overline{a}_y - 3 \overline{a}_z)(\cos \phi \overline{a}_x + \sin \phi \overline{a}_y) \\
&= 2 \cos \phi + \sin \phi \\
\overline{A}_\phi &= \overline{A} \cdot \overline{a}_\phi = (2 \overline{a}_x + \overline{a}_y - 3 \overline{a}_z)(-\sin \phi \overline{a}_x + \cos \phi \overline{a}_y) \\
&= -2 \sin \phi + \cos \phi \\
\Rightarrow \overline{A} &= (2 \cos \phi + \sin \phi) \overline{a}_r + (\cos \phi - 2 \sin \phi) \overline{a}_\phi - 3 \overline{a}_z
\end{align*}
\]

b) **FOR SPHERICAL CO-ORDINATES**

\[
\begin{align*}
\overline{a}_r &= \sin \theta \cos \phi \overline{a}_x + \sin \theta \sin \phi \overline{a}_y + \cos \theta \overline{a}_z \\
\overline{a}_\theta &= \cos \theta \cos \phi \overline{a}_x + \cos \theta \sin \phi \overline{a}_y - \sin \theta \overline{a}_z \\
\overline{a}_\phi &= -\sin \phi \overline{a}_x + \cos \phi \overline{a}_y \\
\overline{A} &= 2 \overline{a}_x + \overline{a}_y - 3 \overline{a}_z \\
&= (2 \sin \theta \cos \phi + \sin \theta \sin \phi - 3 \cos \theta) \overline{a}_r \\
&\quad + (2 \cos \theta \cos \phi + \cos \theta \sin \phi + 3 \sin \theta) \overline{a}_\theta \\
&\quad + (-2 \sin \phi + \cos \phi) \overline{a}_\phi
\end{align*}
\]

1-8) \( \overline{A} = 6 \overline{i} + 6 \overline{j} \)

\( \overline{B} = -6 \overline{i} + 3 \overline{k} \)

\( \overline{C} = 0 \overline{i} - 6 \overline{j} - 3 \overline{k} \)

\( \overline{D} = 0 \overline{i} - 3 \overline{k} \)

\( \overline{E} = -3 \overline{k} \)

\( \overline{A} + \overline{B} + \overline{C} + \overline{D} + \overline{E} = \overline{0} \)
\[
\begin{align*}
1-11) \quad \frac{1}{R} &= \left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{1}{2}} \\
\frac{\partial}{\partial x}\left( \frac{1}{R} \right) &= 2(x-x') \left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{3}{2}} \\
\frac{\partial^2}{\partial x^2}\left( \frac{1}{R} \right) &= -\left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{5}{2}} \\
\n1-12) \quad \psi &= x^2y^z \\
\nabla \psi &= 2xyzt_x + x^2yzt_y + x^2yt_z = 12zt_x + 4zt_y + 12zt_z \\
(\nabla \psi) \cdot \left( \frac{3}{\sqrt{50}} \tilde{\alpha}_x + \frac{3}{\sqrt{50}} \tilde{\alpha}_y + \frac{3}{\sqrt{50}} \tilde{\alpha}_z \right) &= \frac{11}{2}\sqrt{50} \\
1-13) \quad \frac{1}{R} &= \left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{1}{2}} \\
\frac{\partial}{\partial x}\left( \frac{1}{R} \right) &= -(x-x') \left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{3}{2}} \\
\frac{\partial^2}{\partial x^2}\left( \frac{1}{R} \right) &= (x-x') \left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{3}{2}} \\

1-14) \quad \tilde{A} &= x^2\tilde{\alpha}_x + xy^2\tilde{\alpha}_y + 2xy^2z^2\tilde{\alpha}_z \\
\nabla \cdot \tilde{A} &= 2x + 2xy + 72x^2y^2z^2 \\
\int_V \nabla \cdot \tilde{A} \, dV &= \int V \nabla \cdot \tilde{A} \, dx \, dy \, dz \\
\int_0^1 \int_0^2 \int_0^4 (2x + 2xy + 72x^2y^2z^2) \, dx \, dy \, dz \\
\int_0^1 \int_0^2 \int_{-2}^2 [x^2 + x^2y + 24x^2y^2z^2] \, dy \, dz \\
\int_0^1 \int_{-2}^2 \left[ 1 + y + \frac{24y^2z^2}{2} \right] \, dy \, dz \\
\int_{-2}^2 \left[ \frac{3}{2} + 8z^2 \right] \, dz = \left[ \frac{3}{2}z + \frac{8}{3}z^3 \right]_0^1 = \frac{3}{2} + \frac{8}{3} \\
= \frac{9}{6} + \frac{16}{6} = \frac{25}{6} \\
\text{(cont.)}
\end{align*}
\]
\[ \bar{\mathbf{A}} = x^2 \bar{a}_x + x^2 y^2 \bar{a}_y + 2x^2 y^2 \bar{a}_z \]

**Gauss' Law:**

\[ \int_V \nabla \cdot \bar{\mathbf{A}} \, dV = \oint \bar{\mathbf{A}} \cdot d\mathbf{S} \]

\[ \int_V \nabla \cdot \bar{\mathbf{A}} \, dV = \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (2x^2 y^2 + 72x^2 y^2 z^2) \, dx \, dy \, dz = \frac{1}{24} \]

**For side 1:**

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 y^2 \, dx \, dz = \frac{1}{36} \]

**For side 2:**

\[ d\bar{S}_2 = dxdz \bar{a}_y \quad ; \quad y = \frac{1}{2} \]

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 y^2 \, dx \, dz = \frac{1}{36} \]

**For side 3:**

\[ d\bar{S}_3 = dydz \bar{a}_x \]

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x \, dx \, dy \, dz = \frac{1}{4} \]

**For side 4:**

\[ d\bar{S}_4 = dydz \bar{a}_y \quad ; \quad x = \frac{1}{2} \]

\[ -\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \, dy \, dz = -\frac{1}{4} \]

**For side 5:**

\[ d\bar{S}_5 = dxdy \bar{a}_z \]

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} 24x^2 y^2 z^2 \, dx \, dy \, dz = \frac{1}{48} \]

**For side 6:**

\[ d\bar{S}_6 = dxdy \bar{a}_z \quad ; \quad z = \frac{1}{2} \]

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \, dz = \frac{1}{48} \]

\[ \frac{1}{2} \int S \bar{A} \cdot d\bar{S} = \frac{1}{16} + \frac{1}{16} = \frac{1}{8} \]
1.18) \textbf{PROVE:} \( \int_V \nabla \cdot \mathbf{F} \, dV = \oint_S \mathbf{F} \cdot \mathbf{n} \, dS - \int_V \mathbf{F} \cdot \nabla \varphi \, dV \)

Now \( \nabla \cdot (\mathbf{F} \cdot \mathbf{n}) = \varphi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \varphi \)

And \( \int_V \nabla \cdot \mathbf{A} \, dV = \oint_S \mathbf{A} \cdot \mathbf{n} \, dS = \oint_S \mathbf{A} \cdot d\mathbf{S} \)

Let \( \mathbf{A} = \mathbf{F} \cdot \mathbf{n} \)

\( \Rightarrow \int_V \nabla \cdot (\mathbf{F} \cdot \mathbf{n}) \, dV = \oint_S \mathbf{F} \cdot d\mathbf{S} \)

Now \( \int_V \nabla \cdot (\mathbf{F} \cdot \mathbf{n}) \, dV = \int_V \varphi \nabla \cdot \mathbf{F} \, dV + \int_V \mathbf{F} \cdot \nabla \varphi \, dV \)

\( = \int_V \varphi \nabla \cdot \mathbf{F} \, dV + \oint_S \mathbf{F} \cdot d\mathbf{S} - \int_V \mathbf{F} \cdot \nabla \varphi \, dV \)

1.16) \( d\mathbf{S} = dy \, \mathbf{a}_y \)

\( d\mathbf{S} = dx \, \mathbf{a}_x \)

\( d\mathbf{S} = dz \, \mathbf{a}_z \)

\( \mathbf{F} = x \mathbf{a}_x + x^2 y \mathbf{a}_y + y^2 x \mathbf{a}_z \)

\( \int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_0^1 dy \int_0^1 dx \mathbf{F} \cdot d\mathbf{S} \)

\( = \int_0^2 \int_0^1 dy \int_0^1 dx (x^2 y \mathbf{a}_y + y^2 x \mathbf{a}_z) \cdot (dy \, \mathbf{a}_y + dx \, \mathbf{a}_x) \)

\( = \int_0^2 \int_0^1 dy \int_0^1 dx (x^2 y dy + y^2 x dx) \)

\( = \int_0^2 \int_0^1 dx \left( \int_0^1 x^2 y dy + \int_0^1 y^2 x dx \right) \)

\( = \int_0^2 \int_0^1 dx \left( \frac{1}{3} x^3 y + \frac{1}{2} y^2 x \right) \)

\( = \int_0^2 \left( \frac{1}{3} x^3 \int_0^1 dy + \frac{1}{2} y^2 x \int_0^1 dx \right) \)

\( = \int_0^2 \left( \frac{1}{3} x^3 + \frac{1}{2} y^2 x \right) \)

\( = \left[ \frac{1}{3} \left( \frac{1}{3} x^6 + \frac{1}{2} x^3 y^2 \right) \right]_0^2 \)

\( = \left[ \frac{1}{3} \left( \frac{64}{3} + 64 \right) \right] \)

\( = 8 \)

\( \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = (2yx) \mathbf{a}_x - (y^2) \mathbf{a}_y + 2xy \mathbf{a}_z \)

\( d\mathbf{S} = dx \, dy \, \mathbf{a}_z \)

\( \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_0^1 \int_0^1 (2yx \mathbf{a}_x - y^2 \mathbf{a}_y + 2xy \mathbf{a}_z) \cdot (dx \, dy \, \mathbf{a}_z) \)

\( = \int_0^2 \int_0^1 \int_0^1 (2yx - y^2 + 2xy) \, dx \, dy \)

\( = \int_0^2 \int_0^1 (2yx - y^2 + 2xy) \, dy \)

\( = \left[ \frac{1}{2} x^2 y^2 - \frac{1}{3} y^3 + xy^2 \right]_0^1 \)

\( = \left[ \frac{1}{2} x^2 - \frac{1}{3} y^3 + xy^2 \right]_0^1 \)

\( = \frac{1}{2} - \frac{1}{3} + x \)

\( = \frac{1}{2} + \frac{1}{3} \)

\( = \frac{5}{6} \)

\( = 8 \)
\[ \mathbf{\hat{R}} = \mathbf{r}_{a_r} - \mathbf{r}_{a_{q'}} \]

\[ \mathbf{\hat{a}}_r = \sin \Theta \mathbf{\hat{a}}_{q'} + \cos \Theta \mathbf{\hat{a}}_2 \]

\[ \mathbf{\hat{a}}_{q'} = \cos(\phi - \phi') \mathbf{\hat{a}}_q - \sin(\phi - \phi') \mathbf{\hat{a}}_\phi \]

\[ \mathbf{\hat{a}}_2 = \cos \Theta \mathbf{\hat{a}}_r + \sin \Theta \mathbf{\hat{a}}_\phi \]

\[ a_{q'} = \frac{1}{2 \sin \Theta} \frac{a_q}{a_r} \cos \Theta \]

\[ a_{\phi} \]
\[ \begin{align*}
\Delta r &= \Delta r_1 \cos \theta - \Delta r_2 \sin \theta \\
\Delta \theta &= \Delta \theta_1 \cos \theta + \Delta \theta_2 \\
\Delta \phi &= \Delta \phi_1 \cos \theta + \Delta \phi_2 \\
\Delta l &= \Delta l_1 \cos \theta + \Delta l_2 \\
\end{align*} \]

\[ \begin{align*}
\Delta r_1 &= \cos \phi' (\cos \theta \Delta r_1 - \sin \theta \Delta r_2) \\
&+ \sin \theta \left[ \cos (\phi - \phi') (\sin \theta \Delta r_1 + \cos \theta \Delta r_2) - \sin (\phi - \phi') \Delta r_2 \right] \\
\Delta r_2 &= [\cos \theta \cos \phi' + \sin \theta \sin \phi' \cos (\phi - \phi')] \Delta r_1 \\
&+ [\cos \theta \sin \phi' + \sin \theta \cos \phi' \cos (\phi - \phi')] \Delta r_2 \\
&- [\sin \phi'] \Delta r_2
\end{align*} \]

\[ \begin{align*}
\Delta l_1 &= \left[ r - r' \cos \phi' \cos \theta - r' \sin \phi' \sin \theta \cos (\phi - \phi') \right] \Delta r_1 \\
&+ \left[ r' \sin \phi' \cos \theta - r' \cos \phi' \sin \theta \cos (\phi - \phi') \right] \Delta r_2 \\
&+ \left[ r' \sin (\phi - \phi') \right] \Delta \phi \\
&= (x - x') \Delta x + (y - y') \Delta y + (z - z') \Delta z
\end{align*} \]
\overline{J_{x}} = \rho \langle \psi \rangle
DUALITY

CONDUCTOR
\[ \nabla \times \vec{E} = 0 \]
\[ \vec{J} = \sigma \vec{E} \]
\[ \nabla \cdot \vec{J} + \frac{\partial \phi}{\partial t} = 0 \]

ASSUME \( \nabla \cdot \vec{J} = 0 \)

ALSO
\[ \nabla \times \vec{J} = 0 \]
\[ \vec{J} = \sigma \vec{E} = -\sigma \nabla \phi \]
\[ \nabla^2 \phi = 0 \]

\[ J \Leftrightarrow \vec{D} \]
\[ \sigma \Leftrightarrow \epsilon \]

DIELECTRIC
\[ \nabla \times \vec{E} = 0 \]
\[ \vec{D} = \epsilon \vec{E} \]
\[ \nabla \cdot \vec{D} - \rho = 0 \]

ASSUME \( \nabla \cdot \vec{D} = 0 \)

ALSO
\[ \nabla \times \vec{D} = 0 \]
\[ \vec{D} = \epsilon \vec{E} = -\epsilon \nabla \phi \]
\[ \nabla^2 \phi = 0 \]

"CURVILINEAR SQUARES"

IF WE SELECT \( \frac{\Delta \vec{w}}{\Delta z} = 1 \) (SQUARES)
\[ I = -\sigma \Delta \phi \]
(OVER \( \psi := \int \vec{D} \cdot d\vec{s} = -\epsilon \Delta \phi \))
\[ I_{\text{Total}} = -\sigma N_F \Delta \phi \]
\[ \phi_2 - \phi_1 = V = N_F \Delta \phi \]
\[ R = \frac{|V|}{|I_{\text{Total}}|} = \frac{N_F}{\sigma} \]

OR
\[ \psi_{\text{Total}} = \varepsilon N_F \Delta \phi = Q \]
\[ V = N_F \Delta \phi \]
\[ C = \frac{\varepsilon}{V} = \frac{\varepsilon N_F}{N_F} \]

\[ N_F = 12 \]
\[ N_P = 4 \]

**Aluminum:**
\[ R = \frac{4}{(3.5 \times 10^7)(12)} = 0.0095 \text{ m}^2 \text{V}^{-1} \]

**Air:**
\[ C = \frac{10^{-9}}{36 \pi (\frac{12}{4})} = 26.6 \text{ m}^2 \text{V}^{-1} \]

**In a conductor:**
\[ \nabla \cdot J + \frac{\varepsilon_0}{\varepsilon} \frac{\partial E}{\partial t} = 0 \text{ (charge conserved)} \]
\[ \nabla \cdot E = \rho \quad J = \sigma E \]
\[ \varepsilon_0 \nabla \cdot E = \frac{E}{\varepsilon} \nabla \cdot J = \rho \]
\[ \frac{\partial E}{\partial t} + \frac{\partial E}{\partial t} = 0 \quad \rho = \rho_0 e^{-0.4t} \]
\[ J = \frac{E}{\varepsilon} = \text{relaxation time} \]

**For thermal conduction:**
\[ \nabla \cdot q + \frac{\partial}{\partial t} \left( \frac{s^2}{\varepsilon} E \right) = 0 \text{ (heat conserved)} \]
\[ q = -k \nabla T \text{ (see Prob. 5.11)} \]

\[ \begin{array}{ccc}
\text{Material} & \rho & \varepsilon_0 \\
\text{Cu} & 1.5 \times 10^{-19} \text{sec}^{-1} & 1.3 \times 10^{-5} \\
\text{Ag} & 1.3 \times 10^{-5} \text{sec}^{-1} & \\
\text{Sea H}_2\text{O} & 2 \times 10^{-18} \text{sec}^{-1} & \\
\text{Distilled H}_2\text{O} & 10^{-16} \text{sec}^{-1} & \\
\text{Quartz} & 10 \text{ days} & \\
\end{array} \]
\[ \vec{E} = \frac{\mu_0}{4\pi R^2} \left[ I_2 \vec{dl} \times (I_1 \vec{dl} \times \vec{a}_e) \right] \]

\[ = I_2 \vec{dl} \times \vec{d}\vec{B}_{21} \]

\[ d\vec{B}_{21} = \frac{\mu_0}{4\pi R^2} (I_1 \vec{dl} \times \vec{a}_e) \]

A small "amuck" of current

\[ I \vec{dl} \rightarrow q \vec{v} \rightarrow J ds dl = J dv \rightarrow J dw \]

\[ \vec{B} = \frac{\mu_0}{4\pi} \int \frac{I \vec{dl} \times \vec{a}_e}{R^2} \]
\[ \frac{d}{dt} v(t) = -\mathbf{v}(t) \]

\[ -\mathbf{\nabla} \mathbf{v}(\mathbf{r}) = \mathbf{v} \times \left( \frac{\mathbf{E} \times \mathbf{B}}{R} \right) - \frac{1}{R} \mathbf{v} \times (\mathbf{E} \times \mathbf{B}) \]

\[ \mathbf{B} = \frac{k}{\mu_0} \int \left[ \mathbf{v} \times \left( \frac{\mathbf{E} \times \mathbf{B}}{R} \right) - \frac{1}{\mu_0} \mathbf{v} \times (\mathbf{E} \times \mathbf{B}) \right] \]

\[ = \mathbf{v} \times \frac{\mu_0}{\mu_0} \int \frac{\mathbf{E} \times \mathbf{B}}{R} = \mathbf{v} \times \mathbf{A} \]

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{E} \times \mathbf{B}}{R} \] - Magnetic Vector Potential

\[ \text{POTENTIAL} \]
\[ V = V_0 \int_{r_{01}}^{r_{02}} \frac{d}{dr} \left[ \frac{1}{r} \right] dr = \frac{\hbar^2}{8\pi} \int_{r_{01}}^{r_{02}} \frac{V(r)}{r} \, dr \]

\[ \left( \text{boundary: } g(r) \right) \]

\[ = \frac{\hbar^2}{8\pi} \int_{r_{01}}^{r_{02}} \left[ \frac{V(r)}{r} \right] \, dr \]

\[ = \frac{\hbar^2}{8\pi} \int_{r_{01}}^{r_{02}} \left[ -\mu_0 s(r) \right] \, dr \]

\[ = \frac{\hbar^2}{8\pi} \left[ \mu_0 J(r=0) \right] \]

\[ V \cdot \vec{B} = \mu_0 J (x, y, z) \]

\[ \nabla \times \vec{B} = \mu_0 J \left( \text{point} \right) \]
\[ \int_{V} \mathbf{E} \cdot d\mathbf{V} = 0 \quad \nabla \cdot \mathbf{B} = 0 \]

<table>
<thead>
<tr>
<th>Electric Fields (Integral)</th>
<th>Magnetic Fields (Differential)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \nabla \phi = \mathbf{E} ]</td>
<td>[ \oint \mathbf{E} \cdot d\mathbf{s} = q ]</td>
</tr>
<tr>
<td>[ \nabla \times \mathbf{E} = \mathbf{0} ]</td>
<td>[ \oint \mathbf{B} \cdot d\mathbf{s} = 0 ]</td>
</tr>
<tr>
<td>[ \nabla \phi = -\mathbf{V} ]</td>
<td>[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} ]</td>
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<tr>
<td>[ \mathbf{E} = \mathbf{F} + \mathbf{F} ]</td>
<td>[ \mathbf{B} = \mathbf{V} \times \mathbf{A} ]</td>
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<tr>
<td>[ \mathbf{B} = \mathbf{E} / \mu_0 ]</td>
<td>[ \mathbf{A} = \mathbf{A} / \mu_0 ]</td>
</tr>
</tbody>
</table>

\[ \mathbf{H} = \mu_0 \mathbf{H} \]
\[ \mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E} + \nabla \phi \]
\[ \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \]

\[ \nabla \mathbf{B} = 0 \]

or \[ \oint \mathbf{B} \cdot d\mathbf{S} = 0 \]

\[ \Rightarrow (\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{a}_n = 0 \]

\[ H_2 \ell_2 + H_1 \ell_1 + H_8 \ell_8 = NI \]

\[ \mu_0 H_8 = K_m \mu_0 H_2 \Rightarrow H_2 = \frac{H_8}{K_m} \]

\[ \Rightarrow H_8 (\frac{\ell_2}{K_m} + \ell_8) = NI \Rightarrow \mathbf{B} = \mu_0 H_8 = \frac{\mu_0 NI}{\ell_2/K_m + \ell_8} \]
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}
\]

\[
- \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H}
\]

\[
\Phi = \frac{1}{2} \int_{\Sigma} \left( \frac{\partial \phi}{\partial t} \right) \sqrt{\left| \left[ \mathbf{r} - \mathbf{a} \cos(\phi) \mathbf{e}_\phi \right] \right|^2} \, dA
\]

\[
\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 \quad \text{and} \quad \mathbf{E} = \epsilon_0 \nabla \Phi + \mathbf{E}_1
\]

\[
\mathbf{E}_1 = \frac{\partial \mathbf{B}}{\partial t}
\]

\[
\mathbf{D}_1 = \epsilon_0 \nabla \Phi + \mathbf{E}_1
\]

\[
\mathbf{H}_1 = \mathbf{B}_1 / \mu_0
\]
\[ \int_0^1 \int_0^1 (x+y) \, dx \, dy = 2 \]

\[ \bar{y} = \frac{1}{b_x} \left( \bar{a}_y - a_y \bar{a}_y \right) \]

\[ \frac{x-1}{\left( \frac{1}{16} \right)} = \frac{y}{\left( \frac{3}{16} \right)} = \frac{z}{\left( \frac{1}{16} \right)} \]

\[ x-1 = \frac{\frac{3}{16}}{\frac{1}{16}} = 3 \]

\[ \overline{dl} = dx \overline{a}_x + dy \overline{a}_y + dz \overline{a}_z \]

\[ \int \left[ x(a_x + x(y+3)a_y + yz \overline{a}_y) \right] (dx \overline{a}_x + dy \overline{a}_y + dz \overline{a}_z) \]

\[ y = 2 - 2z \quad dy = -2dz \]

\[ 3 = 1 - x \quad dz = -2dx \]

\[ ydx + z[(2-2x)(1-x)](-2dx) + (2-2x)(1-x)(-dz) \]

\[ (9x^2 - 2)dx = -\frac{7}{6} \]

**KURIER  ASSIGNMENT**

**Problem 26, 27, 28, 29**
\[ \int \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{B} \cdot d\mathbf{S} \]

\[ \frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta t} \left[ \int_{\Delta t} \mathbf{B}(t) \cdot d\mathbf{S} - \int_{\Delta t} \mathbf{B}(t-\Delta t) \cdot d\mathbf{S} \right] \]

Applying Gauss' Law over the surface at time \( t \):

\[ \left( \mathbf{B}_0 \cdot \mathbf{E}_0 + \int \mathbf{B}_0 \cdot d\mathbf{S} - \int \mathbf{B}_0 \cdot d\mathbf{S} \right) \cdot d\mathbf{S} = \int \mathbf{E} \cdot d\mathbf{S} \]

\[ \mathbf{d} \times d\mathbf{S} \]

\[ \int \mathbf{B}_0 \cdot d\mathbf{S} \cdot \mu_0 \mathbf{E} \cdot d\mathbf{S} \]
\[ \oint \mathbf{v} \cdot \mathbf{E} \text{ d}S + \int \mathbf{E} \cdot \nabla \phi \text{ d}V \quad (\mathbf{E} = \mu \mathbf{H}) \]
\[ - \oint \mathbf{H} \cdot \mathbf{d}S - \int \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{d}V \]
\[ = \mathbf{n} \cdot \left( \int \mathbf{v} \mathbf{E} \text{ d}S + \int \frac{\partial \mathbf{E}}{\partial t} \text{ d}V - \phi \mathbf{E} \right) \text{ d}V \]
\[ \frac{\partial}{\partial t} \mathbf{E} \cdot \mathbf{n} = \mathbf{n} \times \mathbf{H} = \mathbf{m} \times \mathbf{B} \]
\[ \mathbf{B} = \mu \mathbf{v} \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} - \mathbf{v} \times \mathbf{\mu} \times \mathbf{B} \]
\[ \phi \mathbf{E} \cdot \mathbf{n} = \frac{\partial}{\partial x} \left( \mathbf{v} \right) \cdot \mathbf{d}S \]
\[ = \int \left( \mu \mathbf{v} \mathbf{E} + \frac{\partial \mathbf{E}}{\partial t} - \mathbf{v} \times \mathbf{\mu} \times \mathbf{B} \right) \cdot \mathbf{d}S \]
\[ v = \frac{\sqrt{3} - \sqrt{2}}{b - b_0} \]

\[ V \cdot (F - \mu \times B) = -\frac{2\pi}{\epsilon_0} \]

\[ \mathbf{F} = \mathbf{F} \times \mathbf{B} \]

\[ E = F = \mu \times B \]
To find variation in a plane normal to direction of propagation,

\[ \nabla^2 E = \alpha_x \left( \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) + \alpha_y \left( \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) + \alpha_z \left( \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \]

\[ \text{PROPAGATION CONSTANT} \]

\[ 
E(t) = \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \theta) 
\]

\[ \mathbf{F} + \mathbf{A} \cdot \mathbf{E} = 0 \]

\[ \nabla^2 E = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \]

\[ E_0 = E_0(x, y, z, t) = E(x, y, z) e^{i \omega t} \]

\[ \Rightarrow \nabla^2 E + \omega^2 \varepsilon E = 0 \]

\[ \nabla^2 E + k^2 E = 0 \]

Assume only x component varying with z \( \Rightarrow \nabla^2 E = \frac{\partial^2 E_x}{\partial z^2} \)
Reflection from a Perfect Conductor

\[ \mathbf{E}_i = \mathbf{E}_r (\cos \theta \mathbf{e}_3 - \cos \theta \mathbf{e}_3) \]

\[ \mathbf{E}' = \mathbf{E}_r (\cos \theta' \mathbf{e}_3 + \cos \theta' \mathbf{e}_3) \]

\[ \mathbf{F} = x \mathbf{F} + \mathbf{e}_3 \]

(uniform plane wave)
\[ E = E_0 e^{i(k(-x \cos \theta + y \sin \theta) + z \sin \omega t - \omega t)} \]

\[ E_x = E_0 \cos \theta e^{i(k(-x \cos \theta + y \sin \theta) + z \sin \omega t)} \]
\[ + E'_0 \sin \theta e^{i(k(x \cos \theta + y \sin \theta))} \]

\[ E_y = E_0 \sin \theta e^{i(k(x \cos \theta + y \sin \theta))} \]
\[ - E'_0 \cos \theta e^{i(k(x \cos \theta + y \sin \theta))} \]

\[ H_y = \frac{E_0}{\eta} e^{i(k(x \cos \theta + y \sin \theta))} \]
\[ + \frac{E'_0}{\eta} e^{i(k(x \cos \theta + y \sin \theta))} \]

At \( x = 0 \), \( E_y = 0 \)

\[ 0 = E_0 \cos \theta e^{i(ky \sin \theta)} - E'_0 \cos \theta e^{i(ky \sin \theta)} \]
\[ \sin \theta = \sin \theta' \]
\[ \theta = \theta' \]
\[ \rightarrow E_0 = E'_0 \]

\[ k \approx \frac{n}{\mu} \]
\[ E_y = 2 E_0 \sin \theta \cos (k x \cos \theta) e^{-j k y \sin \theta} \]
\[ E_x = 2 j \mu E_0 \cos \theta \sin (k x \cos \theta) e^{-j k y \sin \theta} \]
\[ H_y = \frac{2}{\eta} E_0 \cos (k x \cos \theta) e^{-j k y \sin \theta} \]
\[ \nu_p = \frac{\omega}{k \sin \theta} = \frac{c}{\sin \theta} \quad \text{PHASE VELOCITY} \]

\[ E_0 = 0 \]

STANDING WAVE

TRAVELING WAVE

\[ k x \cos \theta = n \pi \]

\[ a = \frac{n \pi}{k \cos \theta} \]
\[ \cos \theta = \frac{n \pi}{ka} \]

\[ \sin \theta = \sqrt{1 - \left(\frac{n \pi}{ka}\right)^2} \]

\[ Z = \frac{\varepsilon_0}{\mu_0} \]

\[ k \sin \theta = \sqrt{k^2 - \left(\frac{n \pi}{a}\right)^2} = \beta \]

\[ E_x = \frac{2E_0}{\varepsilon} \cos \left(\frac{n \pi x}{a}\right) e^{-j \beta z} \]

\[ E_y = \frac{2jE_0}{\varepsilon} \frac{n \pi}{ka} \sin \left(\frac{n \pi x}{a}\right) e^{-j \beta z} \]

\[ H_y = \frac{2E_0}{\mu} \cos \left(\frac{n \pi x}{a}\right) e^{-j \beta z} \]

\[ \sigma = \infty \]

\[ \delta = \infty \]

---

ALL MAGNETIC VECTORS LIE IN PLANE TRANSVERSE TO PROPAGATION
ALTERNATE METHOD OF SOLUTION

\[ \frac{\partial^2 E_3}{\partial x^2} + (\kappa^2 - \beta^2) E_3 = 0 \quad \beta \approx \left[ E_3(\alpha) \right] + \delta(\alpha) \theta \]

Let \( \beta^2 = (\kappa^2 - \beta^2) \)

\[ E_3 = A \sin \lambda x + B \cos \lambda x \]

\[ E_3 = 0 \text{ at } x = 0: \]

\[ 0 = A(0) + B \quad \rightarrow \quad B = 0 \]

\[ E_3 = 0 \text{ at } x = a: \]

\[ 0 = A \sin \lambda a \quad \rightarrow \quad \lambda a = n \pi \]

\[ l = \frac{n \pi}{a} \]

\[ \theta = \sqrt{k^2 - \beta^2} = \sqrt{k^2 - (\kappa^2 - \beta^2)} \]
Let \( A = 2jE_0e^{j\beta z} \)

\[
\frac{2E_0}{R} \sin lx e^{-j\beta z}
\]

**From Handy-Dandy Ditto Sheet**

\[
\frac{1}{\pi} \frac{1}{x^2 + y^2} \left( j\alpha \frac{\partial E_z}{\partial x} \right) = -j\beta \frac{2jE_0}{R} \cos lx e^{-j\beta z}
\]

\[
\frac{1}{\pi} \frac{1}{x^2 + y^2} \left( -j\alpha \frac{\partial E_z}{\partial y} \right) = 0
\]

\[
\frac{1}{\pi} \frac{1}{x^2 + y^2} \left( j\omega e \frac{\partial E_z}{\partial x} \right) = 0
\]

\[
\frac{1}{\pi} \frac{1}{x^2 + y^2} \left( j\omega e \frac{\partial E_z}{\partial y} \right) = -j\omega \frac{2jE_0}{R} \cos lx e^{-j\beta z}
\]

\[
= \frac{2E_0}{R} \cos lx e^{-j\beta z}
\]
\[ z = \sqrt{\left( \frac{e}{n} \right)^{2} - \left( \frac{e}{n} \right)^{2} \sqrt{1 - \frac{e}{n}}} \]
\[ \beta^2 = \beta^2 - 1 = \omega^2 \mu \epsilon - \left( \frac{n \pi}{a} \right)^2 \]

\[ \beta^2 = 0 = \omega^2 \mu \epsilon - \left( \frac{n \pi}{a} \right)^2 \]

\[ \omega^n = \left( \frac{n \pi}{a} \right)^2 c^2 \quad w_c = \frac{n \pi c}{a} \]

\[ 2n f_c = \frac{n \pi f_c \lambda_c}{a} \]

\[ \lambda_c = \frac{2a}{n} \quad \text{or} \quad a = \frac{n}{2} \]

\[ \text{evanescent mode} \]
For a while, $\beta$ becomes imaginary,

$$\beta = \frac{1}{\alpha} \left[ 1 - \frac{1}{4} \left( \frac{\omega}{\alpha} \right)^2 \right]$$

$e^{-\lambda x} \cos(\omega t) = e^{-\frac{1}{2} x} \cos(\omega t)$

$$x^2 = \left( \frac{\omega}{\alpha} \right)^2 - \omega^2 \mu_0$$

$E_0 > 0$ would "like" to occur here

$E_0 < 0$ is forced to occur here

Induced $E$ field must oppose causing $\phi_0$
\[ J_0 = \sigma E_0 \]
\[ E = E_0 (xy) e^{-(\alpha + j\beta) \cdot z} \]
\[ J_s = \int_0^\infty J_0 e^{-(\alpha + j\beta) \cdot z} \, dz = \frac{J_0}{(\alpha + j\beta)} = \sigma E_0 \]
\[ P_{\text{loss}} = \frac{1}{2} \int_0^\infty E \cdot E^* \, dz = \frac{\sigma E_0^2}{\alpha} \]
\[ R_s = \frac{\alpha}{2} E_0 \]
\[ \text{Define } R_s \]
\[ P_{\text{loss}} = \frac{1}{2} \int_0^\infty E \cdot E^* \, dz = \frac{\sigma E_0^2}{4\alpha} \]
\[ R_s = \frac{\alpha}{2} E_0 \]
\[ E = E_0 e^{-\sigma z} e^{i(\omega t - k z)} \]

\[ k = \omega \sqrt{\mu \sigma (1 + \frac{\sigma}{j\omega \mu})} \]

\[ \sigma >> 1 \]

\[ \sigma = \sqrt{\frac{\mu \sigma}{j\omega \mu}} = \sqrt{\frac{\mu \sigma}{\omega}} (1 + j) \]

\[ -j k z = -\sqrt{\frac{\mu \sigma}{\omega}} (1 + j) z = -(\alpha + j\beta) z \]

\[ \sigma = \beta = \sqrt{\pi f \mu \sigma} \]

\[ \tau = \frac{1}{\sigma} = \text{"Time constant" for space variation} \]

\[ \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \text{Skin depth} \]

\[ E = E_0 e^{-\sigma z} e^{i(\omega t - k z)} \]
\[ r = \sqrt{\frac{\mu}{\sigma}} \]

\[ \sqrt{\frac{\mu}{\sigma}} (1 + i) \]

\[ \sigma = \sqrt{\frac{\mu}{\gamma}} \]
**Resistivity of Walls**

\[ R = r_x(y) e^{-x} e^{j(wt-\beta x)} dx \]

\[ \bar{R} = r_y(y) e^{-y} e^{j(wt-\beta y)} dy \]

**Power Transfer Past a Point**

\[ P_t = \frac{1}{2} R_0 \int \vec{E} \cdot \vec{n} ds \]

\[ = \frac{1}{2} \int \left[ \vec{E}_x, \vec{E}_y \right] e^{-2\times3} ds \]

**Power Loss Per Unit Length**

\[ \frac{dP_{\text{loss}}}{dy} \]

\[ \frac{dP_{\text{loss}}}{dy} = \frac{P_{\text{out}}(y) - P_{\text{out}}(y+\Delta y)}{\Delta y} = \frac{dP_{\text{out}}}{dy} \]

\[ \text{at} \quad \frac{P_{\text{loss}}}{2P_{\text{out}}} \]
\[ H_y = \frac{2E}{n} \cos \alpha e^{-j \beta z} \]

\[ E_y = \frac{2E}{n} \cos \alpha e^{-j \beta z} \]

\[ P_{\text{loss}} = \frac{1}{2} J^2 R_b (1) = \text{POWER LOSS PER UNIT LENGTH} \]

\[ L = 1 \text{ meter width} \]

\[ |H_0| = |H_y| \quad \hat{a}_n \times (\hat{r}_e - \hat{r}_i) = j \hat{z} \]

\[ P_{\text{loss}} = \left\{ \frac{j E}{n} \cos \alpha \right\}^2 \sqrt{\frac{\mu_0}{\pi}} e^{-2 \pi \beta z} \]

\[ Q = \frac{1}{2} \int \mu_0 H_y \, ds = \frac{1}{2} \int \frac{4 \pi E}{n} \cos^2 \alpha e^{-2 \pi \beta z} \, ds \]

\[ = \frac{E^2}{n} \coth \beta l \]
\[ C^n_P = \frac{n^2}{\gamma^2} e^{-2\chi^2} \]

\[ \frac{2e^3\theta}{n^k} e^{-2\chi^2} \]

\[ \frac{2k}{\beta} \sqrt{\frac{\omega}{2\pi^2}} \frac{1}{\sqrt{2}} \]

\[ \omega = \sqrt{\frac{2\omega_s}{c}} \]

\[ \frac{1}{\sqrt{1 - \left(\frac{A}{c}\right)^2}} \]
\[ E = \omega \sqrt{\frac{\mu}{\varepsilon}} (1 + \frac{3}{32\mu\varepsilon}) \]

\[ \mu + \varepsilon^2 \approx 1 + \frac{3}{2\mu\varepsilon} \]

\[ \eta = \omega \mu \varepsilon^2 (1 + \frac{3}{2\mu\varepsilon}) \]

\[ \eta^2 = \left( -\frac{\omega \mu \varepsilon^2}{2 \sqrt{\frac{\mu}{\varepsilon}}} \right)^2 = -\eta (\pm i \beta) \]}

\[ \omega = \frac{\varepsilon}{\mu} \quad \beta = \omega \mu \varepsilon^2 \]

\[ z \approx \sqrt{\frac{\mu}{\varepsilon}} \quad \approx \sqrt{\frac{\mu \varepsilon^2}{\varepsilon}} \]

\[ E = E_0 e^{-\frac{\sqrt{\mu}}{2\mu} r} e^{j \omega t - \frac{2}{3} \eta z} \]
\[ A = \frac{\mu I_d}{4\pi r} \int \frac{I_d dl}{R} = \frac{\mu I_d}{4\pi} \int e^{i(\omega t - kr)} \frac{dl}{l} \]

\[ A = \frac{\mu I_d d}{4\pi} e^{i(\omega t - kr)} \rightarrow \bar{A}_g = A_g \bar{A}_g \]

\[ A_r = A_g \omega \cos \theta \quad A_\theta = -A_g \sin \theta \]

\[ \vec{B} = \nabla \times \vec{A} \]

\[ B_\phi = \frac{1}{r} \frac{\partial}{\partial r} (r A_g) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \quad B_r = B_\theta = 0 \]

\[ B_\phi = \mu H_\phi = \frac{\mu I_d d}{4\pi} \omega \sin \theta \left[ \frac{1}{r^2} + \frac{k^2}{r^2} \right] e^{i(\omega t - kr)} \]

\[ \nabla \times \vec{H} = j \omega e \vec{E} \]

\[ \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\sin \theta H_\phi) - \delta_\varphi \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) = j \omega e (\varphi F_r + \delta_\varphi E_\phi) \]
\[ e = \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{r} \frac{1}{b} \cos(\omega t + \phi_b) \right] - \frac{I_0}{2\pi} \cos \theta \left[ \frac{1}{r} + \frac{1}{2} + \frac{1}{r^{1.5}} \right] e^{j\omega t} \]

\[ I_0 = \frac{I_0 e^{j\omega t}}{2\pi} \left[ \frac{1}{r^2} + \frac{2e^{j\omega t}}{r} \right] e^{j(\omega t - kr)} \]

\[ \phi_b \rightarrow \text{ELECTROSTATIC DIPOLE FIELD} \]

\[ \phi_s \rightarrow \text{INDUCTION (BIOT-SAVART) FIELD} \]

\[ \phi_r \rightarrow \text{RADIATION FIELD} \]

\[ \left| \phi_r \right| = \frac{\mu_0 I}{2\pi r} \quad \Rightarrow \quad r = \frac{1}{m} = \frac{\gamma}{2\pi} \]

\[ \phi_r = \frac{1}{2} \text{Re} (\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} \text{Re} \left[ (\mathbf{E}_r + \mathbf{E}_o \mathbf{E}_o) \times \mathbf{H}_o \right] \]

\[ = \frac{1}{2} \text{Re} \left[ -\mathbf{E}_o \mathbf{E}_o \mathbf{H}_o^* + \mathbf{E}_o \mathbf{E}_o \mathbf{H}_o \right] \]

\[ \text{PURE IMAGINARY} \]

\[ \phi_b \rightarrow \mathbf{E} = \mathbf{H} \left( \frac{\theta}{2\pi} - \frac{1}{r} \right) \left[ \frac{1}{r^2} - j \frac{1}{r} \right] = \frac{\mathbf{E}}{\mathbf{H}} \]
\[ P = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left( \frac{I_0 k}{4\pi r} \right)^2 \sin^2 \theta \, d\theta \, dr \int_{r_0}^{\infty} \frac{e^{-\eta r}}{r^2} \, dr \]

\[ P = \frac{2\pi I_0^2 (\frac{d}{N})^2}{3} = \frac{1}{2} I_0^2 R_{\text{rad}} \]

\[ R_{\text{rad}} = \frac{2\pi I_0^2 (\frac{d}{N})^2}{3} = 80\pi^2 \left( \frac{d}{N} \right)^2 \quad \text{(FREE SPACE) \quad \eta = 120\pi} \]

\[ \left[ \eta = 0.01, \quad R_{\text{rad}} = 0.08 \right] \]

**For large \( r \),**

\[ E_\theta = \frac{I_0 k d \sin \theta}{4\pi r} e^{j(\omega t - kr)} \]

\[ H_\phi = \frac{I_0 k d}{4\pi r} \sin \theta e^{j(\omega t - kr)} \]

Polar plot of \( |E_\theta| \)
\[ E_0 = \frac{I^2 \pm d_0}{4 \pi r} \cos \theta e^{j(\omega t - kr)} \]

**Magnitude:** \[ |E_0| = \frac{1}{r} \]

**Phase:** \[ \phi = \frac{\pi}{2} - \frac{\theta}{2} \]

\[ E_0 = n H_0 = \int_{\phi}^{\phi + \frac{2\pi}{3}} dE_0 \]

\[ H_0 = \frac{I a b}{4\pi r} \sin \theta e^{j(\omega t - kr)} \left[ \int_{\phi}^{\phi + \frac{2\pi}{3}} e^{j\frac{k}{r} \sin \theta} \sin k(r_1) dr_1 + \int_{\phi}^{\phi + \frac{2\pi}{3}} e^{j\frac{k}{r} \sin \theta} \sin k(r_2) dr_2 \right] \]

\[ \int_{0}^{\infty} \sin(bx+a) \, dx = \frac{e^{ax}}{a^2 + b^2} \left[ a \sin(bx+a) - b \cos(bx+a) \right] \]

\[ E_0 = \frac{ILn \, \sin \theta e^{j(\omega t - kr)}}{2\pi r} \left[ \sin \left( k_0 \cos \theta - k_r \right) \right] \]
\[ E_0 = \frac{1}{2\pi} \left[ \frac{\sin \left( \frac{\theta}{2} \cos \theta \right)}{\sin \theta} \right] e^{j(\omega t - kz)} \]

\[ R_p = \int_0^{2\pi} \left[ \frac{1}{2} \rho \left( \begin{array}{c} E_0^2 \\ 2\pi \end{array} \right) \right] \left[ \begin{array}{c} r^2 \sin \theta \cos \theta \sin \theta \cos \theta \end{array} \right] \sin \theta \cos \theta d\theta \]

\[ = \frac{2\pi}{3} \int_0^{\pi/2} \frac{E_0^2}{2\pi} r^2 \sin \theta \cos \theta d\theta = \frac{n I_m^2}{4\pi} \int_0^{\pi/2} \cos^2 \left( \frac{\theta}{2} \cos \theta \right) d\theta \]

\[ = \frac{1}{2} I_m^2 R_{\text{rad}} \]

\[ R_{\text{rad}} = \frac{\mu_0}{2\pi} (128) = 73.1 \Omega \quad \text{(FREE SPACE)} \]
Antenna Gain

\[
g = \frac{4 \pi R^2 |P|_{\text{max}}}{R}
\]

\[
|P|_{\text{max}} = \frac{E_{\text{om}}^2}{2n}
\]

\[
E_{\text{om}} = \frac{j I_m n}{2 \pi R_0} \quad [1]
\]

\[
P = \frac{1}{2} I_m^2 R_{\text{rod}}
\]

\[
g = \frac{4 \pi R^2 \left( \frac{I_m n}{2 \pi R_0} \right)^2 \frac{1}{2n}}{\frac{1}{2} I_m^2 \left[ \frac{n}{2 \pi} (422) \right]} = \frac{2}{1.22} = 1.64
\]

\[
g(\text{db}) = 10 \log_{10} 1.64 = 2.15 \text{ db}
\]
\[ E_0 = \tilde{a}_0 \cdot K_0(\theta) \cdot I_0 \cdot e^{-j\omega t} \]
\[ \hat{E}_1 = \tilde{a}_0 \cdot K_2(\theta) \cdot I_1 \cdot e^{-j\omega t} \]

For a dipole: \( K(\theta) = \frac{j\omega}{2\pi f} \left[ \frac{\cos(\frac{j\omega}{2} \sin \theta)}{\sin \theta} \right] \)

Suppose:
\[ I_0 = I_0 e^{j\theta} \quad I_1 = c_1 I_0 e^{j\phi} \]

\[ r = r - c \sin \theta \cos \phi \]

\[ \hat{E} = \tilde{a}_0 \cdot K(\theta) \cdot |I_0| \cdot e^{-j\omega t} \left[ 1 + c \cdot e^{j(\omega t + ka \sin \theta \cos \phi)} \right] \]

ANTENNA FACTOR

ARRAY FACTOR
\[ \mathbf{A} = \frac{\mu}{4\pi r_0} e^{i(ut-r_0)} \int \mathbf{I}_a e^{i\mathbf{a}_a \cdot \mathbf{r}} d\mathbf{a} \]

\[ N = \int \mathbf{I}_a e^{i\mathbf{a}_a \cdot \mathbf{r}} d\mathbf{a} = N_1 \mathbf{\hat{a}} + N_2 \mathbf{\hat{b}} + N_3 \mathbf{\hat{c}} \]

\[ C = \text{RADIATION VECTOR (FN. OF } \theta \text{ & } \phi \text{ ONLY, NOT } r \text{)} \]

\[ x\mathbf{A} = \frac{\mathbf{\hat{a}}}{E \mathbf{\hat{n}} \mathbf{\hat{a}}} \left[ \frac{2}{i} \left( R \frac{\partial (N_1 \mathbf{a}_1)}{\partial \theta} - \frac{\partial \mathbf{\hat{a}}}{\partial \theta} \right) + \frac{\partial \mathbf{\hat{a}}}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial R}{\partial \theta} - \frac{2}{\sin^2 \theta} (\mathbf{\hat{n}} \mathbf{\hat{a}}) \right] \right. \]

\[ \left. + \frac{\partial \mathbf{\hat{a}}}{\partial \phi} \left[ \frac{2}{i} \left( R \frac{\partial (N_3 \mathbf{a}_3)}{\partial \phi} \right) - \frac{\partial \mathbf{\hat{a}}}{\partial \phi} \right] \right] \]

For terms which will vary as \( 1/r \):

\[ \mathbf{B} = \nabla \times \mathbf{A} = \frac{\mathbf{\hat{a}}}{E} \frac{2}{i} (\mathbf{\hat{b}} R \mathbf{\hat{a}}) + \frac{\mathbf{\hat{b}}}{6} \frac{2}{i} (\mathbf{\hat{a}} R \mathbf{\hat{a}}) \]

\[ = \frac{\mu}{4\pi r_0} \left( i k e^{-i kr} N_2 \mathbf{\hat{a}} - i k e^{-i kr} N_3 \mathbf{\hat{c}} \right) e^{iut} \]

\[ \mathbf{C} = \frac{\mathbf{B}}{\mu} = \frac{i k}{4\pi r_0} e^{i(ut-r_0)} (N_2 \mathbf{\hat{a}} - N_3 \mathbf{\hat{c}}) \]
\[ E = -\frac{\partial A}{\partial t} + \mathbf{V} \times \mathbf{A} \]

After a laborious expansion and investigation, we obtain:

\[ E = \frac{\mu_0 I}{4\pi r} \phi (N_0 \hat{\mathbf{a}}_x + N_0 \hat{\mathbf{a}}_y) \left\{ \text{for } \frac{1}{c} \text{ terms} \right\} \]

\[ \hat{\mathbf{P}}_{\text{magnetic}} = \frac{\mu_0}{8\pi r^2} \left[ |N_0|^2 + |N_0'|^2 \right] \hat{\mathbf{a}}_r \]

\[ \mathbf{P} = \frac{\mu_0}{8\pi} \int \left[ |N_0|^2 + |N_0'|^2 \right] \sin \theta \, d\theta \, d\phi \]

Recall that from:

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \]

we obtained \( \mathbf{J}_s = \hat{\mathbf{a}}_n \times \mathbf{H} \).

If:

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{D}}{\partial t}, \]

then \( \mathbf{J}_{\text{magnetic}} = -\hat{\mathbf{a}}_n \times \mathbf{E} \)

Magnetic current density

And:

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_s \, d\mathbf{A}}{R} \, e^{j(wt-kR)} \]

\[ \mathbf{F} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_{\text{magnetic}} \, d\mathbf{A}}{R} \, e^{j(wt-kR)} \]

"Electric vector potential"
WE MAY ALSO OBTAIN:

\[
\vec{E} = \frac{\hbar}{\kappa^2} \nabla (\nabla \cdot \vec{A}) - j \omega \vec{A} - \frac{i}{\epsilon} \nabla \times \vec{F}
\]

\[
\vec{H} = \frac{\hbar}{\kappa^2} \nabla (\nabla \cdot \vec{F}) - j \omega \vec{F} + \frac{1}{\mu} \nabla \times \vec{A}
\]

AND FOR \( \frac{1}{\kappa} \) RADIATION FIELDS

\[
\vec{E} = \frac{E}{4 \pi \kappa} e^{j (\omega t - k \vec{r})} \int I_m e^{j \kappa \mu \cos \theta} d\vec{l} =
\]

\[
\vec{H} = \int I_m e^{j \kappa \mu \cos \theta} d\vec{l} = L_r \hat{a}_r + L_\theta \hat{a}_\theta + L_\phi \hat{a}_\phi
\]

\[= \text{magnetic radiation vector}\]

THE FIELDS BECOME:

\[
\vec{E} = \frac{\hbar}{2 \kappa \epsilon_0} e^{j (\omega t - k \vec{r})} \left[ (\kappa N_\theta + L_\phi) \hat{a}_\theta + (\kappa' N_\phi - L_\theta) \hat{a}_\phi \right]
\]

\[
\vec{H} = \frac{1}{\kappa} \left[ E_\theta \hat{a}_\phi - E_\phi \hat{a}_\theta \right]
\]

\[
\vec{D} = \vec{D}\text{avg} = \frac{n_0 L_\phi}{8 \pi \kappa^2} \left[ \left| N_\theta + \frac{L_\phi}{n_0} \right|^2 + \left| N_\phi - \frac{L_\theta}{n_0} \right|^2 \right] \hat{a}_r
\]
WAVE PROPAGATING IN $z$-DIRECTION

$E(t, x, z) = e^{j(\omega t - k z)}$

$A = \omega \sqrt{\mu \varepsilon}$

$\nabla \times \vec{H} = j \omega \varepsilon \varepsilon \vec{E}$

$\nabla \times \vec{H} = -j \omega \mu \vec{H}$

$\frac{\partial H_y}{\partial z} + j \beta H_y = j \omega \varepsilon \varepsilon E_x$

$\frac{\partial H_x}{\partial z} - j \beta H_x - \frac{\partial H_z}{\partial x} = j \omega \varepsilon \varepsilon E_y$

$\frac{\partial H_z}{\partial x} - \frac{\partial H_y}{\partial y} = j \omega \varepsilon \varepsilon E_z$

$\begin{aligned}
\alpha &= \frac{-1}{k^2 - \beta^2} \left( j \beta \frac{\partial E_x}{\partial x} + j \omega \mu \frac{\partial H_y}{\partial y} \right) \\
\beta &= \frac{1}{k^2 - \beta^2} \left( -j \beta \frac{\partial E_y}{\partial y} + j \omega \mu \frac{\partial H_x}{\partial x} \right) \\
\gamma x &= \frac{1}{k^2 - \beta^2} \left( j \omega \varepsilon \varepsilon E_y - j \beta \frac{\partial H_z}{\partial x} \right) \\
\gamma y &= \frac{-1}{k^2 - \beta^2} \left( j \omega \varepsilon \varepsilon E_x + j \beta \frac{\partial H_y}{\partial y} \right) \\
\nabla^2 E + \frac{\partial E_x}{\partial t} = \nabla^2 E - \beta^2 E \\
\frac{\partial E_y}{\partial t} = \nabla^2 E + \left( k^2 - \beta^2 \right) E = 0
\end{aligned}$
"The amount of noise which anyone can bear undisturbed stands in inverse proportion to his mental capacity and may therefore be regarded as a fair measure of it."

—Arthur Schopenhauer
19th Century German Philosopher

A parallel plate capacitor with an inhomogeneous dielectric is shown. Determine (Bond's short)

(a) the polarization charge density in the dielectric,

(b) the surface polarization charge density at \( x = d \),

(c) the capacitance.

For the system shown, determine the self inductance. (Assume the toroid \( \mathbf{H} = \mu \mathbf{J} \) to be complete...only a section is illustrated.)

\[
\psi = \oint_S \mathbf{B} \cdot d\mathbf{s} = \oint_S (B_\phi) (dS_\phi) = \frac{\mu NI \overline{a}_\phi}{\pi \overline{(b^2 - a^2)} ab} \Rightarrow L = \frac{\psi}{\mathbf{J}} = \frac{\mu N ab}{\pi \overline{(b^2 - a^2)}}
\]
The toroid shown has a circular cross section of 2 cm radius and the B-H curve indicated on the page attached. Determine

(a) The magnetic flux density in the gap,

(b) The dipole moment per unit volume in the iron, \( M \).

\[
\phi_H \cdot dA = NI \quad H_1 + H_2 = NI \quad \Rightarrow \quad H_2 = NI - H_1 \quad \Rightarrow \quad H_2 = \frac{(700) - N_1(2\pi(.15) - 002)}{.002} \quad (over) \quad (To \ Back)
\]

Determine the reading of an ideal voltmeter attached to the terminals.

\[
V_{\text{IND}} = \int_{\Gamma} \frac{\phi}{\mu_0} \cdot \vec{B} = \int_{\Gamma} \vec{v} \cdot \vec{B} \cdot d\ell
\]

\[
\vec{v} = (\omega a) \vec{a}_\phi
\]

\[
d\vec{I} = a d\phi \vec{a}_\phi
\]

\[
\Rightarrow \quad \vec{v} = \phi_c (\omega a \vec{a}_\phi) \times (B_0 \vec{a}_z) (a d\phi \vec{a}_\phi) = \phi_c (\omega a B_0 \vec{a}_\phi \cdot a d\phi \vec{a}_\phi) = 0
\]

\[
\vec{M} = M_0 (a-r) \vec{a}_z
\]

For the system shown, determine

(a) all of the equivalent magnetic charges, and (back)

(b) the equivalent electric currents.

In the system shown, a perfectly conducting rectangular loop is rotating on its axis in the field of a DC current \( I \). Determine the voltage induced in the loop.

\[
V_{\text{IND}} = \int_{\Gamma} \frac{\phi}{\mu_0} \cdot dS + \phi_c \vec{v} \times \vec{B} \cdot d\ell
\]

\[
\vec{E} = \frac{q \vec{v} \times \vec{B}}{\gamma} \quad \vec{E} = \vec{E}_q = \frac{\vec{v} \times \vec{B}}{\gamma} = (\omega a) \vec{a}_\phi \times \vec{B} = \frac{\gamma}{\sigma}
\]
$B \ (\text{webers} / \text{m}^2)$
1) \( x=d \), \( e=E \), \( A=\text{AREA} \), \( \rho=\frac{\rho}{\text{D}(x+d)} \), \( V=V_0 \), \( (E, x+d, E) = \nabla x P \)

2) \( D=E \), \( \rho=\nabla P = \nabla V_0 \), \( (E, x+d, E) = \nabla x P \)

3) \( \rho D \), \( \rho \nabla x = 0 \), \( \rho D = 0 \)

4) \( \rho E = 0 \)

5) \( \rho E = 0 \)

b) \( R = \frac{\rho E}{E} = \frac{\int \rho E \cdot dE}{\int \rho E \cdot ds} \)

\( C = \frac{\rho}{V} = \frac{\int \rho E \cdot ds}{V} \)

\( C = \frac{\rho}{V} \)

\( V_0 \)
The toroid shown has a circular cross section of 2 cm radius and the B-H curve indicated on the page attached. Determine

(a) The magnetic flux density in the gap.

(b) The dipole moment per unit volume in the iron, \( M \).

\[ H = \frac{N I}{A} \]
\[ B = \mu_0 H + \mu H_i \]

Determine the reading of an ideal voltmeter attached to the terminals.

\[ V = \int \mathbf{E} \cdot d \mathbf{l} = \frac{d}{d t} \int \mathbf{B} \cdot d \mathbf{S} + \oint \mathbf{E} \cdot d \mathbf{S} \]

For the system shown, determine

(a) all of the equivalent magnetic charges, and

(b) the equivalent electric currents.

In the system shown, a perfectly conducting rectangular loop is rotating on its axis in the field of a DC current \( I \). Determine the voltage induced in the loop.

\[ V_{ind} = \int \mathbf{E} \cdot d \mathbf{l} = \oint \mathbf{E} \cdot d \mathbf{S} = \omega q \mathbf{a} \times \mathbf{B} = \frac{d}{d t} \mathbf{A} = \mathbf{E} = \mathbf{B} \times \mathbf{B} = (\omega q \mathbf{a} \times \mathbf{B}) = \frac{d}{d t} \mathbf{A} \]
\[ \mu_0 H_\parallel = \mu H_1 \]
\[ H_\parallel = \frac{700 - H_1 (3)}{2 \times 10^{-3}} \]
\[ \mu H_1 = \frac{700 - H_1 (3) \pi}{2 \times 10^{-3}} \]
\[ B = \frac{700 - H_1 (94)}{157} \times 10^{-4} \]
\[ = \frac{700}{157} \times 10^{-4} - \frac{(94)}{157} H_1 \times 10^{-4} \]
\[ = 4.46 - 6 \times 10^{-9} H_1 \quad \text{(Why?)} \]

\[ 5a) \quad A = \frac{1}{4 \pi} \int_V \frac{J(x',y';z')}{R} \, dV \]

\[ \bar{J} = \frac{2M}{2 \pi a^2 L} \int_V \frac{\mu_0 (a-r) z^2 a_z}{a_z} \, dV \]

\[ = \frac{\mu_0 M_0 a_z}{2 \pi a^2 L} \int_V (a-r) z^2 r dr d\phi dz \]

\[ = \frac{\mu_0 M_0 a_z}{2 \pi a^2 L} \int_0^a \int_0^{2\pi} \int_0^Q (a-r^2) r^2 dr d\phi dz \]

\[ b) \quad J = \frac{2M}{\pi a^2 L} = \frac{2M_0 (a-r) z^2 a_z}{\pi a^2 L} \]

\[ I = \int_S J \cdot ds = \int_S r d\phi dr \]

\[ I = \int_0^a \int_0^{2\pi} 2M_0 (a-r) z^2 \frac{r d\phi dr}{\pi a^2 L} \]
INHOMOGENEOUS (GIVEN) WAVE EQ.
\[ \nabla^2 \psi(x,t) - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -4\pi f(x,t) \]

VECTOR GENERALIZATION
\[ \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -4\pi \vec{J}(x,t) \]

IN CARTESIAN COORDINATES, MAY BREAK UP
\[ \nabla^2 E_x - \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} = -4\pi \int \delta(x-x') \, f(x',t) \, dx' \]

CONCENTRATE ON ONE DIMENSION:
\[ \nabla^2 \psi(x,t) - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -4\pi f(x,t) \]

WILL FIND GENERAL SOLUTION USING GREEN'S FUNCTION.

DEFINE:
\[ \Box_x = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \Box \text{D'ALAMBERTIAN} \]
(NAMED AFTER D'ALAMBERT)

THEN
\[ \Box_x \psi(x,t) = -4\pi \int \delta(x-x') f(x',t) \, dx' \quad (1) \]

\[ \Box_x \psi(x,t) = -4\pi \int \delta(x-x') \psi(x',t) \, dx' \quad \text{IS HOMOGENEOUS} \]

CONSIDER
\[ \Box_x G(x,t; x',t') = -4\pi \delta^3(x-x') \delta(t-t') \quad (2) \]

IN A HOMOGENEOUS ISOTROPIC MEDIA:
\[ G(x,t; x',t') = G(x-x',t-t) \]

TO SEE HOW \( G \) IS USEFUL, TAKE
\[ \int_{x_1}^{x_2} \int_{t_1}^{t_2} d^3x' \, [G(x,t; x',t) - \psi(x,t)] \times \psi(x,t') \frac{d^3x'}{dt'} \]
\[ \Rightarrow \int_{x_1}^{x_2} \int_{t_1}^{t_2} d^3x' \left[ G \Box_x \psi - \psi \Box_x G \right] \]
\[ = -4\pi \int d^3x' \, G f + 4\pi \int d^3x' \, \psi(\vec{x}', t') \]
\[ \times \delta^3(x-x') \delta(t-t') \]
WE HAVE YET TO SPECIFY BOUNDARY CONDITIONS WHICH WILL SIMPLIFY.

NOW TO FIND $G$.

RECALL

$$
\Box_x' G(x', t') = -\frac{\hbar^2}{2\mu} \delta^3(x-x') \delta(t-t')
$$

$$
\Rightarrow \Box_x' G(x', t') = -\frac{\hbar^2}{2\mu} \delta^3(x-x') \delta(t-t')
$$

$$
= -\frac{\hbar^2}{(2\pi)^4} \int d\omega \int d^4k \ e^{-ik\cdot(x'-x)}
$$

FROM $\delta(t') = \frac{1}{2\pi} \int d\omega \ e^{-i\omega t'} dt'$

AND $\delta(x') = \frac{1}{2\pi} \int d\eta_x \ e^{in\cdotx'} dx'$

$$
\Rightarrow \delta(x') \delta(y') dx' = \int d\eta_x \int d\eta_y \int d\eta_z \ e^{i(n_1 x_1 + n_2 y_1 + n_3 z_1)}
$$

$$
= \int d^3k \ e^{-ik\cdotz}
$$

DEFINE $g(k, \omega) = \int d\omega \int d^3k \ e^{-ik\cdotz}

G(x', t') = \frac{1}{(2\pi)^4} \int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}

L E T $k = |k|$

$$
\Box_x G(k, x) = -\frac{\hbar^2}{2\mu} \delta^3(x-x) \nabla_x (\nabla^2 G(k, x))
$$

$$
\nabla_x G(k, x) = -k^2 G(k, x) \Rightarrow k^2 = k^2 + k^2
$$

$$
\Box_x' \int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$

$$
\int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$

$$
\frac{1}{(2\pi)^3} \int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$

$$
\int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$

$$
\frac{1}{(2\pi)^3} \int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$

$$
\frac{1}{(2\pi)^4} \int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$

$$
\frac{1}{(2\pi)^3} \int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$

$$
\frac{1}{(2\pi)^4} \int d\omega \int d^3k \ e^{i(k\cdot(x'-x) - \omega t')}
$$
For $t' < 0$, $G_R = 0$. For $t > 0$, most evaluate residues, then

$$G_R(x', t') = \frac{-e^{-i\omega t'}}{4\pi^2} \int_0^\infty \frac{dk}{k} e^{ikx'} \left[ \frac{e^{-i\omega t'}}{\omega - ck} + \frac{e^{-i\omega t'}}{\omega + ck} \right]$$

$$= \frac{-ic^2}{2\pi^2} \int_0^\infty d^3 k \frac{e^{ikx'}}{ck} (\omega - 2ck) (\omega + 2ck)$$

$$= \frac{c}{2\pi^2} \int_0^\infty \frac{d^3 k e^{ikx'}}{ck} \frac{\omega - ck}{\omega + ck}$$

To evaluate this, use spherical coordinates in $k$ space. Choose it such that the polar axis lies along $x'$.

Thus

$$G_R = \frac{c}{2\pi^2} \int_0^\infty k^2 dk \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi e^{ik|x'|} \cos \frac{d\phi}{\sqrt{x'^2 - k^2}}$$

$$= \frac{c}{\pi} \int_0^\infty k^2 dk \frac{e^{ik|x'|} \cos x'}{\sqrt{x'^2 - k^2}} \left[ \frac{1}{\sqrt{x'^2 - k^2}} \right]$$

$$= \frac{c}{\pi} \int_0^\infty \frac{k^2 dk}{\sqrt{x'^2 - k^2}} e^{ik|x'|} \frac{\cos x'}{\sqrt{x'^2 - k^2}}$$
Plugging back in gives

\[
\psi(x, t) = \int_V d^3x' \int dt' \frac{\frac{\partial}{\partial t} \psi(x', t')}{|x - x'|} + \frac{1}{4\pi c} \int_V d^3x' \left[ G \frac{\partial}{\partial t} - \psi \frac{\partial G}{\partial t} \right] \frac{t - \frac{|x - x'|}{c}}{t' - \frac{|x - x'|}{c}} + \text{other terms}
\]

where

\[
t'_{\text{RET}} = t - \frac{|x - x'|}{c}
\]

\(t'_{\text{RET}}\) is the time the radiation takes to get from the source to observer.

Reviewed Green's Functions is kind of equivalent to Huygen's principle.

From our result, we may derive Kirchoff's formula for scalar optics. Assume

\[
\int = 0 \quad (\omega, \text{ no sources})
\]

\[
\frac{\partial \psi}{\partial t}, \psi \rightarrow 0 \quad \text{as } t \rightarrow 0
\]

So we are left with only the third term.
DIGRESSION:

IF, IN

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{1}{c^2} \int \frac{\partial f(x, \omega)}{\partial \omega} e^{-i\omega t} \, d\omega$$

WE ASSUME

$f(x, \omega) = \int f(x, \omega) e^{-i\omega t} \, d\omega$,

THEN WE GET

$$\nabla^2 \tilde{\psi}(x, \omega) + (k^2 - \frac{1}{c^2}) \tilde{\psi}(x, \omega) = \frac{1}{c^2} \int \frac{\partial f(x, \omega)}{\partial \omega} e^{-i\omega t} \, d\omega$$

WHICH IS THE INHOMOGENEOUS FORM OF HELMHOLTZ'S EQUATION.

THE GREENS FUNCTION FOR THIS IS Merely THE FOURIER TRANSFORM OF THE GREENS FUNCTION OF EQ. 1: $G_R = \frac{1}{R} e^{-iKR}$. EQ. 2 IS THE FOURIER TRANSFORM OF EQ. 1.

DIVIDE UP S

LET $S = S_1 + S_2$

SOURCE

$S_2$

AT INFINITY, $\psi$ BEHAVES AS

$\psi(0, \omega) \sim \frac{1}{\omega}$

AND $\nabla \psi$ AS

$\psi(1 - \frac{4\pi}{kR})$

AS $S_2 \to \infty$, THE SURFACE INTEGRAL OVER $S_2$ VANISHES AND

$$\psi(x) = \frac{1}{4\pi} \int_{S_1} \frac{e^{iKR}}{R} \cdot \left[ \nabla \tilde{\psi} + i (1 + \frac{4\pi x}{kR}) \tilde{\psi} \right] dS$$
Plug 'em in $\Psi$:

\[ \psi(x) = \frac{1}{4\pi} \int_{S_1} \frac{e^{ikr}}{r} \left[ 1 - \frac{\hat{n} \cdot (x' - x)}{r} \right] e^{ik \hat{n} \cdot x'} \cdot \hat{n} \cdot \left[ \nabla \phi + k \left( 1 + \frac{\hat{n} \cdot (x' - x)}{r} \right) \right] \frac{R_1}{R} \psi ds' \]

$\hat{n} \cdot (x' - x) \to \text{NO}\Rightarrow$ \text{REJECT}

$\phi \to \text{NAKED}$

$\hat{n} \cdot (x' - x) \to \text{NOE}$

We will look at field

only near $z$ axis

gives

\[ \psi(x) = \frac{e^{ikr}}{4\pi r} \int_{S_1} e^{i(k_x x' + k_y y')} \cdot \left[ \frac{\partial \phi}{\partial x'} + iky \right] dx'dy' \]

where

$k_x = k \hat{n}_x \text{in direction}$

$k_y = k \hat{n}_y \text{in direction}$

Support incident wave is a

quasi plane wave:

\[ \psi \text{ incident} = f(x, y) e^{ik_z z} \approx f(x, y) e^{ik_z \zeta} \]

plug in $\psi$:

\[ \psi(x') = e^{iky} \int_{A} e^{-i(k_x x' + k_y y')} \phi(x', y') dx'dy' \]

The far field is thus the two-dimensional Fourier transform of the aperture.

May similarly use lens:

\[ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \]
\[ \nabla \times E = -\frac{1}{c^2} \frac{\partial B}{\partial t} \quad \text{(FARADAY's LAW)} \implies \text{A MAXWELL EQ., IN GAUSSIAN CGS} \]

**EQUIVALENTLY**

\[ \nabla \times E = -\mu_0 \nabla \phi \]

Or,

\[ \nabla \times \left[ E + \frac{1}{c^2} \frac{\partial A}{\partial t} \right] = 0 \]

**THUS**

\[ E + \frac{1}{c^2} \frac{\partial A}{\partial t} = -\nabla \phi \]

\[ \therefore B = \nabla \times A \]

\[ E = -\nabla \phi - \frac{1}{c^2} \frac{\partial A}{\partial t} \quad \text{(L \& F. FUNDAMENTAL)} \]

**NOTE:** **NEED 4 SCALAR FUNCTIONS TO SPECIFY** \[ B \neq \frac{1}{c} \frac{\partial A}{\partial t} \]

**IN GAUSSIAN CGS UNITS IN FREE SPACE**

\[ B = \frac{\mu_0}{c} H; \quad D = E \]

\[ \nabla \times B = \frac{1}{c} \frac{dE}{dt} + \frac{4\pi}{c} J \]

\[ \nabla \times \nabla \times B = \frac{1}{c^2} \left( -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \right) + \frac{4\pi}{c} J \]

**IDENTITY**

\[ \nabla \left( \nabla \cdot B \right) = \nabla \times \nabla \times B = \nabla \phi + \frac{1}{c^2} \frac{\partial E}{\partial t} = \frac{4\pi}{c} J \]

**OR**

\[ \nabla \phi + \nabla \cdot \nabla \times B = \frac{4\pi}{c} J \]

**RECALL:** \[ \nabla \cdot D = \nabla \cdot \mathbf{E} = 0 \]

**SINCE** \[ E = D, \quad \nabla \cdot E = 4\pi \rho \]

**THEN**

\[ \nabla \cdot \left( -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \right) = 4\pi \rho \]

\[ = \nabla \cdot \phi + \frac{1}{c^2} \frac{\partial E}{\partial t} = \phi \left( \frac{1}{c^2} \frac{\partial A}{\partial t} + \nabla \phi \right) \]

\[ \nabla \cdot \phi - \frac{1}{c^2} \frac{\partial A}{\partial t} = 4\pi \rho \]
WE HAD
\[ \nabla \phi \cdot \mathbf{A} + \frac{\nabla \times \mathbf{A}}{c} \times \mathbf{E} = \frac{4\pi}{c^2} \mathbf{J} \]
\[ \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = \frac{4\pi}{2} \right) \]
\[ \nabla \phi \cdot \mathbf{A} + \frac{\nabla \times \mathbf{A}}{c} \times \mathbf{E} = \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = \frac{4\pi}{2} \right) \]

LET'S GO TO A NEW GAUGE, WANNA GET RID OF \( \chi \):
\[ \nabla \mathbf{A}' + \nabla \chi' = \frac{4\pi}{c^2} \mathbf{J} \]
\[ \mathbf{A}' = \mathbf{A} + \nabla \chi, \quad \phi' = \phi - \frac{1}{c^2} \frac{\nabla \chi}{c} \]
\[ \nabla \phi' \cdot \mathbf{A}' + \frac{\nabla \times \mathbf{A}'}{c} \times \mathbf{E} = \nabla \cdot \mathbf{D}, \quad \chi' = \nabla \mathbf{A}' + \frac{1}{c^2} \frac{\nabla \chi}{c} \]

LET'S TRY TO CHOOSE \( \chi \) TO MAKE \( \chi' = 0 \)
\[ \chi' = \nabla \mathbf{A}' + \frac{1}{c^2} \frac{\nabla \chi}{c} = \nabla \left( \mathbf{A} + \nabla \chi \right) + \frac{1}{c^2} \left( \phi - \frac{1}{c^2} \nabla \chi \right) \]
\[ = \chi + \nabla \chi \]

IF WE CHOOSE \( \chi + \nabla \chi = -\chi \), THEN \( \chi' = 0 \).

THAT IS
\[ \nabla^2 \chi + \frac{\nabla \chi}{c^2} = \frac{4\pi}{c^2} \mathbf{J} \]

LORENZ

THIS IS AN INHOMOGENEOUS (DRIVEN) WAVE EQUATION. USUALLY, THERE'S NO SOLUTION MAKING THIS SOLUTION GIVES

\[ \begin{cases} \nabla \phi = -\frac{4\pi}{c^2} \mathbf{J} \\ \nabla \phi = -\frac{4\pi}{c^2} \mathbf{J} \end{cases} \]

EQUIVALENT TO MAXWELL'S EQUATION.

\( \mathbf{J} \) DRIVES \( \mathbf{A} \), \( \phi \) DRIVES \( \mathbf{B} \)

FOR DRIVEN WAVE EQUATIONS

EX: SINGLE CHARGED PARTICLE
\[ \mathbf{J}(x,t) = q \delta(x - \mathbf{v} \cdot t) \]
\[ \mathbf{J}(x,t) = q \delta(x - \mathbf{v} \cdot t) \]
2-3-75 (TUES)

OR BY ZEWONK

SMTHE - "STATIC & DYNAMIC ELECTRICITY"
SUPPLEMENTARY: JACKSON "CLASSICAL ELECTRODYNAMICS"

GRADING:
HOMEWORK - 50 POINTS
TEST - 20 POINTS
FINAL - 30 POINTS

A 100-90 B 90-80 C 80-70

COULOMB'S LAW

\[ \mathbf{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \]

\( C = \text{CAPACITYITY} \)

\[ E_{\text{vacuum}} = 8.85 \times 10^{-12} \text{ Nm}^2/\text{C} \]

\[ K = \frac{E}{E_y} \]

\[ q \rightarrow \mathbf{E} \left( \mathbf{E} \cdot \mathbf{r} \right) = \mathbf{E} \frac{q}{r} \]

\[ E_p = -\frac{1}{4\pi\varepsilon_0} \sum \frac{q_i}{r_i^2} \rightarrow \text{NOTE: NO TEST CHARGE} \]

POTENTIAL

\[ \mathbf{dV} = -E dS \]

\[ V_p - V_0 = \int V \mathbf{dV} \]

\[ = \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{r_p} - \frac{1}{r_0} \right) \]

AS \( r_0 \rightarrow \infty \), \( V_p \rightarrow \frac{q}{4\pi\varepsilon_0 r_p} \)

\[ \text{NOTE THAT RESULT IS IND. OF PATH} \]

\[ \oint \mathbf{E} \cdot d\mathbf{S} = 0 \]
\[ d\mathbf{S} = \lambda E \]

\[
\frac{dx}{Ex} = \frac{dy}{Ey} = \frac{dz}{Ez}
\]

\[
E_x = \frac{q}{4\pi \epsilon_0} \left[ \frac{x-a}{\sqrt{y^2 + (x-a)^2}} \pm \frac{x+a}{\sqrt{y^2 + (x+a)^2}} \right]
\]

\[
E_y = \frac{q}{4\pi \epsilon_0} \left[ \frac{y}{\sqrt{y^2 + (x-a)^2}} \pm \frac{y}{\sqrt{y^2 + (x+a)^2}} \right]
\]

\[
\frac{dy}{dx} = \frac{E_y}{E_x}
\]

**Solve this differential eq.**

\[
U = \frac{x+a}{y}, \quad V = \frac{x-a}{y}
\]

**Gives**

\[
(x+a) \left[ (x+a)^2 + y^2 \right]^{-\frac{1}{2}} \pm (x-a) \left[ (x-a)^2 + y^2 \right]^{-\frac{1}{2}} = C
\]

**Different C's will give different lines of the E field.**

**Homework:**

Find this same eq, using Gauss' theorem
GAUSS' ELECTRIC FLUX THEOREM

\[ dN = \varepsilon E_n \, dS = \text{flux out of } ds \text{ PERPENDICULAR} \]

\[ E_n = \frac{q \delta}{4\pi \epsilon_0 r^2} \]

Then

\[ dN = \frac{q \delta \, dS}{4\pi r^2} = \frac{q \, dS}{4\pi} \]

INTEGRATE OVER WHOLE SURFACE

\[ 4\pi \oint dN = q \int d\Phi \]

\[ N = \oint dN = q \]

\(-\) CAUSS FLUX THEOREM FOR NO CHARGE, NO FLOW

WE MAY EQUVALENTLY WRITE

\[ \oint \hat{n} \cdot \vec{E} \, dS = q \]

\[ = \oint \hat{n} \cdot \vec{D} \, dS = q \]

\[ \Rightarrow \vec{D} = \varepsilon \vec{E} \]

\[ \begin{align*}
  [E] &= \text{V/m} \\
  [D] &= \text{Coul/m}^2
\end{align*} \]
2-5-75 (THURS)

Doubling changes does not change lines.
Potential on line, though, will be doubled.

Generalize

Doubling $q_1, q_2, q_3$ will double $V \propto P(q)$

$$\Rightarrow \frac{Q}{V} = \text{CONST} = \text{CAPACITANCE}$$

$$C = \frac{Q}{V}$$

$\equiv$ Capacitor

$$C = \frac{Q}{V_1 - V_2}.$$ 

$$C = \sum_{i=1}^{n} C_i \ (\text{PAR}) \quad \frac{1}{C} = \sum_{i=1}^{n} \frac{1}{C_i} \ (\text{SERIES})$$
CYLINDRICAL CAPACITOR

\[ l \gg b ; \quad a < r < b \]

\[ \oint \mathbf{D} \cdot d\mathbf{s} = Q \]

\[ \Rightarrow r\mathbf{E} \cdot d\mathbf{E} = Q \]

\[ E = -\frac{Q}{\varepsilon_0} \]

\[ V_a - V_b = \frac{Q}{2\pi\varepsilon_0} \ln \frac{b}{a} \]

\[ \sigma = \text{Area Charge Density} = 2\pi b \ell \]

\[ \Rightarrow V_a - V_b = \frac{Q}{2\pi\varepsilon_0} \ln \frac{b}{a} \]

\[ C = \frac{2\pi\varepsilon_0}{\ln (b/a)} \]

BACK TO SPHERICAL CAPACITOR, LET \( r(b) \to \infty \)

\[ \Rightarrow V = \frac{Q}{\varepsilon_0} \]

\[ \frac{r}{Q} \]

\[ V = \frac{Q}{4\pi\varepsilon_0 r} \]

\[ \text{NOTE: CANNOT DO THIS FOR CYLINDER} \]

\[ \text{SINCE WE RESTRICTED} \quad l \gg b \]
Look at energy

\[ W_j = q_j V_j = q_j \sum_{i=1}^{n} \frac{q_i}{4\pi \epsilon R_{ij}} \quad i \neq j \]

\[ q_j \rightarrow \frac{q_j q_j}{q_j} \]

\[ W = \sum_{j=1}^{n} W_j \] \leq \text{DOUBLED SUM}

1.
\[ W = 1 \left( 2 + 3 + 4 \right) = 12 + 13 + 14 \]

2.
\[ W = 2 \left( 1 + 3 + 4 \right) = 12 + 23 + 24 \]

3.
\[ W = 3 \left( 1 + 2 + 4 \right) = 13 + 23 + 34 \]

4.
\[ W = 4 \left( 1 + 2 + 3 \right) = 14 + 24 + 34 \]

\[ \Rightarrow W = \frac{1}{2} \sum_{j=1}^{n} q_j V_j = \frac{1}{2} \sum_{j=1}^{n} q_j \sum_{i=1}^{n} \frac{q_i}{4\pi \epsilon R_{ij}} \]

\[ = \frac{1}{2} \sum_{j=1}^{n} q_j V_j \] \quad \text{FOR CONDUCTING MATERIAL}

\[ = \frac{1}{2} V \sum_{j=1}^{n} q_j \]

\[ = \frac{1}{2} V Q = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \alpha V^2 \]

\[ V = V_1 - V_2 \]

\[ \Rightarrow W = \frac{1}{2} Q \left( V_1 - V_2 \right) \]
GREEN'S RECIPROCATION THEOREM

\[
\sum_{i=1}^{n} Q_i V_i' = \sum_{j=1}^{n} Q_j V_j'
\]

CONSIDER \( n = 1 \)

\[
\frac{Q_1'}{V_1'} = \frac{Q_1'}{V_1'} \Rightarrow \frac{Q_1'}{V_1'} = \frac{Q_1'}{V_1'} = C
\]

PROOF:

\[
Q_1' V_1 = 0 + \frac{Q_2}{4\pi \epsilon R_{12}} + \frac{Q_3}{4\pi \epsilon R_{13}} + \ldots + \frac{Q_n}{4\pi \epsilon R_{1n}}
\]
\[
Q_2' V_2 = \frac{Q_1'}{4\pi \epsilon R_{12}} + 0 + \frac{Q_3}{4\pi \epsilon R_{23}} + \ldots
\]
\[
\Rightarrow \sum_{i=1}^{n} Q_i' V_i' = C + \frac{Q_i}{4\pi \epsilon R_{ij}} ; i \neq 3
\]
\[
= \sum_{i=1}^{n} Q_i' V_i'
\]
Example (from Text)

Find displacement on surface of plane conductor of zero thickness \( \frac{1}{2} \) on the sheet of conducting material of thickness \( d \), same geometry. For two cases,

1) Both conductors contain same charge \( Q \).
2) Both conductors have same \( Q \).

Is it possible that \( D \) is not parallel to \( \vec{E} \), if yes, why? Example from Chapter 2: 4, 13, 14, 15, 490

\[
\frac{1}{k} = 0 \quad 0 = \frac{Q}{k} \quad D \text{ on surface}
\]

2 cases

\( d = 0 \)

\( d \neq 0 \)

\( S_1 \), \( S_2 \) \( (= S_1) \)

\( S_0 \)
**Apply to Problem**

\[ P \rightarrow r \]

\[ V_P' = \frac{Q'}{4\pi \varepsilon_0} \quad V' = \frac{Q'}{4\pi \varepsilon_0} \]

Gives \( Q = \text{induced charge on sphere} \)

\[ Q = -\frac{q}{r} \]

**What is induced charges \( Q_1 \) \& \( Q_2 \)**

\[ q = (Q_1 + Q_2) \]

Charge only inner surface:

\[ q V_P' + Q_1 V_1' + Q_2 V_2' = 0 \]

Gives

\[ Q_1 = \frac{V_2' - V_P'}{V_1' - V_2'} q \]

\[ Q_2 = \frac{V_1' - V_P'}{V_2' - V_1'} q \]

May take surfaces as spheres for homework?
WITH APPROPRIATE GROUNDING

\[ \frac{Q}{C_{ef}} \text{ and } V_{ef} \] \[ \frac{Q}{C_{ef}} = C_{ef} V_{ef} \]

VOLTAGE ON S

\[ \frac{Q}{C_{ef}} = C_{ef} V_{ef} \]

\[ V_{S} \]

\[ S \]

\[ \theta_{S} \]

\[ \theta_{L} \]

\[ \theta_{R} \]

SURFACE I CANNOT INDUCT ON 3 OR 4

RECALL \( E = \frac{1}{2} \nabla \cdot D \)

\[ \frac{dW}{dV} = \frac{1}{2} E \cdot D \Rightarrow W = \frac{1}{2} \int D \cdot E dV \]

RECALL, FROM ADDING IN \( Q \) FROM \( \theta \) TO \( \theta_{S} \):

\[ W_{L} = V_{L} \cdot Q_{L} \Rightarrow \left[ W = \frac{1}{2} \sum_{L} V_{L} \cdot Q_{L} \right] \]

MAY USE MATRIX RELATIONSHIP TO MAKE THIS ALL VOLTAGE OR ALL CHARGE:

\[ F_{\vec{F}} = -\frac{\vec{E}}{s} \]

\[ F_{\vec{F}} = \int_{s} \frac{d\vec{E}}{2} \rho \cdot \nabla \cdot dS \]
STOKES' THEOREM
\[ \mathbf{A} = \nabla \times \mathbf{F} \]

\[ \oint_{C} \mathbf{A} \cdot d\mathbf{s} = \iint_{S} \nabla \cdot (\mathbf{n} \times \mathbf{F}) \, dV \]

\[ \nabla \cdot (\mathbf{n} \times \mathbf{F}) = \mathbf{F} \cdot \nabla \times \mathbf{n} - \mathbf{n} \cdot \nabla \times \mathbf{F} \]

\[ \mathbf{F} \cdot \nabla \times \mathbf{n} = 0 \]

\[ (\mathbf{n} \times \mathbf{F}) \mathbf{n} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \mathbf{a}_y & \mathbf{a}_z & \mathbf{a}_x \\ 0 & 0 & 0 \end{vmatrix}, \mathbf{n} = 0 \]

2-12-76 (THURS.)

RECALL GAUSS' THEOREM
\[ \oint_{\partial V} \mathbf{A} \cdot d\mathbf{s} = \iint_{S} \nabla \cdot \mathbf{A} \, dV \]

LET \[ \mathbf{A} = \mathbf{n} \times \mathbf{F} \]

\[ \oint_{\partial \mathbf{V}} \mathbf{n} \times \mathbf{F} \cdot \mathbf{n} \, d\mathbf{s} = \iint_{V} \nabla \cdot \mathbf{n} \times \mathbf{F} \\ dV \]

\[ \nabla \cdot \mathbf{n} \times \mathbf{F} = -\mathbf{n} \cdot \nabla \times \mathbf{F} \]

\[ (\mathbf{n} \times \mathbf{F}) \mathbf{n} = 0 \]

\[ \Rightarrow \quad \mathbf{n} \times \mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{n}' \times \mathbf{n} \quad \text{(NOTE \( \mathbf{n}' \perp \mathbf{n} \))} \]

\[ dV = h \, ds \quad ; \quad ds' = h \, dl \]

\[ \mathbf{n} \times \mathbf{n} \, dl = -\mathbf{n} \times \mathbf{n}' \, dl = d\mathbf{\hat{e}} \]
\[ \oint_{\partial V} \vec{n} \cdot d\vec{n} \, ds = \nabla \cdot \vec{A} \, dv. \]

**Rewrite as**
\[
\sum_{j=1}^{n} \beta_{j} \vec{n} \cdot d\vec{n} \, ds_{j} = \int_{V} \nabla \cdot \vec{A} \, dv
\]

**For Dielectrics**

\[
\oint_{\delta_{p}} (\vec{A}_{p} - \vec{A}_{p}'' - \vec{n}_{p}''') \, ds_{p} \sum_{j=1}^{m} \delta_{p} \vec{n} \, ds_{j} + \oint_{\delta_{p}} (\vec{A}_{p} + \vec{A}_{p}' + \vec{A}_{p}''') \, ds_{p} = \int_{V} \nabla \cdot \vec{A} \, dv
\]

\[ \vec{A} = \vec{\nabla} \phi \text{ or } \vec{A} \cdot \vec{n} = \phi \in \nabla \phi \]

**Combining gives Green's First Theorem:**

\[
\sum_{j=1}^{n} \delta_{j} \vec{n} \cdot \frac{\partial \phi}{\partial x_{j}} \, ds_{j} + \int_{V} \beta \frac{\partial \phi}{\partial x_{p}} \delta_{p} \, dv + \int_{V} \gamma \delta_{p} \frac{\partial \phi}{\partial x_{p}} \, dv = \oint_{\partial V} \vec{n} \cdot \nabla \phi \, dv
\]

\[ + \int_{V} \vec{A} \cdot \nabla (\varepsilon \nabla \phi) \, dv \]
RECALL GREEN'S THEOREM OR STOKES' THEOREM

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \]

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

\[ \mathbf{E} = \mathbf{E}' \]

\[ \mathbf{E}' = -\nabla \phi' \]

\[ \mathbf{E} + \mathbf{E}' = 0 \]

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} \]

\[ \nabla \cdot \mathbf{D} = \rho \]

\[ \nabla \cdot \mathbf{D} = \varepsilon_0 \rho \]

Thus, if \( \mathbf{E} \) and \( \mathbf{D} \) have discontinuities, \( \mathbf{E}' \) must not interfere.

FURTHERMORE, THERE IS NO CHARGE \( \neq 0 \) POISSON'S EQUATION MUST BE SATISFIED

\[ \Rightarrow \oint (\mathbf{E} + \mathbf{E}') = 0 \]

\[ \Rightarrow \oint \mathbf{E}' = 0 \]

\[ \Rightarrow \oint \nabla \phi' = 0 \]

\[ \Rightarrow \int_S \nabla \phi' \cdot d\mathbf{S} = 0 \]

\[ \Rightarrow \int_S \mathbf{E}' \cdot d\mathbf{S} = 0 \]

\[ \Rightarrow \int_S \varepsilon_0 \nabla \phi' \cdot d\mathbf{S} = 0 \]

\[ \Rightarrow \int_S \varepsilon_0 \frac{\partial \phi}{\partial n} dS = 0 \]
\[ \Phi \left[ \frac{1}{4\pi \epsilon} \frac{\phi}{\sqrt{r}} - \phi \frac{\phi}{4\pi \epsilon r} \right] ds \\
= \frac{1}{4\pi \epsilon} \Phi \left[ \frac{1}{\sqrt{r}} \frac{\phi}{\sqrt{r}} + \frac{1}{4\pi \epsilon r} \right] ds \\
= \frac{1}{4\pi \epsilon} \int_0^1 \frac{\phi}{r^{3/2}} G \, ds \\
G = \phi + \frac{1}{4\pi \epsilon r} = \text{GREEN FUNCTION} \\
\text{SECOND TERM IN SUM GIVES} \\
= \int_0^1 \frac{1}{4\pi \epsilon \phi} \left[ \frac{1}{\sqrt{r}} - \phi \frac{\phi}{4\pi \epsilon r} \right] ds' \\
= -\int_0^1 \phi \frac{1}{\sqrt{r}} \frac{\phi}{4\pi \epsilon r} ds' \\
= -\int_0^1 \phi \frac{1}{\sqrt{r}} \, p d\nu = -\frac{\phi}{\epsilon} p \\
\text{SUM OF THESE TWO IS} \\
\Rightarrow \frac{1}{4\pi \epsilon} \int \frac{\phi}{\sqrt{r}} G \, ds = \phi_0 = \frac{1}{4\pi \epsilon} \int_0^1 \frac{\phi}{r} ds \]
INTRODUCING A METAL BALL INTO
A SYSTEM REDUCES ENERGY

\[ W = \frac{E}{2} \int_V E^2 \, dV \]

\[ w - w' = \frac{E}{2} \int_{V} E^2 \, dV - \frac{E}{2} \int_{V'} E'^2 \, dV \]

ASSUME \( Q_j = Q_j' \)

\[ w - w' = \frac{E}{2} \int_{V - V'} E^2 \, dV + \frac{E}{2} \int [(E - E')^2] \, dV \]

\[-2E' (E' E) \]

APPLY FIRST GREEN'S THEOREM WITH

\[ \gamma = V', \quad \phi = V - V' \]

SINCE \( Q_j = Q_j' \), WE HAVE \( \nabla^2 \phi = 0 \)

\[ \nabla^2 V = \nabla^2 V' = -\frac{\rho}{\varepsilon} + \frac{\rho}{\varepsilon} = 0 \] (SINCE \( \rho = 0 \))

THEN

\[ \sum_{d, j} \int_{S_d} \frac{\partial \phi}{\partial n_j} \, dS_v = \int_V \left[ \nabla \cdot (\nabla \times \nabla \phi - \nabla \phi \nabla \phi) + \nabla \times (\nabla \phi) \times \nabla \phi \right] \, dV \]

\[ = \int_V \nabla \cdot \nabla \phi \cdot \nabla (V - V') \, dV \]

\[ = \int_V \nabla \cdot \nabla \phi \cdot (E - E') \, dV \] (FROM \( E = \nabla \phi \))

\[ = \sum_{d, j} \int_{S_d} \nabla \cdot [\frac{\partial V'}{\partial n_j} - \frac{\partial V}{\partial n_j}] \, dS_v \]

\[ = \frac{1}{\varepsilon} \sum_{d, j} \int_{S_d} (\sigma - \sigma') \, dS_v \] (FROM GAUSS THEOREM)

\[ = 0 = -2E' (E' E) \]
\[ \gamma = \nabla \phi; \quad \phi = V \]
\[ \nabla \cdot (\nabla \phi) = \gamma \]

**Second Theorem**

\[ \int_V \nabla \cdot (\nabla \phi) \, dV = -\int_V \partial V \, dV \]

\[ \sum_{j=1}^{m} \left( \int_{S_j} \frac{\delta V}{\nabla \phi} \cdot n_j \, dS_j \right) = \sum_{j=1}^{m} \delta V_j \cdot \int_{S_j} \sigma \, dS_j \]

\[ \Rightarrow \int_V \nabla (\delta V) \cdot \nabla \phi \, dV = \sum_{j=1}^{m} \delta V_j \cdot Q_j + \int_V \delta V \, dV \]

\[ = 26W \quad (p_8, 59) \]

\[ \Rightarrow \delta W = -\frac{1}{2} \int_V \delta V \, dV \]

For \( \delta \phi > 0 \Rightarrow \delta W < 0 \)

*Fewer lines of force are generated*
RECALL \( w = (M - \nabla) \psi \)

\[
\begin{align*}
\frac{dV}{dn} &= \frac{\partial V}{\partial n} = 4\pi e \left\{ \frac{S}{2\pi} \left( \frac{1}{r} \right) \right\} \\
\text{DIPLOLE STRENGTH IS THEN} \quad \sigma ds \\
\text{FOR A DOUBLE LAYER,} \quad \phi = \frac{M}{S} \quad \text{(pg. 14)} \\
V_p &= \frac{1}{4\pi e} \int \phi \frac{S}{2\pi} \left( \frac{1}{r} \right) ds \\
&= \frac{1}{4\pi e} \int \phi \frac{n \cdot r}{r^3} ds \\
&= \frac{1}{4\pi e} \int \phi dS \\
\text{GREEN'S STRATUM} \\

\begin{align*}
y &= r^{-1}, \quad \phi &= \psi, \quad \nabla^2 \phi &= -\frac{\rho}{\varepsilon} \\
\text{USE GREEN'S 2nd THEOREM} \\
(\text{WHAT DO THIS STUFF MEAN?}) \\
\int_5 \frac{\partial V}{\partial n} ds &= \int_5 V \frac{S}{2\pi} \left( \frac{1}{r} \right) = \frac{1}{4\pi} \int \frac{\partial V}{\partial n} ds = \frac{1}{4\pi} \int \frac{\partial V}{\partial n} ds + 4\pi \int_5 V \frac{S}{2\pi} \left( \frac{1}{r} \right) ds + 4\pi \int_5 V \frac{S}{2\pi} \left( \frac{1}{r} \right) ds \\
V_p &= \frac{1}{4\pi e} \int \frac{\partial V}{\partial n} ds = \frac{1}{4\pi} \int \frac{\partial V}{\partial n} ds + 4\pi \int_5 V \frac{S}{2\pi} \left( \frac{1}{r} \right) ds + 4\pi \int_5 V \frac{S}{2\pi} \left( \frac{1}{r} \right) ds \\
\text{VOLUME} \\
\text{SURFACE}
\end{align*}
\]
2-20-76 (THURS)

TWO DIMENSIONAL POTENTIAL \[ \nabla^{2} \phi \] HAS DIMENSIONS OF \( \frac{q}{m} \).

RECALL CYLINDRICAL CAPACITOR.

\[
\begin{align*}
\oint \mathbf{E} \cdot d\mathbf{s} &= q \times 1 \text{ meter} \\
2\pi r \varepsilon_{0} E &= q \Rightarrow E &= \frac{q}{2\pi r \varepsilon_{0}} \\
\text{Similar laws hold:} \quad E &= -\frac{V}{\sqrt{\varepsilon_{0} r}} \\
\Rightarrow V &= -\frac{q}{2\pi r \varepsilon_{0}} + C
\end{align*}
\]

WE CHOOSE \( C \) TO FIT BOUNDARY CONDITION (LIKE \( V @ r = \infty = 0 \)).
\[
\frac{\Delta^2 \Theta}{\Delta \Theta^2} = -n^2 \cdot 0
\]

\[\Rightarrow \Theta = A_n \cos n \theta + B_n \sin n \theta\]

\[r \frac{\Delta^2 R}{\Delta r^2} + r^2 \frac{\Delta^2 R}{\Delta r^2} = n^2 \frac{R}{r^2}\]

\[\Rightarrow R = C_n r^n + D_n r^{-n}\]

For \( n = 0 \):

\[\Theta_0 = A \Theta + B\]

\[R_0 = C \ln R + D\]

**This gives**

\[V = \sum_n \Theta_n R_n\]

**Consider:**

\[\sum_n A_n \cos n \theta + B_n \sin n \theta = \sum_n (C_n \cos n \theta + D_n \sin n \theta)\]

\[\Rightarrow A_n = C_n; B_n = D_n\]

**Consider:**

\[\Theta_1 = A_1 \cos \theta + B_1 \sin \theta + A_5 \cos 5 \theta + B_5 \sin 5 \theta\]

\[\Theta_2 = C_2 \cos 2 \theta + D_2 \sin 2 \theta + C_6 \cos 6 \theta + D_6 \sin 6 \theta\]

**Equate:** \( \Theta_1 = \Theta_2 \)

\[\Rightarrow A_1 = B_1 = C_2 = D_2 = 0\]

\[A_5 = C_6; B_5 = D_6\]
BOUNDARY CONDITIONS
\[
\begin{align*}
&\theta r = 0, \quad V_0 = V_1, \\
&\theta r = b, \quad E_0 \frac{\partial V_0}{\partial r} = E \frac{\partial V_1}{\partial r} \\
&\Rightarrow \frac{\partial V_0}{\partial r} = k \frac{\partial V_1}{\partial r}
\end{align*}
\]

SPHERE (CONDUCTOR) \text{ HAS } \quad V_2 = 0

\Rightarrow V_b (a) = 0

\begin{align*}
\text{FOR } n \neq 1 \quad (A_0, C_0) \\
\Rightarrow A_0 = B_0 = C_0 = 0 \\
A_1 = -EB^2 \frac{(k+1) a^2 + (k-1) b^2}{(k+1)b^2 + (k-1)a^2} \\
B_1 \quad \text{BUCK (p. 67)} \\
C_1
\end{align*}

POTENTIAL'S SHAPE:
\[
\begin{align*}
V_0 &= \left( E (r + \frac{a^2}{r}) \right) \cos \theta \\
V_1 &= \left( B_1 r + \frac{C_1}{r} \right) \cos \theta
\end{align*}
\]

FOR JUST CONDUCTOR
(SHARP) \text{ LET } \quad k = 1

\Rightarrow V_0 = E \left( r + \frac{a^2}{r} \right) \cos \theta

JUST DIELECTRIC \Rightarrow \quad \text{IL} \quad \theta = 0
\[
\begin{align*}
V_0 &= E \left( r - (k+1) \frac{b^2}{r} \right) \cos \theta \\
V_1 &= k + r \cos \theta
\end{align*}
\]
\[ V_0 = \frac{q}{4\pi \epsilon} \left[ \sum \frac{1}{n} \left( \frac{r}{b} \right)^n + \frac{n}{r_n} \cos \theta - 1 \right] \]

Turns out potential looks like

\[ q' = \frac{1-k}{1+k} q \quad \text{(both line charge)} \]

Use method of images.
Eq. 403

\[ V_o = E \left( r - \frac{K-1}{K+1} \frac{b^2}{r} \right) \cos \theta \]

**Before**

\[ V = Er \cos \theta \]

\[ V_o = Er \cos \theta - E \cos \theta \left( \frac{K-1}{K+1} \frac{b^2}{r} \right) \]

Is impossible to have different polarities on \( V \neq V_o. \)

Change coordinate system

\[ V = Er \cos \theta \ (\text{Before}) \quad V = Er \cos \theta \]

\[ V = Er \cos \theta + A \frac{(r-r_o)^2}{r} \cos \theta \]

**Boundary Conditions:** \( V = 0 \)

\[ E(r_o + a) = -A \frac{(r_o + a - r_o)}{r} \]

\[ \Rightarrow A = -Eq (r_o + a) \]

And

\[ V = E \left[ r - \frac{a^2}{r-r_o} - \frac{a r_o}{r-r_o} \right] \cos \theta \]

For \( \# 2, \) \( V = Er_o \cos \theta \)

\[ E(r_o + a) = A \frac{(r_o + a - r_o)}{r} = E(r_o) \]

\[ \Rightarrow A = -Eq^2 \]

\[ V = E \left[ r - \frac{a^2}{r-r_o} \right] \cos \theta \]
\[ W = U + jV \]
\[ W = \ln \frac{z + \theta}{z - \theta} \]
\[ W = \ln \frac{r e^{i\theta}}{r e^{-i\theta}} \quad (r \text{ is complex here}) \]
\[ = \ln e^{i2\theta} \]
\[ = i2\theta = i2 \tan^{-1} \frac{\theta}{z} = j2 \cot^{-1} \frac{z}{\theta} \]
\[ \therefore W = 2j\theta = j2 \tan^{-1} \frac{\theta}{z} \]

**Z Plane**

\[ z = a \cos \frac{U + jV}{\frac{V}{z}} = a \cos \frac{U}{2} - jV \]
\[ = -a \sin \left( \frac{U}{2} \right) + a \sin V \]
\[ = \frac{1}{2} (e^{j(\frac{U}{2})} - e^{-j(\frac{U}{2})}) + \sin V \]
\[ = a \left( e^{jU/2} + e^{-jU/2} \right) - \cos V \]
\[ = a \cosh U + \sin V \]
\[ X = a \cosh U - \cos V \]
\[ Y = a \sinh U \]
\[ Z = X + jY \]
FROM \( q_1 = q_2 = q \), WE FOUND EQUPOENTIAL CYLINDERS \( \mathbb{A}(x,y) = \text{const} \ U \).

Now, we will assume \( \mathbb{A}(x,y) \)
and find \( q_1 = -q_2 \Rightarrow q_1 = -q_2 \)
and then find capacitance.

\[ R_1 = q / \cosh U_1 \]
\[ R_2 = q / \cosh U_2 \]
\[ D = a \left[ | \cosh U_1 | / \cosh U_2 \right] \]

"+" Cylinders separate, "-" inside each other.

May find capacitance from \( (U_2 - U) \)

\[ C = 2 \pi \epsilon \left[ \cosh^{-1} \left( \pm \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right) \right] \]

(Capacitance per unit length)

Transformation used was

\[ W = \ln \frac{B+dQ}{2-dQ} \text{ vs. } \text{USC for point charges} \]

(Powerful when used with Method of Images)
3/2/76 (TUES)

CHAPTER 5

5.04, METHOD OF IMAGES

\[ \sigma = - \frac{q}{2\pi} \ln \left( \frac{a}{r} \right) \]

5.07.

\[ q'' = \frac{2q}{k+1} \quad q = \frac{a}{b} q \]

5.09.

- \( r' = \frac{q'^2}{r} \)
- \( r' r = k^2 \)

\[ F(r, \theta, \phi) \rightarrow F \left( \frac{k^2}{r}, \theta, \phi \right) \]

Here, \( r = r' \)

\[ k = 2q, \quad r' = \frac{k^2}{r} \rightarrow \frac{4q^2}{r^2} = \frac{4q^2}{2q} = 2q \]
3.9.7c (Tues)

\[
\frac{1}{r} = \left( a^2 + b^2 - 2ab \cos \theta \right)^{1/2}
\]

\[
\mu = \cos \theta
\]

For \( b > a \), \( \frac{1}{r} = \frac{1}{b} \left[ 1 + \mu(\frac{a}{b}) + \frac{3\mu^2}{2} \left( \frac{a}{b} \right)^2 + \ldots \right]
\]

\[= \frac{1}{b} \left[ P_0(\mu) + \frac{a}{b} P_1(\mu) + \left( \frac{a}{b} \right)^2 P_2(\mu) + \ldots \right]
\]

5.17. ENTER

WE CAN WRITE: \( V = \sum_{n=1}^{\infty} \left( A_n r^n + B_n \frac{1}{r^{n+1}} \right) P_n(\mu) \)

For Ring

\( V_r = \frac{q}{2\pi \Delta r} \sum_{n=0}^{\infty} \left( \frac{q}{r} \right)^{2n+1} \mu^n P_{2n}(\mu) \); \( a < r \)

For Sphere: \( B_n = 0 \Rightarrow V_s = \sum_{n=0}^{\infty} A_n r^n P_n(a \cos \theta) \)

\( (r > a) \)

\( V = V_r + V_s \)

\( V(a) = 0 \) TO FIND \( A_n \)

(ALL OTHER VALUES DROP OUT)

\( V_r(a) = -V_s(a) \)

GIVES \( A_{2n} b^n = -\frac{q}{4\pi \Delta r} \left( \frac{a}{b} \right)^{2n+1} P_{2n}(a) \)

\( A_{2n+1} b^n = 0 \)

RETURN
3/11/76 (Thurs)
Do pros 60-61 in chap. 5

3/16/76 (Tues)

\[
\begin{align*}
\text{(a)} & \quad Z \\
\text{(b)} & \quad C_b \\
\text{(c)} & \quad b
\end{align*}
\]

Without electric

\[
V - \phi_1(k) = \frac{q}{4\pi\epsilon_0} \int_0^\infty J_0(kr) e^{-kr} \, dr
\]

For \( -\infty < Z < a \)

\[
V_1 = -\int_0^\infty \phi_1(k) J_0(kr) e^{-kr} \, dk
\]

For \( a < Z < b \)

\[
V_2 = \frac{q}{4\pi\epsilon_0} \int_0^a \psi(k) J_0(kr) \, dk
\]

(fest nxt. tues)
\[ \nabla^2 w_1 = 0 \quad \nabla^2 w_2 = 0 \]

1. \[ B = \nabla \times A \]
   \[ = \nabla \times \left( \nabla \times w \right) \]
   \[ = \nabla \times \left[ \nabla \times (u w_1 + u \times \nabla w_2) \right] \]
   \[ = \nabla \times \left[ \nabla \cdot \nabla w_1 \right] + \nabla \times \nabla \times (u \times \nabla w_2) \]
   \[ \nabla \times \left( \nabla \cdot \nabla w_1 \right) = -\nabla \times (r \times \nabla w_1) \]
   \[ = r \frac{\partial}{\partial r} (\nabla w_1) \]
   \[ = \nabla \times (\nabla w_1) \]

2. \[ \nabla \times (\nabla (u \times \nabla w_2)) \]
   \[ = \nabla \times \left[ \nabla \cdot \nabla w_2 - \nabla (u \cdot \nabla w_2) - \nabla w_2 \right] = 0 \]

Assume \( \overrightarrow{w} = k \overrightarrow{w}_1 \) (in \( z \) direction)
\[ \nabla^2 w_1 = 0 \]

Find \( \nabla \times \overrightarrow{w} = \begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \overrightarrow{w}_1 \]
\[ = 2 \frac{\partial}{\partial y} w_1 - 2 \frac{\partial}{\partial x} w_1 + k \overrightarrow{0} \]

This will be useful for boundary conditions. Note perpendicular components of \( \nabla \)
\[ A = \frac{1}{4\pi} \int_0^\infty \frac{\text{d}V}{r^2} = \frac{1}{4\pi} \int_0^\infty \frac{\text{d}V}{r^2} \]

A will only have \( \phi \) component for \( \nabla^2 W_i = 0 \)

\[ W_i = \frac{1}{2\pi} \int_0^{2\pi} \phi (z + \text{Re}(\rho \sin \theta)) \text{d}\theta \]

\[ \nabla^2 W_i = -\frac{1}{4\pi} \left( \frac{\text{d}^2 W_i}{\text{d}r^2} + \frac{2}{r} \frac{\text{d} W_i}{\text{d}r} + \frac{W_i}{r^2} \right) \]

Now \( \phi (z) \) is a real function of \( z \).

\[ W_i = \frac{k}{\rho} W_i \]

\[ \nabla^2 W_i = -\frac{1}{\rho} \frac{\text{d} W_i}{\text{d}\rho} \]

\[ \nabla^2 (\rho \phi) = \rho \nabla^2 \phi \]

\[ \phi (z) = \sum_{n=0}^{\infty} \frac{1}{n! (2n+1)!} \frac{z^{2n+1}}{2} \]

\( B = 0 = B \phi (z) = \frac{\varepsilon r \phi (z)}{\varepsilon 0} \)
STREAM FUNCTION \( \psi = \sum_{n}^{\infty} \left( \Theta, \phi \right) \)

FROM CHART 6: \( i = \frac{1}{r} \frac{dI}{d\phi} \)

IN SPHERICAL COORDINATES

\[
\begin{align*}
\hat{r} &= \hat{r} = \frac{1}{\sin \theta} \hat{\phi} \\
\hat{\phi} &= \frac{1}{\sin \theta} \hat{\phi}
\end{align*}
\]

APPLY AMPERE'S LAW (HOLD \( \theta \) CONST)

\[
\oint B \, dl = \mu I
\]

\( I = \Delta \Theta \hat{r} \), \( B_{\text{outside}} - B_{\text{inside}} = B_0 \)

\( \Rightarrow (B_0 - B_0) \Delta \Theta = \mu \Delta \Theta \hat{r} \)

LET \( W = r W_1 \)

GIVES \( B_0 = \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{S^2(r W_1)}{r \sin \phi} \right) \)

\[
\begin{align*}
\frac{S^2(r W_1)}{r \sin \phi} &= \frac{S^2(r W_1)}{r \sin \phi}
\end{align*}
\]

WHEN WE HOLD \( \Theta \) CONSTANT, WE GET

\[
\frac{\partial}{\partial \phi} \left( \frac{S^2(r W_1)}{r \sin \phi} \right) = -\frac{1}{r \sin \phi} \frac{S^2(r W_1)}{r \sin \phi}
\]

\[
\Rightarrow \mu \psi = \frac{S}{r} (r W_1 - r W_1)
\]

BOUNDARY CONDITIONS: \( r = q \Rightarrow W_1(r) = W_0(q) \)

\[
\Rightarrow \mu \psi = q \left( \frac{S}{r} \frac{W_0}{r} - \frac{S}{r} W_1 \right)
\]

\[\nabla^2 W_1 = 0\]
7.12

**PARALLEL WIRES**

**ANSWER NOT DEPEND ON** \( \phi \). **WE HAVE**

\[
\chi(\theta) = \sum_{n=1}^{\infty} C_n P_n(\cos \theta)
\]

**WE KNOW** \( \hat{A} = [0, 0, A\hat{\phi}] \)

**SINCE, FROM AMPELEPS'S LAW, \( \hat{A} \) IS**

**SAME DIRECTION AS** \( \hat{A} \).

\[
\hat{A} = \frac{d}{d \phi} \left( \sum_{n=1}^{\infty} \frac{C_n}{\lambda} \frac{n}{2n-1} \int \frac{d}{d \phi} P_n(\cos \phi) \right)^2
\]

Choose

\[
W_{1o} = \sum_{n=1}^{\infty} \left( A_n r^n + B_n r^{n-1} \right) P_n(\cos \theta)
\]

**THIS GIVES**

\[
W_{1o} = \sum_{n=1}^{\infty} \frac{n}{2n+1} C_n \frac{(\alpha r)^n}{n!} P_n(\cos \theta)
\]

\[
W_{1d} = \sum_{n=1}^{\infty} \frac{\alpha}{2n+1} \frac{(\alpha r)^n}{n!} P_n(\cos \theta)
\]

**NOW:** \( \hat{A} = [0, 0, A\hat{\phi}] \)

\[
B = [B_r, B_{\theta}, 0]
\]

\[
B_r = -\frac{\sigma^2}{\varepsilon_0 r^2} \left( \sum_{n=1}^{\infty} \frac{C_n}{\lambda} \frac{n}{2n+1} \int \frac{d}{d \phi} P_n(\cos \phi) \right)\]

\[
P_{\theta i} = -\frac{1}{\varepsilon_0 r} \left( \sum_{n=1}^{\infty} C_n \frac{(\alpha r)^{n-1}}{2n-1} P_n(\cos \phi) \right)
\]
Thus, we have
\[ C_n = \frac{(2n+1)}{2n(n+1)} \int \sin \alpha \, P_n'(\cos \alpha) \, \sin \theta \, d\theta \]

Thus,
\[ A_{p_n} = \mathcal{R} \frac{\alpha \mathcal{L}}{2} = \frac{\alpha \mathcal{L}}{n(n+1)} \left( \frac{r}{\mathcal{R}} \right)^n P_n'(\cos \alpha) P_n(\cos \alpha). \]

Steps:
- Determine B.C. with stream functions
- Stream via spherical harmonics
- Choose \( w \), representative of soln.

Examples in two dimensions:
Find \( \mathbf{B} \) in da hole

7.16.

Assume uniform current \( \mathbf{I} \):
- \( \mathbf{H} \), \( \mathbf{H} = \text{constant} \)

1. Find field for \( \Theta \)
2. \( U \)
3. Subtract currents to give zero current in \( b \)

Cont \( \rightarrow \)
\[ A_x = \frac{\mu I}{2\pi} \left[ \int_{-\infty}^{\infty} \frac{dx}{r_1} - \int_{-\infty}^{\infty} \frac{dx}{r_2} \right] \]
\[ = \frac{\mu I}{2\pi} \left[ \int_{0}^{\infty} \frac{dx}{\sqrt{a_2^2 + x^2}} - \int_{-\infty}^{0} \frac{dx}{\sqrt{a_2^2 + x^2}} \right] \]
\[ = \frac{\mu I}{2\pi} \ln \frac{a_2}{a_1} \]

Try a couple problems from or a chapter
ASSUME ONLY $i_z$ COMPONENTS $(0, 0, i_z)$

FIND POTENTIAL @ POINT P

$A_z = \frac{\mu_0}{2\pi} \ln R = \mu_i a \ln R$

INTEGRATE AROUND $\alpha$ FOR MANY WIRES

$A_z = \mu_i a \int i \ln R \, d\alpha$

$= \frac{\mu_i}{2\pi} \int_0^{2\pi} i \ln R \, d\alpha$

WE CAN EXPRESS $i_z$ AS

$i_z = \sum_{n=0}^{\infty} [C_n \cos n\alpha + D_n \sin n\alpha]$

AND

$\ln R = \ln q - \frac{1}{m} \left( \frac{r}{a} \right)^m \left[ \cos m\theta \cos m\alpha + \sin m\theta \sin m\alpha \right]$

PUTTING IT TOGETHER:

$A_z = \mu_i a \sum_{n=0}^{\infty} \left[ C_n \ln q + \frac{1}{2m} \left( \frac{r}{a} \right)^m \right] \left[ \cos m\theta \cos n\alpha + \sin m\theta \sin n\alpha \right]$

IF WE HAVE UNIFORM $i_z$ (IT DOES NOT DEPEND ON $\alpha$), THEN $C_n = 0$

$A_z = \mu_i a i \ln q$ \text{ CONSTANT EVERYWHERE}

$\Rightarrow B = 0$
\n\n\[ \n\n\vec{\nabla} (d^2 \vec{A}) = (\vec{A} \cdot \nabla) d^2 + \vec{A} \times (\nabla \times d^2) \\
+ (d^2 \cdot \nabla) \vec{A} + d^2 \nabla \times (\nabla \times \vec{A}) \\
\]

Thus, we can rewrite the previous force as:

\[ F \cdot \int (d^2 \cdot \nabla) \vec{A} + d^2 \nabla \times (\nabla \times \vec{A}) \]

Leaving:

\[ F = \int \oint d^2 \times \vec{B} \]

\[ \text{FORCE \perp TO THESE VECTORS} \]

\[ \Rightarrow \vec{B} \]
A second simple case (7.19)

\[ \mathbf{F} = \frac{\mu I I}{2\pi a^2} \int d\mathbf{B} \times \mathbf{r} \times \mathbf{a} \]

Due to symmetry, force will only be in \( z \) direction.

Thus, we need only to look at \( \mathbf{B}_0 \).

\[ \Rightarrow \mathbf{F} = I \int \mathbf{B} \times d\mathbf{L} \mathbf{B}_0(a, b, c) \]

\[ = I \int \mathbf{B}_0(a, b, c) \sum_{\theta} b d\theta \]

\[ = 2\pi I \int \mathbf{B}_0(a, b, c) \sum_{\theta} b d\theta \]

Now

\[ \mathbf{B}_0 = -\frac{5A \mathbf{d}}{8 \pi^2} = \frac{\mu I C}{2\pi^3} \left[ \frac{1}{(a^2 + bc)^2 + c^2} \right]^{1/2} \]

where \( m(a, b, c) \)
4.15-76 \text{(THUNS)}

\( \mathbf{t} = \mu \nabla \times \mathbf{M} \)

\( \mathbf{M} = \mathbf{B} \left( \frac{\mathbf{E}}{\mu} - \frac{I}{\sigma} \right) \)

\( \oint \mathbf{E} \cdot d\mathbf{l} = \mathbb{I} \)

\( \oint \frac{\mathbf{B}}{\mu} \cdot d\mathbf{s} = \mathbb{I} \)

\[ \mathbf{A} = \frac{\mu}{4\pi} \int_V \frac{1}{r} \mathbf{v} \, dV + \frac{\mu}{4\pi} \int_S \mathbf{M} \times \mathbf{n} \, ds \]

\[ V = \frac{1}{\mu \varepsilon_0} \left[ \int_V \rho \, dV + \oint \frac{\mathbf{q}}{\mathbf{n}} \, ds \right] \]

**USE SAME BOUNDARY CONDITIONS AS BUI**

\[ V_i = V_0 \]

\[ \mathbf{E}_i = \mathbf{E}_0 \]

\[ \mathbf{H}_i = \mathbf{H}_0 \]

\[ \mathbf{B}_i = \mathbf{B}_0 \]

so for vector \( \mathbf{A} \)

\[ \mathbf{A}_0 = \mathbf{A}_i \]

\( \mathbf{A} \text{ is} \]

\[ \nabla \times (\mathbf{A}) = \mathbf{M} \]

\[ \frac{\varepsilon_0}{\varepsilon \mu} \frac{\delta \mathbf{A}_0}{\delta t} - \frac{\varepsilon_0}{\varepsilon \mu} \frac{\delta \mathbf{A}_0}{\delta t} = \mu \nabla \times (\mathbf{M} \times \mathbf{H}) \]

or, \text{equivalently,}

\[ \text{since} \quad \mathbf{M} = \mathbf{B} \left( \frac{\mathbf{E}}{\mu} - \frac{I}{\sigma} \right) \]

\[ \frac{\varepsilon_0}{\varepsilon \mu} \frac{\delta \mathbf{A}_0}{\delta t} - \frac{\varepsilon_0}{\varepsilon \mu} \frac{\delta \mathbf{A}_0}{\delta t} = \mu \nabla \times (\mathbf{M} \times \mathbf{H}) \]
\[
\begin{align*}
\text{ouals} & \quad \frac{dw}{dv} |_{A} = \frac{B}{2} \\
\frac{sw}{dv} & = \frac{B}{2} \\
\n\nF & = -\frac{sw}{5} \\
F & = \frac{sw}{5}
\end{align*}
\]

\[
M = \left(\frac{1}{2} \cdot \frac{1}{\rho} \right) B \\
w_{B} = \frac{1}{2} \int M \cdot B \, dv
\]
\[ \frac{dA}{dt} = \frac{2S}{SA} A_1 - A_2 \]

\[ A_1 = f(x, y, z) \]

For \( t > 0 \), \( t \to 0 \), \( z = 0 \)

Boundary Conditions:

- \( t = 0 \): \( A_1 = f(x, y, z) \)
- \( t > 0 \), \( z = 0 \)

10.10

1.29-76 (others)
GREENS FUNCTIONS (FROM HAGLER LECTURE)
\[ \nabla^2 \psi(x', t') - \frac{\delta^2}{c^2 \epsilon^2} \psi(x', t') = -4\pi \int \delta(x - x') \delta(t - t') \delta (t - t') \equiv DRIVEN WAVE EQ. \]
\[ \nabla_x \psi = \nabla_x^2 \psi = \text{D'ALAMBERTIAN} \]
\[ G = G(x', t'; x, t) = \text{GREENS FUNCTION} \]
\[ \nabla_x \psi = -4\pi \frac{\delta^2}{c^2 \epsilon^2} \delta (x - x') \delta (t - t') \]
\[ \}\begin{cases} \nabla_x \psi = \text{SOURCE COMPONENTS} \\ x, t \Rightarrow \text{OBSERVER COMPONENTS} \end{cases} \]
\[ G(x', t'; x, t) = G(x' - x; t' - t) \]

CONSIDER:
\[ \int dt' \int d^3 x' \left[ G(n_x \cdot \nabla_x - \nabla_x) \psi \right] = -4\pi \int dt' \int d^3 x' G f + 4\pi \int dt' \int d^3 x' \psi (x, t') \delta (x - x') \delta (t - t') \]
\[ = -4\pi \int dt' \int d^3 x' G f + 4\pi \int dt' \int d^3 x' G f \]
REARRANGE:
\[ \psi (x, t) = \int dt' \int d^3 x' G f + \frac{1}{4\pi} \int dt' \int d^3 x' (G \nabla_x \cdot \nabla_x - \nabla_x \cdot \nabla_x) \psi \]
\[ + \frac{1}{4\pi} \int dt' \int d^3 x' \left[ G \nabla_x \cdot \nabla_x - \nabla_x \cdot \nabla_x + \int d^3 x' \delta (t - t') \delta (x - x') \right] \]
BUT \[ G \nabla_x \cdot \nabla_x - \nabla_x \cdot \nabla_x G = \nabla_x \cdot (G \nabla_x \cdot \nabla_x) G \]
THRU THE DIVERGENCE THEOREM
\[ \int_V \nabla \cdot \bar{M} d^3 x = \int_S \bar{n} \cdot \bar{M} ds \]
\[ \Rightarrow \psi (x, t) = \int dt' \int d^3 x' G f \]
\[ + \frac{1}{4\pi} \int dt' \int_S ds' \bar{n} \cdot \left[ G \nabla_x \cdot \nabla_x - \nabla_x \cdot \nabla_x \right] \]
\[ + \frac{1}{4\pi} \int_V d^3 x \left( -\frac{1}{c^2} \right) [G \frac{\delta}{c^2} - \nabla_x \nabla_x G] \]

THESE TERMS PHYSICALLY CORRESPOND TO:
1) CONTRIBUTIONS FROM SOURCES WITHIN V
2) CONTRIBUTIONS EXTERNAL TO V
3) DUE TO FINITE DELAY
To find \( G \):

\[
G = G\left(\xi', \xi, t', t\right) = G\left(\xi; \gamma\right)
\]

\[
\Delta^2 G = \frac{4\pi}{c^2} \delta^3\left(\xi\right) \delta(\gamma)
\]

\[
= \frac{4\pi}{(2\pi)^4} \int d\omega \int d^3k \ e^{i(k \cdot \xi' - \omega \gamma)}
\]

**Define \( \hat{G}(\vec{k}, \omega) \) as the Fourier Transform of \( G \):**

\[
G = \int d\omega \int d^3k \ \hat{G}(\vec{k}, \omega) \ e^{i(k \cdot \xi - \omega \gamma)}
\]

Then

\[
\Delta^2 \hat{G} = \int d\omega \int d^3k \ \hat{G}(\vec{k}, \omega) \left[ \frac{1}{c^2} \ e^{i(k \cdot \xi - \omega \gamma)} \right]
\]

\[
= \int d\omega \int d^3k \ \hat{G}(\vec{k}, \omega) \left[ \frac{\omega^2}{c^2} - k^2 \right] \ e^{i(k \cdot \xi - \omega \gamma)}
\]

**Equating with previous expression for \( \Delta^2 G \) gives**

\[
\int d\omega \int d^3k \left[ \left( \frac{\omega^2}{c^2} - k^2 \right) \hat{G}(\vec{k}, \omega) + \frac{1}{c^2} \right] \ e^{i(k \cdot \xi - \omega \gamma)} = 0
\]

**Thus:**

\[
\hat{G}(\vec{k}, \omega) = -\left[ \frac{1}{4\pi^3 \left( \frac{\omega^2}{c^2} - k^2 \right)} \right]^{-1}
\]

**Regain \( G \) by an inverse Fourier Transform:**

\[
G(\xi, \gamma) = \frac{-c^2}{4\pi^3} \int d^3k \int d\omega \ \frac{e^{i(k \cdot \xi - \omega \gamma)}}{\left( \omega - k \omega \right) \left( \omega + k \omega \right)}
\]

**Use contour integration:**

\[
\int_0 \text{only for } \gamma < 0
\]

\[
\int_0 \text{only for } \gamma > 0
\]

Poles are divided to yield causal "retarded" Green's function.

**Gives for \( \gamma > 0 \):**

\[
G_R(\xi, \gamma) = \frac{-c^2}{4\pi^3} \int d^3k \ e^{i(k \cdot \xi)} \left[ \left. \frac{e^{i\omega \gamma}}{\omega + k \omega} \right|_{\omega = k \omega} + \left. \frac{e^{-i\omega \gamma}}{\omega - k \omega} \right|_{\omega = -k \omega} \right]
\]

\[
= \frac{c}{2\pi^2} \int d^3k \ e^{i(k \cdot \xi)} \frac{\sin \left( c k \gamma \right)}{k} ; \gamma > 0
\]
To evaluate this, use spherical coordinates in $k$-space:
\[
G_R(x, \gamma) = \frac{C}{2\pi^2} \int_0^\infty k^2 dk \int_0^\pi \sin \theta d\theta \int_0^{2\pi} e^{ik|\vec{x}| \cos \theta \sin \phi} \frac{\sin c k \gamma}{k} \left( \frac{1}{ik|\vec{x}|} \right) \sin c k \gamma
\]
\[
= \frac{C}{\pi |\vec{x}|^2} \int_0^\infty dk \sin k|\vec{x}| \sin c k \gamma
\]
Since the integrand is even,
\[
G_R(x, \gamma) = \frac{C}{\pi |\vec{x}|^2} \int_{-\infty}^{\infty} dk \sin k|\vec{x}| \sin c k \gamma
\]
Using trig identity,
\[
G_R(x, \gamma) = \frac{C}{\pi |\vec{x}|^2} \int_{-\infty}^{\infty} dk \left[ \cos \{k (|\vec{x}| - c \gamma)\} \cos \{k |\vec{x}| + c \gamma\} \right]
\]
We may add to the integrand even functions, which are even and will integrate to 0. This gives
\[
G_R(x, \gamma) = \frac{C}{\pi |\vec{x}|^2} \int_{-\infty}^{\infty} dk \left[ e^{ik (|\vec{x}| - c \gamma)} - e^{ik (|\vec{x}| + c \gamma)} \right]
\]
\[
= \frac{C(2\pi)}{2\pi |\vec{x}|^2} \left[ \delta \{ |\vec{x}| - c \gamma \} - \delta \{ |\vec{x}| + c \gamma \} \right]
\]
Since $\gamma > 0$, $\delta \{ |\vec{x}| + c \gamma \} = 0$ and
\[
G_R(x, \gamma) = \frac{C}{|\vec{x}|^2} \delta \{ |\vec{x}| - c \gamma \} \text{ for all } \gamma
\]
\[
= \frac{C}{|\vec{x}|^2} \delta \left[ \frac{|\vec{x}' - \vec{x}|}{c} - (t - \xi') \right]
\]
\[
\Leftrightarrow \text{ Greens function}
\]
Plug into first term on bottom of pg4:
This is the steady state term:
\[
\Psi_s(x, t) = \int_V d^3x' \int dt \frac{f(x', t')}{|\vec{x}' - \vec{x}|^{-1}} \delta \left[ \frac{|\vec{x}' - \vec{x}|}{c} - (t - \xi') \right]
\]
\[
= \int_V d^3x' \frac{f(x, t_{\text{RET}})}{|\vec{x}' - \vec{x}|}
\]
\[
t_{\text{RET}} = t - \frac{1}{c} \frac{|\vec{x} - \vec{x}'|}{c} = \text{ time from source to observer.}
\]
I. BASIC IDEAS

\[ F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \implies \text{COULOMB'S LAW} \]

\[ E = \frac{-1}{4\pi\varepsilon_0} \frac{q}{r^3} \implies \text{ELECTRIC FIELD INTENSITY} \]

\[ V_p = \frac{1}{4\pi\varepsilon_0} \int \frac{q}{r} \implies \text{ELECTROSTATIC POTENTIAL} \]

\[ E = \nabla V \]

\[ V_p = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho dV}{r} + \frac{1}{4\pi\varepsilon_0} \int \sigma dS \]

\[ m = \frac{q}{\hbar} \implies \text{DIPOLE MOMENT} \]

\[ V = \frac{1}{4\pi\varepsilon_0} \int \frac{n dS}{r} \implies \text{ELECTROSTATIC POTENTIAL FROM DIPOLE MOMENT} \]

\[ q = \int_S \frac{\varepsilon_0 E \cdot n dS}{E} \implies \text{GAUSS' FLUX THEOREM} \]

\[ \psi(E) = \frac{1}{2} \implies \text{TENSION IN (HOMO/ISO) MEDIUM TWIXT FORCE LINES} \]

\[ \phi(E) = \frac{1}{2} \implies \text{FORCE TWIXT FORCE LINES IN (HOMO/ISO) MEDIUM ON A CONDUCTOR, } \sigma = D = E = \text{SURFACE CHARGE DENSITY} \]

II. CAPACITORS, DIELECTRIC, SYSTEMS OF CONDUCTORS

\[ C = \frac{Q}{V} \implies \text{CAPACITANCE} \]

\[ \varepsilon_0 \text{ FOR TWO CONCENTRIC SPHERES} \]

\[ \frac{2\pi\varepsilon_0}{\ln(b/a)} \text{ FOR TWO CONCENTRIC CYLINDERS} \]

\[ \varepsilon_0 \text{ FOR TWO PARALLEL PLATES} \]

\[ W = \frac{1}{2} \varepsilon_0 Q_1 V_1 \implies \text{ENERGY IN CHARGED CAPACITOR} = \frac{1}{2} CV^2 \]

\[ \varepsilon_0 Q_1 V_1' \implies \text{GREEN'S RECIPROCATION THEOREM} \]

\[ Q_i \implies V_i \implies Q_i + Q_i' \implies V_i + V_i' \implies \text{FIELD SUPERPOSITION} \]

\[ Q = \frac{V}{V_i} q \implies \text{CHARGE, } Q_i \text{ INDUCED BY PT. CHARGE } q \]

\[ V = \frac{q}{\varepsilon_0} \implies \text{POT. TO WHICH } E \text{ IS RAISED/PLACED 1 COUL @ S} \]

\[ Q = \frac{C}{V} \implies \text{CAPACITANCE} \]
III. General Theorems
\[ \oint_S \mathbf{A} \cdot \mathbf{n} \, ds = \int_V \nabla \cdot \mathbf{A} \, dV \ \Rightarrow \text{Gauss' Theorem} \]
\[ \oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S (\mathbf{n} \times \mathbf{F}) \cdot d\mathbf{s} \ \Rightarrow \text{Stokes' Theorem} \]
\[ \nabla \cdot \mathbf{V} = 0 \ \Rightarrow \text{Poisson's Eq.} \]
\[ \nabla \cdot \mathbf{E} = 0 \ \Rightarrow \text{LaPlace's Eq.} \]

IV. Two Dimensional Potential Distributions
\[ E = \frac{q}{4\pi r} \quad V = \frac{-q}{2\pi \ell} \cdot r + c \]
\[ V = R(r) \Phi(\phi) \Omega(\varphi) = R(r) \Omega(\varphi) \]
\[ \Theta_n = \left\{ \begin{array}{ll}
\Lambda \theta + B & n \neq 0 \\
\Lambda \theta & n = 0
\end{array} \right. \]
\[ R_n = \left\{ \begin{array}{ll}
C r^n + D r^{-n} & n \neq 0 \\
C \ell n r + D & n = 0
\end{array} \right. \]

Conductor
\[ q \quad \begin{array}{c}
q' = -q \quad q' = \frac{r_0 - r}{r_0 + r} \quad q' = \frac{1 - \frac{1}{k}}{1 + \frac{1}{k}} q
\end{array} \]

Image
\[ q' = -q \quad q' = \frac{2}{1 + k} q \]

\[ W = \mathbf{V} + \mathbf{V} \]
\[ -\frac{\partial W}{\partial r} \text{ gives } x \frac{\partial}{\partial x} \text{ components of } \mathbf{E} \]

Gauss' Flux Theorem (V is Potential)
\[ Q = \text{Flux} = -\oint_S E \cdot d\mathbf{n} = -\oint_S \mathbf{u}_1 \cdot d\mathbf{n} ds = \int_{S_2} \mathbf{u}_2 \cdot d\mathbf{s} = \mathbf{E}(V_2 - V_1) \]
\[ C = V_2 - V_1 = \int_{S_2} \mathbf{E} \cdot d\mathbf{s} = E(U_2 - U_1) \]
V. THREE DIMENSIONAL POTENTIAL DISTRIBUTIONS

\[ q' = \frac{-aq}{b} \]

THREE DIMENSIONAL INVERSION

\[ R' = k^2 \quad k = \text{radius of inversion} \]

\[ f(r, \theta, \phi) \leftrightarrow f\left(\frac{k^2}{r}, \theta, \phi\right) \]

\[ \frac{r'}{r} = \frac{q}{q'}, \quad \frac{\theta'}{\theta} = \frac{k^2}{r'^2} \quad q_{\lambda', \mu'} \leftrightarrow q_{\lambda', \mu'} \]

LEGENDRE POLY. EXPANSION OF \( \frac{1}{R} \)

\[ \frac{1}{R} |_{b > a} = \frac{1}{k^2} \sum_{n=0}^{\infty} \frac{(a/b)^n}{n!} P_n(\mu) \]

\[ \mu = \cos \theta \]
ADVANCED FIELDS I, TEST #2 GRAM SHEET.

VI. ELECTRIC CURRENT
\[ I = \frac{dQ}{dt}, \quad \mathbf{j} = \frac{d\mathbf{E}}{dt} \]
\[ \mathbf{j} = \nabla \mathbf{E} = \frac{1}{\mu} \nabla \mathbf{E} \iff \text{OHM'S LAW} \]
\[ \rho = I^2 R \iff \text{JOULE'S LAW} \]
\[ \mathbf{R} \mathbf{C} = \mathbf{V} \mathbf{E} \]

VII. MAGNETIC INTERACTION OF CIRCUITS.
\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mathbf{\mu} \mathbf{I} \]
\[ \mathbf{A} = \frac{\mathbf{\mu}}{4\pi} \int \mathbf{I} d\mathbf{s} / \mathbf{r}, \quad \mathbf{B} = \nabla \times \mathbf{A} \]
\[ \mathbf{B} = \mathbf{\mu} I / 2\pi Q \iff \text{BIOT AND SAVART'S LAW} \]
\[ \mathbf{F} = I \int \mathbf{d}s \times \mathbf{B} \]
\[ \mathbf{M} = (\mathbf{\mu}_0 - \mathbf{\mu}) \mathbf{B} = \mathbf{K} \mathbf{B} \iff \text{MAGNETIZATION} \]

VIII. ELECTROMAGNETIC INDUCTION
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \iff \text{FARADAY'S LAW} \]
\[ \mathbf{W} = \frac{1}{2} \mathbf{B}^2 \iff \text{ENERGY DENSITY IN B FIELD} \]
\[ \mathbf{M}_{12} = \frac{1}{4\pi} \int \mathbf{B} \cdot \mathbf{d}s_1 \times \mathbf{d}s_2 / r \iff \text{MUTUAL INDUCTANCE} \]
\[ L_{11} = \frac{N^2}{I_1} \iff \text{SELF INDUCTANCE} \]

IX. MAGNETISM
\[ \mathbf{M} = \mathbf{K} \mathbf{H}, \quad \mathbf{K} = \text{SUSCEPTIBILITY} \]

X. EDDY CURRENTS
\[ \nabla \times \mathbf{B} = \mathbf{\mu} \mathbf{j} \iff \text{AMPERE'S LAW} \]

XI. PLANE ELECTROMAGNETIC WAVES
MAXWELL'S EQ.
LORENTZ 1/4 COULOMB GAUGES
HERZ I VECTOR
POYNTING VECTOR
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1.05. Electric Field Intensity
1.06. Electrostatic Potential
1.07. Electric Dipoles
1.07.1. Interaction of Dipoles
1.08. Lines of Force
1.09. Equipotential Surfaces
1.10. Gauss' Flux Theorem
1.12. Potential of a Double Layer
1.13. Tubes of Force
1.14. Stresses $\times$ Tension
1.15. Gauss' Law for Non-Homo Isotropic Media
1.16. Boundary Conditions $\times$ Stresses on Conductor Surface

II. Capacitors, Dielectrics, Systems of Conductors

2.00. Uniqueness Theorem
2.01. Capacitance
2.03. Spherical Capacitor
2.04. Cylindrical "
2.05. Parallel Plate "
2.07. Energy of a Charged Capacitor
2.08. Energy in an $E$ Field
2.12. Green's Reciprocation Theorem
2.13. Field Superposition
2.14. Induced Charges on Earthed Conductors
2.15/16. Self and Mutual Elastance/Capacitance
2.17. Electric Screening
2.19. Energy in a Charged System
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3.01. STOKES' THEOREM
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3.03. ORTHONORMAL CURVILINEAR COORDINATES
3.05. LAPLACE'S EQUATION IN VARIOUS COORDINATE SYSTEMS

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4.01. CIRCULAR HARMONICS
4.02. HARMONIC EXPANSION OF LINE CHARGE POTENTIAL
4.03. CONDUCTOR OR DIELECTRIC CYLINDER IN UNIFORM FIELD
4.04. DIELECTRIC CYLINDER, METHOD OF IMAGES
4.05. IMAGE IN CONDUCTING CYLINDER
4.06. "" IN PLANE FACE OF DIELECTRIC OR CONDUCTOR (INTERSECTING PLANE)
4.09. COMPLEX QUANTITIES
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5.04. Method of Images. Conducting Plane
5.06. Image of Spherical Conductor
5.09. Inversion in 3D. Geometrical Properties
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5.15. Surface Zonal Harmonics. Legendre's Eq.
5.151. Series Solution of Legendre's Eq.
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5.153. \"\" Coefficients. Inverse Distance
5.154. Recurrence Formula's for Legendre Poly.
5.155. Integral of Product of Legendre Poly.
5.156. Expansion of Function in Legendre Poly.
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5.16. Potential of a Charged Ring
5.17. Charged Ring in Conducting Sphere

5.29. Laplace's Eq. in Cylindrical Coordinates
5.291. Bessel's Eq. \& Bessel Functions
5.293. Solution of Bessel's Eq.
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6.03. HEATING EFFECTS OF ELECTRIC CURRENT
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7.02. MAGNETIC VECTOR POTENTIAL. UNIFORM FIELD
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7.09. " " OF CIRCULAR LOOP
7.11/2. FIELD/ZONAL CURRENTS IN SPHERICAL SHELL
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7.15. FIELD IN A HOLE IN A CYLINDER
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7.20. VECTOR POTENTIAL ½ MAGNETIZATION
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8.01. MUTUAL ENERGY OF TWO CIRCUITS
8.02. ENERGY IN A MAGNETIC FIELD
8.03. MUTUAL INDUCTANCE
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9.01. MAGNETIC SUSCEPTIBILITY
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10.00. Induced Currents in Extended Conductors
10.01. Solution for Vector Potential for Eddy Cur.
10.02. Steady State Skin Effect
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XI. Plane Electromagnetic Waves
11.00. Maxwell's Field Equations.
11.02. Poynting Vector.
BASIC IDEAS OF ELECTROSTATICS

1.05 Electric Field Intensity
\[ \vec{E}_p = \frac{-1}{4\pi \varepsilon} \sum_{i=1}^{n} \frac{q_i}{r_i^3} \vec{r}_i \]

1.06 Capacitance of Homogeneous Material
\[ dV = \vec{E} \cdot d^2 \]

"In a Coulomb Field:
\[ dV = \frac{-q_{\text{total}} \Theta}{4\pi \varepsilon r_0^2} \, ds \]
\[ V_p = \int_{0}^{r_p} dV = \frac{-q_{\text{total}}}{4\pi \varepsilon} \int_{r_0}^{r_p} \frac{dr}{r^2} = \frac{q_{\text{total}}}{4\pi \varepsilon} \left( \frac{1}{r_p} - \frac{1}{r_0} \right) = \frac{q_{\text{total}}}{4\pi \varepsilon r_0} \quad (r_0 \gg r_p) \]

For a number of charges,
\[ \vec{E} = -\vec{\nabla}V \]

For charge densities:
\[ V = \frac{1}{4\pi \varepsilon} \left[ \int_S \frac{\rho}{r} \, dS + \int_V \frac{\rho}{r} \, dV \right] \]

1.07 Electric Dipoles
\[ P(x, y, z) \]

\[ \vec{p}(x, y, z) = -q \cdot \hat{n} \]
\[ (x_0, y_0, z_0) \]
\[ (x, y, z) \]
\[ q \cdot \hat{m} \]
\[ \vec{m} \]

Let \( q \to \infty \) and \( \hat{n} \to 0 \) \( \Rightarrow q \cdot \hat{n} = \frac{m}{m' \cdot r} \) remains constant
\[ \Rightarrow V = \frac{m}{4\pi \varepsilon} \cdot \frac{\delta}{\delta \hat{r}} \left( \frac{1}{r} \right) \text{coulomb} \quad \frac{q_{\text{total}}}{4\pi \varepsilon r^3} \]

Dipole
\[ \vec{p} \]

Translational Force: \( \vec{E} \) acting on charges
\[ \vec{F} = (\vec{m} \cdot \vec{\nabla}) \vec{E} \quad (=0 \text{ in uniform field}) \]

Torque on Dipole:
\[ \vec{T} = \vec{m} \times \vec{E} \]
1.071 INTERACTION OF DIPOLES

Potential Energy of Dipole Work to Bring Charges into Plane

\[ V_2 - V_1 \Rightarrow W = q \left( V_1 - V_2 \right) \]

\[ \kappa \neq 0 \]

\[ \frac{q}{r_1 r_2} \cdot \frac{\delta V}{\delta s} = 1 \text{MV} \cdot \frac{\delta V}{\delta s} = \left( \frac{\delta}{\delta s} \right) V \]

1.08 LINES OF FORCE

\[ ds = \lambda \vec{E} \Rightarrow dx = \lambda E_x ; \quad dy = \lambda E_y ; \quad dz = \lambda E_z \]

\[ \frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \]

EXAMPLE:

\[ q \quad \frac{q}{-a} \quad \frac{q}{q} \]

\[ 4\pi \epsilon \vec{E}_x = \left[ \frac{q (x-a)}{r^3} + (x-a)^2 \right]^{\frac{1}{2}} = \left[ \frac{-q}{r^3 (1+v^2)^{\frac{3}{2}}} \right]^{\frac{1}{2}} \]

\[ \frac{4\pi \epsilon E_y}{-y^3} = \frac{y}{y^2 (1+v^2)^{\frac{3}{2}}} = \frac{y^2}{y^2 (1+v^2)^{\frac{3}{2}}} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{E_y}{E_x} = \frac{(1+v^2)^{\frac{3}{2}}}{(1+u^2)^{\frac{3}{2}}} \]

\[ \frac{dv}{du} - \frac{udv}{dv} = \frac{U(1+u^2)^{\frac{3}{2}} - V(1+u^2)^{\frac{3}{2}}}{X+q} \]

\[ \Rightarrow \frac{dv}{du} = \left( \frac{1+u^2}{1+u^2} \right)^{\frac{3}{2}} \Rightarrow \frac{U}{V} = \frac{X-a}{X-a} = \frac{X-a}{X-a} \]

OR \[ \sqrt{(x+a)^2 + y^2} = \sqrt{(x-a)^2 + y^2} = C \]
1.090 EQUAPOTENTIAL SURFACES

\[ \nabla \mathbf{V} = \text{Const (perpendicular to F lines)} \]

If \( F \) lines are from \( F(x, y) = C \),

and \( V \) lines from \( V(x, y) = C' \),

then \( F(x, y) = \frac{\mathbf{V}}{\nabla V(x, y)} \)

\[ \Rightarrow \text{EQUILIBRIUM POINTS (LINES)} \]

Place at which an equapotential surface crosses itself at least twice.

\( \nabla \mathbf{V} \) (and thus \( \mathbf{E} = \frac{\mathbf{F}}{F} \)) vanish there.

1.10 GAUSS'S FLUX THEOREM

\[ d\Sigma = ds \cos \alpha \]

\[ E_n = \frac{q}{4\pi \epsilon r^3} = \frac{q \cos \alpha}{4\pi \epsilon r^2} \]

\[ d\mathbf{N} = \epsilon E_n ds = \frac{q \cos \alpha ds}{4\pi \epsilon r^2} = \frac{q d\Sigma}{4\pi \epsilon r^2} \]

\[ \therefore d\mathbf{N} = \frac{q d\Omega}{4\pi} \Rightarrow 4\pi \int_S d\mathbf{N} = q \int_S d\Sigma \]

\[ \Rightarrow \epsilon \int_S E \cdot n ds = q \]

1.120 POTENTIAL OF ELECTRIC DOUBLE LAYER.

One may obtain the potential of a dipole moment by differentiating the potential of a single charge in the direction of the dipole moment.

\[ dV = \frac{dr}{4\pi \epsilon} ds = \text{potential due to} \ ds \]

\[ \therefore \frac{dr}{4\pi \epsilon} ds \frac{1}{r} = \text{equivalent dipole potential} \]

\[ \Rightarrow V = \frac{1}{4\pi \epsilon} \int_S \frac{d\phi}{r} ds = \frac{1}{4\pi \epsilon} \int_S \frac{\mathbf{P} \cdot \mathbf{r}}{r^3} ds \]

Is potential due to dipole double layer with strength \( \mathbf{P} \).

Now \( d\mathbf{N} = \frac{\mathbf{P} \cdot \mathbf{r}}{r^3} ds \Rightarrow V = \frac{1}{4\pi \epsilon} \int_S \mathbf{P} \cdot d\mathbf{N} \)

If \( \mathbf{P} \) is uniform, with strength \( \mathbf{P} \):
1.13 TUBES OF FORCE

\[ \nabla \Phi = \mathbf{F} \]

\[ \nabla \Psi = \mathbf{F} \]

\[ \Rightarrow N = \text{FLUX LINES} = \mathbf{DS}_1 = \mathbf{DS}_2 \]

1.14 STRESSES AND TENSION (WIXT LINES IN \( \mathbf{E} \) FIELD)

\[ \Phi(E) = \text{TENSION} = \frac{eE^2}{2} \]

\[ \Psi(E) = \text{FORCE} = -\frac{eE^2}{2} \]

Since \( \frac{d\Phi}{d\Psi} \) depend only on \( \Phi, E \), the origin or shape of the field is immaterial

1.15 GAUSS'S LAW FOR NON-HOMOGENEOUS ISOTROPIC MEDIA

\[ \int_{S'} \mathbf{E}' \cdot \hat{n}' dS' = \int_S \mathbf{E} \cdot \hat{n} dS \]

\[ \Rightarrow \int_S \mathbf{E} \cdot \hat{n} dS = \int_{S'} \mathbf{E}' \cdot \hat{n}' dS' \]

1.16 BOUNDARY CONDITIONS T STRESSES ON SURFACE OF CONDUCTORS

\[ \Rightarrow \text{on and in a conductor, the potential is constant} \]

\[ \nabla^2 \Phi = 0 \]

\[ \Rightarrow \text{TENSION (FORCE)} = \frac{D^2}{2E} = \frac{\sigma^2}{2E} \]

(INDEPENDENT OF CHARGE POLARITY)
CAPACITORS, DIELECTRICS, SYSTEMS OF CONDUCTORS

2.00 UNIQUENESS THEOREM

Only one distribution of charge will give 1) A specified potential to every conductor in \( \vec{E} \) field
2) A specified total charge to each conductor in an \( \vec{E} \) field

2.01 CAPACITANCE

\[ C = \text{CAPACITANCE (FARADS)} \]
\[ S = \text{ELASTANCE (DARRAFES)} \]
\[ Q = CV \text{ \ AND \ } V = SQ \]

2.03 SPHERICAL CAPACITORS

Apply Gauss's flux theorem to sphere of radius \( r \):

\[ \int_S \vec{E} \cdot d\vec{s} = (4\pi r^2) \vec{E} = Q \]
\[ \Rightarrow \vec{E} = -\frac{Q}{8\pi r} = \frac{Q}{4\pi\varepsilon_0 r^2} \]

Thus \( \Delta V = V_a - V_b = \frac{Q}{4\pi\varepsilon_0} \int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0 b - a} \)
\[ \Rightarrow C = \frac{\Delta V}{Q} = \frac{4\pi\varepsilon_0 b - a}{Q} \]

We can let \( b \to \infty \) to get \( C \) for a single hollow sphere: \( C = 4\pi\varepsilon_0 \)

Note: if \( b \) was not grounded, this extra capacitance would be added.

2.04 CYLINDRICAL CAPACITORS

\[ Q = \int_S \vec{E} \cdot d\vec{s} = (2\pi r) \vec{E} \]
\[ \Rightarrow \vec{E} = -\frac{Q}{2\pi r} \]
\[ \Delta V = V_a - V_b = -\frac{Q}{2\pi\varepsilon_0} \int_a^b \frac{dr}{r} = -\frac{Q}{2\pi\varepsilon_0 \ln(b)} \]
\[ \Rightarrow C = \frac{\Delta V}{Q} = \frac{2\pi\varepsilon_0}{\ln(b/a)} \]
2.05 @ PARALLEL-PLATE CAPACITOR

\[ \phi = \frac{Q}{A}, \quad D = \varepsilon E = \sigma = \varepsilon \frac{dV}{dx} \]
\[ \Delta V = \frac{\sigma}{\varepsilon} \int_{0}^{q} dx = \frac{\sigma - q}{\varepsilon} \Rightarrow C = \frac{\varepsilon A}{q} \]

2.07 @ ENERGY OF A CHARGED CAPACITOR

CONSIDER A FIELD OF N POINT CHARGES.

THE WORK REQUIRED TO PUT THE \( j \)TH CHARGE IN PLACE IS

\[ W_j = q_j V_j = \frac{q_j^2}{4\pi \varepsilon} \sum_{i=1}^{j-1} \frac{q_i}{r_{ij}}; \quad i \neq j \]

\[ \Rightarrow W = \frac{1}{2} \sum_{j=1}^{N} q_j V_j \]

IF CHARGES ARE ALL ON SAME CONDUCTOR, \( W = \frac{1}{2} Q V_0 \)

ON A CONDUCTOR WITH CAPACITANCE \( C \), \( W = \frac{1}{2} \frac{Q^2}{C} - \frac{1}{2} CV^2 \)

ON A CAPACITOR @ Q @ Q; \( W = \frac{1}{2} Q (V_1 - V_2) \)

2.08 @ ENERGY IN AN \( E \) FIELD

\[ \oint ds (\varepsilon_0 E_s) = -Eds \]
\[ \Rightarrow \nabla \cdot \mathbf{E} = \frac{\varepsilon_0}{\varepsilon} \sum_{j=1}^{N} q_j \]

2.12 @ GREEN'S RECIPROCATION THEOREM

\[ \frac{\delta}{\delta V_i} = \sum_{i=1}^{N} q_j \]

\( q_i \) \& \( V_i \) ARE THE CHARGE \& POTENTIAL

OF THE \( i \)TH CONDUCTOR

2.13 @ FIELD SUPERPOSITION

\( q_i \rightarrow V_i \) THEN \( q_i + q \rightarrow V_i + V_i' \)

EX. \( q_i \rightarrow q_i \)

FIND VOLTAGE ON 5TH SPHERE

1. VOLTAGE INSIDE A SHELL IS

\[ V_{in} = \frac{Q}{4\pi \varepsilon r} \]

2. OUTSIDE: \( V_o = \frac{Q}{4\pi \varepsilon r_o} \)

\[ \Rightarrow V_r = \frac{1}{4\pi \varepsilon} (Q_1 + Q_2 + \ldots + Q_S) r_s^{-1} + \frac{Q_{S+1} + \ldots + Q_n}{4\pi \varepsilon r_o} \]


2.40 INDUCTIVE CHARGES ON EARTHEO CONDUCTORS

\[ q = \frac{V_p}{V_p' \Rightarrow q = \frac{V_p}{V_p'}} \]

EX. CONSIDER \( q = r \)

\[ V' = \frac{q}{\text{area}} \quad V_p' = \frac{q}{\text{per}} \Rightarrow q = \frac{q}{r} \]

EX.

\[ \frac{q}{r} \]

\[ \frac{q}{r} \]

\[ Q_1 + Q_2 + V_1' + qV_p' = 0 \]

\[ \frac{Q_1}{V_1'} = \frac{V_2' - V_1'}{V_2' - V_1'} = \frac{Q_2}{V_2'} \]

- CYLINDERS: \( Q_1 = Q_2 \cdot \left( \frac{r_1}{r_2} \right) \)

- SPHERES: \( Q_1 = \frac{r_1}{r_2} \cdot \left( \frac{r_2}{r_1} \right) \)

- PLATES: \( Q_1 = \frac{b-a}{a+b} \); \( Q_2 = \frac{a-b}{a+b} \)

2.15 SELF \& MUTUAL ELASTANCE

\[ \begin{bmatrix} V_1' \\ V_2' \\ V_p' \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix} \begin{bmatrix} Q_1' \\ Q_2' \\ Q_n' \end{bmatrix} \quad S_{pp} = S_{pp'} > 0 \]

\( S_{pp} \) = POTENTIAL TO WHICH \( V \) IS RAISED WHEN A UNIT CHARGE IS PLACED ON S MUTUAL ELASTANCE.

\( S_{pp} \) = SELF ELASTANCE

2.16 MUTUAL AND SELF CAPACITANCE

\[ \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \]

\( C_{pp} = \text{SELF CAPACITANCE} > 0 \)

\( C_{pp} = C_{pp} = \text{MUTUAL CAPACITANCE} \neq 0 \)
2.19 Energy in a Charged System

\[ W = \frac{1}{2} \left( c_{1i} V_i^2 + 2c_{12} V_1 V_2 + c_{22} V_2^2 + \ldots \right) \]

\[ W_\rho = \frac{1}{2} \int \rho \mathbf{E} \cdot d\mathbf{V} \]

2.20 Forces and Torques on Charged Conductors

\[ \mathbf{F} = \int \mathbf{E} \cdot d\mathbf{V} \]
III. GENERAL THEOREMS

3.00. GAUSS'S THEOREM

$$\int_S \mathbf{A} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{A} \, dV$$

3.01. STOKES'S THEOREM

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

3.02. POISSON'S LAPLACE EQUATIONS

FROM GAUSS:

$$\iiint_V \nabla \cdot \mathbf{U} \, dV = q = \iiint_V \rho \, dV$$

$$\Rightarrow \nabla \cdot \mathbf{U} = \frac{\partial q}{\partial V} = \rho$$

$$\mathbf{U} = \mathbf{E} = -\varepsilon \nabla V \Rightarrow \nabla \cdot \varepsilon \nabla V = -\rho \leq \text{POISSON'S EQ}$$

FOR NO. CHARGE $$\Rightarrow \nabla \cdot \varepsilon \nabla V = 0 \leq \text{LAPLACE'S EQ}$$

3.03. ORTHOGONAL CURVILINEAR COORDINATES

$$ds = h_i du_i$$

$$\frac{\partial s}{\partial u_i} = h_i$$

3.05. LAPLACE'S EQ. IN VARIOUS COORDINATE SYSTEMS

- SPHERICAL POLAR COORDINATES

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

- CYLINDRICAL COORDINATES

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{\partial z} \left( \frac{\partial}{\partial z} \left( \varepsilon \frac{\partial V}{\partial z} \right) \right) = 0$$
IV. TWO DIMENSIONAL POTENTIAL DISTRIBUTIONS

4.00 FIELD POTENTIAL IN TWO DIMENSIONS

\[ E = \frac{Q}{2\pi \epsilon \text{ln} r} \]

\[ V = \frac{-Q}{2\pi \epsilon \text{ln} r} + C \]

4.01 CIRCULAR HARMONICS

\[ V = R(\rho) \Theta(\phi) \frac{\partial}{\partial \rho} (\rho \frac{\partial}{\partial \rho} \Theta) \]

LAPLACE'S EQUATION BECOMES:

\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \Theta + \frac{\partial^2}{\partial \phi^2} \Theta \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \Theta = -n^2 \Theta \]

Let \( \frac{\partial^2}{\partial \phi^2} = -n^2 \); \( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} = n^2 \frac{R}{\rho^2} \)

GIVES:

\[ \Theta_n = \begin{cases} \lambda \cos n\theta + B \sin n\theta & ; n \neq 0 \\ \lambda \theta + B & ; n = 0 \end{cases} \]

\[ R_n = \begin{cases} C r^n + D r^{-n} & ; n \neq 0 \\ c \ln r + D & ; n = 0 \end{cases} \]

\( n \) IS "DEGREE OF THE HARMONIC"

\( \{ \)

\( n = 0 \Rightarrow V = (A \theta + B) (C \ln r + D) \)

\( \{ \)

\( n \neq 0 \Rightarrow V = (A \cos n\theta + B \sin n\theta) (C r^n + D r^{-n}) \)

SOLUTIONS TO LAPLACE'S EQUATION

\[ V = \sum \Theta_n R_n \quad \text{OR} \quad V = \int \mathcal{F}(n) \Theta_n R_n \, dn \]

4.02

\[ 4\pi V = -2q \ln r \quad A = -q \ln \left[ r^2 + r_0^2 - 2rr_0 \cos (\Theta - \Theta_0) \right] \]

\[ = -q \ln r - q \ln \left[ 1 - \frac{r_0}{r} e^{i \Theta - \Theta_0} \right] \]

IT TURNS OUT THAT

\[ V = \begin{cases} \frac{Q}{4\pi \epsilon} \sum_n n \left( \frac{r_0}{r} \right)^n (\cos n\Theta_0 \cos n\Theta + \sin n\Theta_0 \sin n\Theta) \ln r & ; r_0 > r \\ \frac{Q}{4\pi \epsilon} \sum_n n \left( \frac{r_0}{r} \right)^n (\cos n\Theta_0 \cos n\Theta + \sin n\Theta_0 \sin n\Theta) \ln r & ; r_0 \leq r \end{cases} \]
4.03 Conducting or Dielectric Cylinder in Uniform Field

Initially: \( V = E x = E r \cos \theta \)

The field \( \mathbf{\nabla} \cdot \mathbf{E} = 0 \) must be zero:

\[ V_0 = E r \cos \theta + \sum_{n=1}^{\infty} A_n r^{-n} \cos n\theta \]

Inside dielectric: \( V_d = \sum_{n=1}^{\infty} (B_n r^{-n} + C_n r^{-n}) \cos n\theta \)

Boundary conditions @ \( r = b \) are:

\[ \frac{\partial V_0}{\partial r} = K \frac{\partial V_d}{\partial r} \quad \text{and} \quad V_0 = V_d. \]

This requires \( A_n = B_n = 0 \) \( \forall \ n \neq 1 \)

\[ A_1 = -\frac{E b^2}{(k+1) b^2 + (k-1) q^2} \]

\[ B_1 = 2 \frac{E b^2}{(k+1) b^2 + (k-1) q^2} \]

\[ C_1 = -\frac{2 E a^2 b^2}{(k+1) b^2 + (k-1) q^2} \]

\[ \Rightarrow V_0 = (E r + A_1) \cos 1\theta \quad ; \quad V_d = (B_1 r + C_1) \cos 1\theta \]

Take \( k = 1 \)

\[ V_0 = E (r - \frac{a^2}{r}) \cos 2\theta \]

\[ V_d = \frac{2 E}{k+1} \cos 2\theta \]

Let \( a \to 0 \Rightarrow \) Dielectric Cylinder

\[ V_0 = E (r - \frac{k+1}{k+1} \frac{b^2}{r}) \cos 2\theta \]

\[ V_d = \frac{2 E}{k+1} \cos 2\theta \]
4.04 DIELECTRIC CYLINDER: METHOD OF IMAGES

\[
\begin{align*}
\text{WITHOUT DIELECTRIC: } V &= \frac{q}{4\pi \varepsilon} \left[ \frac{\sin \frac{\theta}{b}}{\frac{\sin \frac{\theta}{b}}{\frac{\theta}{b}}} \frac{\cos n\theta}{b} \right] - \frac{q}{\pi \varepsilon \varepsilon_0} \frac{1}{\frac{\theta}{b}} \frac{\cos n\theta}{b} \frac{\cos \theta}{b} \frac{\cos \theta}{b} \\
\text{FIELD SYMMETRIC WITH X AXIS: } V_0 &= \frac{q}{4\pi \varepsilon} \left[ \sum_{n=1}^{\infty} \frac{\sin \frac{\theta}{b}}{\frac{\sin \frac{\theta}{b}}{\frac{\theta}{b}}} \frac{\cos n\theta}{b} \right] \\
\text{FIELD INSIDE DIELECTRIC MUST BE FINITE: } V_i &= \frac{q}{2\pi \varepsilon \varepsilon_0} \left( \sum_{n=1}^{\infty} B_n r^n \cos n\theta + C_1 \right) \\
1. V_0 &= V_i \text{ AT } r=0 \\
&\Rightarrow \frac{1}{n} \left( \frac{a}{b} \right)^n = \frac{A_n}{a^n B_n} \quad ; \quad C_2 = -b + C_1 \\
2. \frac{\partial V_0}{\partial r} &= \frac{\partial V_i}{\partial r} \Rightarrow 1 < \frac{\partial V_i}{\partial r} = \frac{\partial V_i}{\partial r} \text{ AT } r=0
\end{align*}
\]

\[
\begin{align*}
\text{COMBINING GIVES: } A_n &= \frac{1 - k}{1 + k} \frac{a^n b^n}{n^2} \quad ; \quad B_n = b \left( 1 + k \right) n b^n \\
\text{LET } C_1 &= 0 = -\frac{1 - k}{1 + k} \ln r + \frac{1 + k}{1 + k} \ln \frac{q}{b} \\
\text{THE POTENTIAL OUTSIDE } (V_o) \text{ MAY BE FOUND VIA }
\end{align*}
\]

\[
\begin{align*}
q' &= \frac{1 - k}{1 + k} q \\
q'' &= \frac{2}{1 + k} q
\end{align*}
\]
4.05 Image in Conducting Cylinder

May get image, let $k \to 0$

\[ a - \frac{q^2}{b} = \frac{-q}{a} \]

4.06 Image in Plane Face of Dielectric or Conductor

(Intersecting Conducting Planes)

Let radius ($a$) of cylinders $\to \infty$ keeping $d = b - a$ constant.

\[ a = \frac{b^2 - a^2}{b} \to d \]

For a Conductor:
\[ \begin{array}{ccc} -q & +q & 0 \\ -b & a & b \\ \end{array} \]

For a Dielectric:
\[ \begin{array}{ccc} +q'' & -q'' & +q'' \\ -b & 0 & b \\ \end{array} \]

For conducting planes intersecting

@ origin @ angle \[ \frac{\pi}{m} \]

\[ \Theta = \frac{2\pi}{m} + \Theta_0, \frac{4\pi}{m} + \Theta_0, \ldots, \frac{2(m-1)\pi}{m} + \Theta_0 \]

\[ \frac{2\pi}{m} = \Theta_0, \frac{4\pi}{m} - \Theta_0, \ldots, 2\pi - \Theta_0 \]
4.09 COMPLEX QUANTITIES
\[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \Leftrightarrow \text{LAPLACE'S EQ. IN TWO DIMENSIONS} \]
\[ U = \Phi(x+jy) + \Psi(x-jy) \text{ is a REAL SOLUTION} \]
\[ \Rightarrow \Phi(x+jy) = U + jV \quad \Psi(x-jy) = U - jV \]
\[ U = 2U ; \quad V = 2V, \quad W = U + jV \]
\[ W = U + jV \text{ satisfies LAPLACE's EQ. (} W = f(z) \text{)} \]
U AND V ARE "CONJUGATE FUNCTIONS"

4.10 THE STREAM FUNCTION
\[ \frac{\partial U}{\partial x} = -\frac{\partial V}{\partial y} \quad \text{AND} \quad \frac{\partial V}{\partial x} = \frac{\partial U}{\partial y} \]
\[ \Rightarrow V = \text{const} \quad \frac{\partial U}{\partial x} = \text{const. intersect orthogonally} \]

4.11 ELECTRIC FIELD INTENSITY/ELECTRIC FLUX
\[ \frac{\partial W}{\partial z} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - j \frac{\partial U}{\partial y} \]
\[ \text{IF } V \text{ IS POTENTIAL, THEN} \]
a. \[ \frac{\partial W}{\partial z} \] gives \( x \) and \( y \) components of \( E \) field
b. \[ \frac{\partial W}{\partial z} \] gives \( |E| \)
\[ \left( \frac{\partial W}{\partial z} \right) = \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - j \frac{\partial U}{\partial y} \] (for potential \( U \))
\[ \left( \frac{\partial W}{\partial z} \right) = \frac{\partial V}{\partial x} = \frac{\partial U}{\partial y} \] (for potential \( V \))

CONSIDER GAUSS' FLUX THERE; (HERE, \( V \) IS POTENTIAL)
\[ \text{FLUX} = -e \int_{S} E \cdot d\mathbf{n} = -e \int_{U_{2}}^{U_{1}} \frac{\partial U}{\partial y} ds = e \int_{U_{2}}^{U_{1}} \frac{\partial U}{\partial x} ds \]
\[ = e(U_{2} - U_{1}) \]
IF OUR SURFACE WAS CLOSED, \( e(U_{2} - U_{1}) = Q \)

DETERMINING CAPACITANCE
\[ Q \]
\[ V_{2} \]
\[ Q = \text{FLUX PER UNIT LENGTH} \]
\[ C = \frac{Q}{V_{2} - V_{1}} = \frac{e(U)}{[V_{2} - V_{1}]} \]
\[ \text{FIELD ENERGY} = \frac{1}{2} C \left| V_{2} - V_{1} \right|^{2} \]
\[ = \frac{1}{2} e \left| U_{2} - U_{1} \right| \left| V_{2} - V_{1} \right| \]
V. THREE DIMENSIONAL POTENTIAL DISTRIBUTIONS

5.04. METHOD OF IMAGES. Conducting planes.

\[ \frac{q}{V} = \frac{q'}{V'} \]

\[ \sigma = \frac{-aq}{\pi r^2} \]

5.06. IMAGE IN SPHERICAL CONDUCTOR

\[ q' = \frac{-aq}{b} \]

If the sphere is ungrounded (at potential \( V \)), we may add a charge \( q = 4\pi \varepsilon_0 V \) at the origin.

5.09. INVERSION IN THREE DIMENSIONS, GEOMETRICAL PROPERTIES

\[ r r' = K^2 \]; \( K \) = radius of inversion

\[ f(r, \theta, \phi) \leftrightarrow f\left(\frac{K^2}{r}, \theta, \phi\right) \]

5.10. INVERSE OF POTENTIAL AND IMAGING SYSTEMS.

\[ \frac{r r'}{r} = K^2 \]

\[ \frac{V}{V'} = \frac{q r'}{q' r} \]

\[ \sigma' = \frac{K^3}{r'^3} \]

\[ q' \leftrightarrow q_i \]

\[ q' \leftrightarrow \rho_i \]
5.12. **Spherical Harmonics**

In spherical co-ordinates, Laplace's equation is

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{\sin \theta}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

Assume solution: \[ V = R(\theta, \phi) \quad \Rightarrow \quad R = A r^n + B r^{-(n+1)} \]

and \[ V = (A r^n + B r^{-(n+1)}) \sin \theta \]

5.14. **Differential Eqs. for Surface Harmonics**

Surface harmonic differential equation, \( (s = \Theta \Phi) \)

\[ \frac{\sin \theta}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d \Phi}{d \theta} \right) + \frac{1}{\sin \theta} \frac{d^2 \Phi}{d \Phi^2} + n(n+1) \sin^2 \theta = 0 \]

Let: \[ \frac{1}{\sin \theta} \frac{d^2 \Phi}{d \Phi^2} = -K = \sin^2 \theta \Rightarrow \Phi(\phi) = C \cos m \phi + D \sin m \phi (m \neq 0) \]

\[ \Rightarrow \sin \theta \frac{d}{d \theta} \left( \sin \theta \frac{d \Theta}{d \theta} \right) + \left[ n(n+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0 \]

5.15. **Surface Zonal Harmonics, Legendre's Eq.**

For \( \mu = \cos \theta \), above eq. becomes

\[ \frac{d}{d \mu} \left[ (1 - \mu^2) \frac{d \Theta_n}{d \mu} \right] + n(n+1) \Theta_n = 0 \]

5.15.1 **Series Solution of Legendre's Eq.**

\[ \Theta_n = a_n \Phi_n + b_n \Phi_{n+1} \]

\[ n \Phi'_{n-1} + (n+1) \Phi'_{n+1} = (2n+1) \Phi_n \]

\[ \Phi_{n-1} - \Phi_{n+1} = (2n+1) \mu \Phi_n \]

\[ (n+1) \Phi'_{n-1} + n \Phi'_{n+1} = -n(n+1)(2n+1) \Phi_n \]

5.15.2. **Legendre Polynomials, Rodrigues' Formula.**

\[ P_n(\mu) = \left( \frac{-1}{2} \right)^n \frac{n!}{(n/2)!^2} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n \quad (n \text{ odd}) \]

\[ = \frac{1}{2^{n+1} (n+1)!} \sum_{s=0}^{n/2} \binom{n}{s} \frac{(2n-2s)!}{s! (n-s)!} (\mu^2 - 1)^{n-2s} \]

\[ = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n \quad \text{RODRIGUES'S FORMULA} \]
05.153. Legendre Coefficients. Inverse Distance

\[ b > a \quad \mu = \cos \theta \]

\[ \frac{1}{R} = \frac{1}{b} \left[ \frac{p_0(\mu)}{b} + \left( \frac{a}{b} \right) p_1(\mu) + \left( \frac{a}{b} \right)^2 p_2(\mu) + \cdots \right] \]

\[ = \frac{1}{b} \sum_{n=0}^\infty \left( \frac{a}{b} \right)^n P_n(\mu) \]

05.154. Recurrence Formulas for Legendre Polynomials

\[ n P_{n-1} + (n+1) P_{n+1} = (2n+1) \mu P_n \]

\[ P_{n+1}' - P_{n-1}' = (2n+1) P_n' \]

\[ P_{n+1} = \mu P_n' + (n+1) P_n \]

\[ P_n' = \frac{n(n+1)}{1-\mu^2} \int P_n(\mu) d\mu \quad \int P_n(\mu) = \frac{P_{n+1} - P_{n-1}}{2n+1} \]

05.155. Integral of Product of Legendre Polynomials

\[ \int_{-1}^1 P_n(\mu) P_m(\mu) d\mu = \frac{2}{2n+1} \delta_{nm} ; \quad \mu = \cos \theta \]

05.156. Expansion of Function in Legendre Polynomials

\[ f(\mu) \in [-1, 1] \]

\[ f(\mu) = \sum_{n=0}^{\infty} a_n P_n(\mu) \]

\[ a_n = \frac{1}{2} (2n+1) \int_{-1}^1 f(\mu) P_n(\mu) d\mu \]

\[ = \frac{2^{m+1}}{2^m n! \cdot m!} \int_{-1}^1 \frac{d^m f(\mu)}{d\mu^m} (1-\mu^2)^m d\mu \leq \text{Rodrigues} \]

05.157. Table of Legendre Polynomials (pp. 147-8)
5.16. Potential of a Charged Ring

- If \( V \) is symmetrical about the \( x \) axis, and if this potential can be expressed in a series of \( x^m \) (\( m \) an integer), then, replacing \( x^m \) by \( r^m \) \( P_n (\cos \theta) \) will give \( V \) everywhere.

**Example:** Ring with charge \( Q \)

\[
V_R = \frac{Q}{4\pi \varepsilon_0 R} = \frac{Q}{4\pi \varepsilon_0} \sqrt{x^2 + z^2} = 2\pi \varepsilon_0 \frac{Q}{2} \]

\[
= \begin{cases} 
\frac{Q}{4\pi \varepsilon_0} \frac{x}{r} P_n (\cos \theta) ; x > \xi \\
\frac{Q}{4\pi \varepsilon_0} \frac{x}{r} P_n (\cos \theta) ; x < \xi 
\end{cases}
\]

Thus \( \Rightarrow \)

\[
V_P = \frac{Q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{x}{r} \right)^n P_n (\cos \theta) P_n (\cos \theta) ; r > \xi \\
= \frac{Q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{x}{r} \right)^n P_n (\cos \theta) P_n (\cos \theta) ; r < \xi
\]

5.17. Charged Ring in Conducting Sphere

From above, potential due to ring \( (\alpha = \frac{\pi}{2}) \) is

\[
V_P = \frac{Q}{2\pi \varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{a}{b} \right)^{2n+1} P_{2n} (0) P_{2n} (\mu)
\]

This is good for \( \alpha < \rho \)

For sphere:

\[
V_s = \sum_{n=0}^{\infty} R_s (r) P_n (\mu) = \sum_{n=0}^{\infty} A_n r^n P_n (\mu)
\]

The total potential is the sum of these,

\( V = V_P + V_s \)

We require that \( V(b) = 0 \Rightarrow V_P (b) = -V_s (b) \)

\[
\frac{Q}{2\pi \varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{a}{b} \right)^{2n+1} P_{2n} (0) P_{2n} (\mu) = \sum_{n=0}^{\infty} A_n b^n P_n (\mu)
\]

\[
\Rightarrow A_{2n} = \frac{(-1)^n}{2n!!} b^{2n} \frac{2n+1}{b} \left( \frac{a}{b} \right)^{2n+1}
\]

QED.
5.29. Laplace's Eq. in Cylindrical Coordinates
\[ \frac{d^2v}{dr^2} + \frac{1}{\rho} \frac{dv}{d\rho} + \frac{1}{\rho^2} \frac{d^2v}{d\phi^2} + \frac{1}{\rho^2 \sin^2 \phi} \frac{d^2v}{dz^2} = 0 \]

\[ x = \rho \cos \phi \quad y = \rho \sin \phi \]

Let \( v = R \Phi Z \)
\[ \Rightarrow \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2} \frac{d^2\Phi}{d\phi^2} + \frac{1}{\rho^2 \sin^2 \phi} \frac{d^2Z}{dz^2} = 0 \]

5.29. Bessel's Eq. and Bessel Functions
\[ \frac{d^2\Phi}{d\phi^2} = -n^2, \quad \frac{1}{\rho^2 \sin^2 \phi} \frac{d^2Z}{dz^2} = \lambda^2 \]

\[ \Phi = A \cos n\phi + B \sin n\phi \]

\[ Z = C \cosh \lambda z + D \sinh \lambda z \]

Then, for \( V = kl \rho \), \( R \) must satisfy
\[ \frac{d^2R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left( 1 - \frac{n^2}{\lambda^2} \right) R = 0 \iff \text{Bessel's Eq.} \]

\[ V = R_n(k\lambda) \Phi(n\phi) Z(\lambda z) \]

\[ \text{Note, for } k = 0 \]

\[ V = (M n^2 + N (-n)) (cz + d) (A \cos n\phi + B \sin n\phi) \]
\[ (M \cos (n\lambda \phi) + N \sin (n\lambda \phi)) (cz + d) \]
\[ \times (A \cosh n\lambda \phi + B \sinh n\lambda \phi) \]

For \( n = k = 0 \)

\[ V_{00} = (M n^2 + N) (cz + d) (A \phi + B) \]
5.293. SOLUTION OF BESSEL'S EQ

\[ J_n(v) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{v^{r+1}}{r+1} \]

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt; x > 0, \Gamma(n+1) = n! \]

For \( n \neq \text{integer}, \) another independent sol. is

\[ Y_n(v) = \frac{\pm i v}{\pi} \left[ J_n(v) \cos \nu \pi - J_{-n}(v) \right] \]

For \( n = \text{integer}, \) this becomes

\[ Y_n(v) = \frac{2}{\pi} \int_0^\infty J_n(v) J_n(u) \sin u v \frac{du}{u} = \frac{2}{\pi} \left( J_{-n+1}^2 - J_{-n}^2 \right) \]

Thus, for \( n = \text{integer}, \) a complete soln. of Bessel's

\[ R_n(v) = A J_n(v) + B Y_n(v) \]

5.294. RECURRENCE FORMULAS FOR BESSEL EQ'S

\[ \frac{d}{dv} v^n J_n(v) = v^n J_{n-1}(v) \]

\[ J_n' - \frac{n}{v} J_n = J_{n-1} \]

\[ \frac{1}{2} \left( J_{n-1} - J_{n+1} \right) \]

\[ \frac{2n}{v} J_n = J_{n-1} + J_{n+1} \]

\[ Y_n' - \frac{n}{v} Y_n = Y_{n-1} \]

\[ -\frac{1}{2} (Y_{n-1} - Y_{n+1}) \]

\[ \frac{2n}{v} Y_n = Y_{n-1} + Y_{n+1} \]

\( \left\{ \begin{array}{l}
\int v^n J_{n-1}(v) dv = v^{-n} J_n(v) \\
\int v^{-n} Y_{n+1}(v) dv = -v^n Y_n(v) \\
\int v^n J_{n-1}(v) dv = v^n J_n(v) \\
\int v^n Y_{n-1}(v) dv = v^n Y_n(v)
\end{array} \right\} \)
5.295. VALUES OF BESSEL FUNCTIONS

\[ J_n(v) \xrightarrow{v \to \infty} \left( \frac{2}{\pi v} \right)^{\frac{1}{2}} \cos \left( v - \frac{\pi}{4} \right) \]
\[ Y_n(v) \xrightarrow{v \to \infty} \left( \frac{2}{\pi v} \right)^{\frac{1}{2}} \sin \left( v - \frac{\pi}{4} \right) \]

5.297. EXPANSIONS IN SERIES OF BESSEL FUNCTIONS

Let \( f(v) = \sum_{n=0}^{\infty} A_n J_n(\mu_n v) \)

possible boundary conditions:

(a) \( f(a) = 0 \Rightarrow f(a) \) is potential, and \( a \) is grounded.

(b) \( f'(a) = 0 \Rightarrow \) boundary is a line of force.

(c) \( a f'(a) + b f(a) = 0 \Rightarrow (a) \) for \( b = 0 \), (b) if \( b \neq 0 \)

For these respective boundary conditions, choose:

(a) \( J_n(\mu_n a) = 0 \)  (b) \( J_n(\mu_n a) = 0 \)  (c) \( \mu_n a J_n'(\mu_n a) + B J_n(\mu_n a) = 0 \)

Coefficients given by

\[ A_n = \frac{\int_{a}^{b} v f(v) J_n(\mu_n v) dv}{\int_{a}^{b} v^2 J_n^2(\mu_n v) dv} \]
\[ \int_{a}^{b} v J_n^2(\mu_n v) dv = \frac{\mu_n^2}{2} \left[ J_{n+1}(\mu_n a) - \mu_n J_n(\mu_n a) J_{n+1}(\mu_n a) \right] \]

For the given boundary conditions, we have:

(a) \( A_n = \frac{2}{a^2} J_{n+1}^2(\mu_n a) \int_{a}^{b} v f(v) J_n(\mu_n v) dv \)

(b) \( A_n = \frac{2}{a^2 - b^2} J_n^2(\mu_n a) \int_{a}^{b} v f(v) J_n(\mu_n v) dv \)

(c) \( A_n = \frac{2}{a^2 + (\mu_n^2 - a^2)} J_n^2(\mu_n a) \int_{a}^{b} v f(v) J_n(\mu_n v) dv \)
0.5.29B, Green's Function for a Cylinder. Inverse Distance

\( V = \sum_{p=1}^{\infty} \sum_{s=1}^{\infty} a_{rs} e^{-\mu r^2} J_s(\mu r) \cos s(\phi - \phi_0) \)

The plane \( z = 0 \) has only lines of force.

This gives, with some cranking, the Green's function for a circular cylinder:

\( V = \frac{1}{2\pi c a^2} \sum_{p=1}^{\infty} \sum_{s=1}^{\infty} (2 - \delta_{s,0}) e^{-\mu_{p+1} r^2} J_s(\mu_{p+1} r) J_s(\mu r) \cos s(\phi - \phi_0) \)
VI. ELECTRIC CURRENT

6.00 ELECTRIC CURRENT DENSITY, EQUATION OF CONTINUITY.

\[ I = \frac{dQ}{dt} = \text{CURRENT} \]

\[ j = \frac{dI}{dS} \]

\[ \nabla \cdot I = 0 \Rightarrow \text{CONTINUITY EQ. (NO SOURCES OR SINKS)} \]

6.01 ELECTROMOTANCE

\[ \mathcal{E} = \oint E \cdot dS = \text{ELECTROMOTANCE} \]

\[ E = E' + E'' \]

\[ E' = \text{ELECTROSTATIC} = -\nabla V \]

\[ E'' = \text{ELECTROMOTANCE} \]

6.02 OHM'S LAW, RESISTIVITY.

\[ R_{AB} = \frac{V_A - V_B}{I_{AB}} = \frac{E_{AB}}{I_{AB}} \]

\[ \gamma = \frac{\Delta E}{\gamma} = \gamma \nabla \mathcal{E} \]

\[ \oint j \cdot ds = \gamma \mathcal{E} \]

\[ \gamma = \text{RESISTIVITY} \quad \gamma = \text{CONDUCTIVITY} \]

6.03 HEATING EFFECT OF ELECTRIC CURRENT

\[ P = I^2 R \leftarrow \text{JOULES LAW} \]

6.04 STEADY CURRENTS IN EXTENDED MEDIAHOMOGENEOUS MEDIA

\[ \nabla \cdot \left( \frac{1}{\gamma} \nabla V \right) = 0 \]

\[ \nabla^2 V = 0 \]

\[ \gamma \text{ PLAYS SAME ROLE AS } \mathcal{E} \text{ IN ELECTROSTATIC CASE.} \]

BOUNDARY CONDITIONS (AND DUALS)

\[ \mathcal{E} = \frac{1}{\gamma} \nabla V = -\frac{1}{\gamma} \frac{dV}{d\mathbf{n}} \]

\[ \mathcal{D} = \epsilon \mathcal{E} = -\epsilon \frac{dV}{d\mathbf{n}} \]

\[ \frac{1}{\gamma} \frac{dV'}{d\mathbf{n}} = \frac{1}{\gamma} \frac{dV''}{d\mathbf{n}} \]

\[ \epsilon' \frac{dV'}{d\mathbf{n}} = \epsilon'' \frac{dV''}{d\mathbf{n}} \]

\[ V' = V'' \]

\[ \frac{1}{\gamma} = \gamma \]

\[ \frac{c'}{c_v} = K_1 \frac{\delta V_1}{\delta \mathbf{n}} - K_2 \frac{\delta V_2}{\delta \mathbf{n}} = -(K_1 \gamma_1 - K_2 \gamma_2) i_n \]
6.05 GENERAL THEOREMS

1. GIVEN \( V \) OVER ALL CONDUCTOR BOUNDARIES
   LOCATION OF SOURCES \& SINKS INSIDE
   SPECIFIES \( V \) INSIDE THE BOUNDARY

2. GIVEN \( \frac{dV}{dx} \) OVER CONDUCTOR BOUNDARIES
   LOCATION OF SOURCES AND SINKS INSIDE
   SPECIFIED \( V_b - V_q = \) POTENTIAL DIFFERENCE

3. AS \( \gamma \) INCREASES, THE TOTAL RESISTANCE
   INCREASES OR STAYS THE SAME

4. AS \( \gamma \) DECREASES, THE TOTAL RESISTANCE
   DECREASES OR STAYS THE SAME

5. THE CURRENT DENSITY IS DISTRIBUTED
   SUCH THAT MINIMUM POWER IS LOST

6.06 CURRENT FLOW IN TWO DIMENSIONS (INFINITESHEET)

\[ W = U + jV = \int (x + jy) = \text{POTENTIAL/STREAM FUNCTION} \]

\[ \begin{align*}
    \frac{d}{dx} \left( \frac{dU}{dx} \right) &= \frac{1}{\rho} \frac{\delta U}{\delta x} = -\frac{1}{\rho} \frac{\delta U}{\delta x} \quad \text{FOR } V \text{ IS POTENTIAL} \\
    \frac{d}{dy} \left( \frac{dU}{dy} \right) &= \frac{1}{\rho} \frac{\delta U}{\delta y} = -\frac{1}{\rho} \frac{\delta U}{\delta y} \quad \text{FOR } U \text{ IS POTENTIAL} \\
\end{align*} \]

\[ \begin{align*}
    \text{CURRENT FLOWING THRU A IS} \\
    \begin{cases}
        U_1 & \text{V1} \\
        V_2 - V_1 & \text{V2} \\
    \end{cases} \\
    I = \frac{U_2 - U_1}{V_2 - V_1} \\
    R = \frac{1}{I} = \frac{1}{V_2 - V_1} \quad \Leftrightarrow \quad C = \frac{1}{I} \\
    RC = \gamma \in \\
\end{align*} \]

\[ R = \frac{1}{2\pi} \cosh \left( \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right) \]

\[ R = \frac{1}{2\pi} \cosh \left( \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right) \]
6.06. **Current Flow in Two Dimensions:** \( W = U + jV \)

\[ I = \frac{1}{T} \left| \frac{dW}{dz} \right| = \frac{1}{T} \left| \frac{6V}{6z} \right| = -\frac{1}{T} \frac{6V}{6z} \text{ for } V \text{ the potential} \]

\[ I = \frac{1}{T} \left| \frac{6U}{6z} \right| = \frac{1}{T} \frac{6U}{6z} \text{ for } U \text{ the potential} \]

For a conductor bounded by potentials \( U_1 \) \& \( U_2 \).

And by lines of force \( V_1 \neq V_2 \):

\[ I = \int_{V_1}^{V_2} \frac{1}{T} \int_{V_1}^{V_2} \frac{6U}{6z} \, ds = \frac{1}{T} \int_{V_1}^{V_2} \frac{6V}{6z} \, ds = \frac{1}{T} \int_{U_1}^{U_2} \frac{6U}{6z} \, ds = \frac{V_2 - V_1}{T} \]

\[ R = \text{conductors' resistance} = \frac{1}{T} \frac{V_2 - V_1}{V_2 - V_1} \]

If \( U_1 \neq U_2 \) are closed curves, then

\[ C = \oint_{U_2 - V_1} V \text{ is integral around } U_2 \text{ on } U_1 \]

\[ \Rightarrow R = \frac{NC}{C} \]

The resistance twixt two electrodes is

\[ R = \frac{1}{2\pi} \cos^{-1} \left( \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right) \]

\( D \), \( R_1 \) \& \( R_2 \) are Radial Distance

"++" \( \Rightarrow \) Outside each other, "-" \( \Rightarrow \) One is in the other.
6.07. LONG STRIP WITH ABRUPT CHANGE IN WIDTH

In the $z_1$ plane, we are bending the real axis as shown. Origin of $z_1$ goes to $z = -i\infty$.

Recall the Schwartz xform $(4.18(y))$:

$$\frac{dz}{dz_1} = c_1 (z_1 - u_1)^{\alpha_1 - 1} (z_1 - u_2)^{\alpha_2 - 1} \ldots$$

Here, $u_1 = -1, u_2 = -a, u_3 = 0, u_4 = a, u_5 = 1$

$$\alpha_1 = \frac{\pi}{2}, \alpha_2 = \frac{3\pi}{2}, \alpha_3 = 0, \alpha_4 = \frac{3\pi}{2}, \alpha_5 = \frac{\pi}{2}$$

$$\Rightarrow \frac{dz}{dz_1} = c \frac{(z_1 - a^2)^{\frac{3\pi}{2}}}{z_1(z_1^2 - 1)^{\frac{3\pi}{2}}} = \frac{c}{z_1(z_1^2 - 1)^{\frac{3\pi}{2}}}$$

We must now evaluate the $c$'s. Set $r_1 = \text{const}$ so that

$$\frac{dz}{dz_1} = j r_1 e^{j\theta} \frac{d\theta}{dz_1} = j \frac{dz}{dz_1} d\theta,$$

$$\Rightarrow dz = j c \left( \frac{z_1 - a^2}{z_1^2 - 1} \right)^{\frac{3\pi}{2}} \frac{d\theta}{d\theta} = j c \left[ \frac{(r_1 e^{2j\theta} - a^2)/(r_1 e^{2j\theta} - 1)^2} {z_1} \right] d\theta$$

* For $r_1 \ll 1$, as $0 < \theta < \pi$, $y = i\infty$, $-h < x < 0$

$$\int_{-h}^{0} dz = j c \left[ \int_{0}^{\pi} \left[ \frac{r_1 e^{2j\theta} - a^2}{r_1 e^{2j\theta} - 1} \right] d\theta \right] r_1 e^{j\theta}$$

* For $r_1 \gg 1$, as $0 < \theta < \pi$, $+h < x < -k$

$$\int_{-h}^{0} dz = \pm j c \int_{0}^{\pi} d\theta,$$

$$\Rightarrow q = h/K, \quad c = \pm \frac{jK^2}{\pi}$$

Gives

$$z = \frac{2}{\pi} \left\{ k \tan^{-1} \left[ \frac{(z_1^2 - a^2)}{(1 - z_1^2)} \right] + h \tan^{-1} \left[ a \sqrt{z_1^2 - a^2} \right] \right\}^{\frac{3}{2}}$$
0.6.07 (cont)

Take \( W = \ln z_1 \), \((z_1 = e^w)\) \( w = U + iV \)

\[
\begin{align*}
I &= \frac{\pi}{2} \quad U \\ V &= \pi \\ V &= 0
\end{align*}
\]

Thus \( z = \frac{2}{\pi} \left( k \tan^{-1} \left( \frac{e^{2w} - a_2}{1 - e^{2w}} \right) + h \tanh^{-1} \left( \frac{1 - e^{2w}}{e^{2w} - a_2} \right) \right) \)

Consider a one meter cut:

\( \xi \) is resistance twixt opposite sides.

\[
\begin{align*}
\left[ \frac{1}{2k} \right] & \quad Y_k \\ \frac{\xi}{\xi Y_k} & \quad R_k = \frac{\xi}{2k}
\end{align*}
\]
6.080 CURRENT FLOW IN THREE DIMENSIONS

ELECTRODES

\( \nabla \cdot \mathbf{E} = 0 \) \quad \text{(UNIFORM ISOTROPIC CONDUCTING MEDIA)} \quad \Rightarrow \text{SOLVE } \nabla^2 V = 0

MAY SOLVE BY FINDING CAPACITANCE

- ELECTROSTATIC BOUNDARY CONDITIONS (ON ELECTRODE)
  \[ V = V_0 \quad Q_a = - \oint_s \frac{\mathbf{E}}{\mathbf{n}} \cdot dS_a \]

- CORRESPONDING CURRENT BOUNDARY CONDITIONS
  \[ V = V_0 \quad \mathbf{I}_a = - \oint_s \frac{\mathbf{E}}{\mathbf{n}} \cdot dS_a \]

THE EQUIVOLUTION SURFACES IN BOTH OF THESE CASES CORRESPOND EXACTLY,

\[ R_C = \frac{V}{C} \quad C \text{ IS CAPAC IN VACUUM.} \]

6.090 SYSTEMS OF ELECTRODES. TWO SPHERES. DISTANT ELECTRODES

FOR \( n \) PERFECTLY CONDUCTING ELECTRODES

IN A HOMOGENEOUS ISOTROPIC MEDIUM

- WRITE \( I_s \) INSTEAD OF \( Q_s \)
- MULTIPLY CAPACITANCES BY \( \frac{1}{\varepsilon} \)

EX. TWO SPHERES (INTERNAL OR EXTERNAL)

SEPARATED BY A DISTANCE \( C \)

\[ V_1 - V_2 = \varepsilon \varepsilon_0 \left( S_{11} - 2S_{12} + S_{22} \right) \mathbf{I}_1 \]

\[ \Rightarrow R = \left| \frac{V_1 - V_2}{\mathbf{I}_1} \right| = \varepsilon \varepsilon_0 \left( S_{11} - 2S_{12} + S_{22} \right) \]
6.16. LIMITS OF RESISTANCE

1) LOWER LIMIT: "INSERT" THIN SHEETS OF CONDUCTOR TO COINCIDE WITH EQUIPOENTIAL LINES

2) UPPER LIMIT: "INSERT" THIN INSULATING SHEETS ALONG THE LINES OF FLOW.

EXAMPLE:

\[
\begin{align*}
& \text{Upper Limit: Make Insulating Surface} \\
& \text{Length: } 2C + \pi (b + x) \frac{1}{2} \text{ Area: } x \, dx \\
& dR_0 = \frac{V_2}{A} = \gamma [2C + \pi (b + x)] \\
& \text{Everything is in parallel, so that} \\
& R_0 = \left( \int dR_0 \right)^{-1} = \left( \frac{1}{\gamma} \int_0^{2C + \pi (b + x)} x \, dx \right)^{-1} \\
& = \pi^2 \gamma \left[ \pi a - (2C + \pi b) \ln \frac{2C + \pi (a + b)}{2C + \pi b} \right]^{-1} \\
& \text{Lower Limit: Put Perfect Conductors } M \frac{1}{N} \\
& \Rightarrow \text{Current is uniformly distributed in each leg, } R = \frac{4C}{a^2} \\
& \text{In semicircular part, } W = C \ln \frac{C}{z} \\
& U = \frac{bC}{a} \text{, } V = 0 \text{ \ (V is potential)}
\end{align*}
\]

The resistance in \( dy \) is

\[
\begin{align*}
& dR = \gamma \left| \frac{V_2 - V_1}{V_2 - V_1} \right| = \gamma \pi \left( \ln \frac{a + b}{b + 2y} \right)^{-1} \\
& R = \left( \int dR \right)^{-1} = \frac{\pi \gamma}{b \ln (a + b) - b \ln b - a} \\
& \text{And the lower limit resistance is} \\
& R_L = \gamma \left[ \frac{4C}{a^2} - \frac{\pi \gamma}{b \ln (a + b) - b \ln b - a} \right]
\end{align*}
\]
VII. MAGNETIC INTERACTION OF CURRENTS

7.010 DEFINITION OF THE AMPERE IN TERMS OF THE MAGNETIC MOMENT

\[ \mu_0 = 4\pi \times 10^{-7} \]

7.01 MAGNETIC INDUCTION AND PERMEABILITY

\[ \mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \]

\[ \mathbf{\nabla} \cdot \mathbf{B} = 0 \]

\[ \mathbf{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \]

7.02 MAGNETIC VECTOR POTENTIAL. UNIFORM FIELD.

Let \( \mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \Rightarrow \mathbf{\nabla} \cdot \mathbf{A} = 0 \)

\( \mathbf{A} = \text{VECTOR POTENTIAL} \)

\[ \Rightarrow \mathbf{\nabla}^2 \mathbf{A} = -\mu_0 \mathbf{J} \]

\[ A_x = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_x \, dv}{r} \]

\[ A_y = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_y \, dv}{r} \]

\[ A_z = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_z \, dv}{r} \]

IF \( \mathbf{B} \) IS IN X DIRECTION ONLY

\[ A_y = -\alpha \mathbf{B} \]

\[ A_z = (1-\alpha) \mathbf{B} \]

7.030 UNIQUENESS THEOREMS FOR MAGNETOSTATICS
7.04 @ ORTHOGONAL EXPANSIONS FOR VECTOR POTENTIAL

Let \( \mathbf{A} = \mathbf{\nabla} \times \mathbf{\hat{W}} \)

\[ \mathbf{\hat{W}} = \mathbf{\hat{U}} \mathbf{W}_1 + \mathbf{\hat{U}} \times \mathbf{\nabla} \mathbf{W}_2 \]

\( \mathbf{\hat{U}} \) IS AN ARBITRARY VECTOR

Note: Components of \( \mathbf{\hat{W}} \) are perpendicular (in magnetostatics, \( \mathbf{W}_2 \) offers nothing to \( \mathbf{B} \))

7.05 @ VECTOR POTENTIAL IN CYLINDRICAL COORDINATES

7.06 @ VECTOR POTENTIAL IN SPHERICAL COORDINATES

7.07 @ VECTOR POTENTIAL IN TERMS OF MAGNETIC INDUCTION ON AXIS \( \rightarrow \) MAGNETIC LENSES

7.09 @ VECTOR POTENTIAL AND FIELD OF BIFILAR CIRCUIT

\( \mathbf{\hat{A}} \) IS IN DIRECTION OF \( \mathbf{I} \)

From 7.02,

\[ \mathbf{A} = \frac{\mu}{4\pi} \oint \frac{\mathbf{I} \hat{r}}{r} \, ds \]

\[ \Rightarrow A_x = \frac{\mu I}{4\pi} \int_{r_1}^{r_2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \, dx = \frac{\mu I}{2\pi} \ln \frac{a_2}{a_1} \]

\[ B_x = 0 \]

\[ B_z = \frac{\mu I}{2\pi} \left( \frac{Y + \frac{1}{2}a}{a_2^2} - \frac{Y - \frac{1}{2}a}{a_1^2} \right) \]

\[ B_y = \frac{\mu I}{2\pi} \left( \frac{l}{a_2^2} - \frac{l}{a_1^2} \right) \]
7.10 Vector Potential of Field of Circular Loop

A will be \( \phi \) independent

Let \( P \) be on \( xz \) plane (i.e., \( \phi = 0 \))

In general (from 7.02):

\[
A = \frac{\mu}{4\pi} \int_S T ds
\]

Here:

\[
A_\phi = \frac{\mu I}{4\pi} \oint r^{-1} d\phi = \frac{q^2 \mu I \sin \theta}{4r^2}
\]

On axis, \( B_\phi = B_\theta = 0 \), \( B_z = \frac{1}{2} \mu_0 q^2 \sqrt{a^2 + z^2} \)

7.1.1 Field of Currents in Spherical Shell

7.1.2 Zonal Currents in Spherical Shell

\( \psi = \text{stream function} \)

\( \text{If current flows in "latitude"} \]

\[
\psi = \sum_{n=1}^{\infty} C_n \frac{P_n(\cos \theta)}{n}
\]

\[
i \phi = -\sum_{n=1}^{\infty} \frac{1}{a} C_n \frac{P_n'(\mu)}{n}
\]

For \( n \geq 2 \)

\[
A = \frac{\phi}{\mu} \sum_{n=1}^{\infty} \frac{C_n}{n(n+1)} \left( \frac{q}{r} \right)^{n+1} P_n(\mu)
\]

\[
B_r = -\frac{\mu I}{2\pi} \sum_{n=1}^{\infty} \frac{n C_n}{2n+1} \frac{P_n(\mu)}{n(n+1)}
\]

\[
B_\theta = -\frac{\mu I}{2\pi} \sum_{n=1}^{\infty} \frac{n C_n}{2n+1} \left( \frac{q}{r} \right)^{n+2} P_n(\mu)
\]

In order to make field uniform inside:

\[
N = 2\pi r \sin \theta \times A_\phi
\]
7.13 Field of Circular Loop in Spherical Harmonics

Current density non-zero only when $\Theta = \Theta_c$

\[ I = -\frac{1}{\mu} \sum C_n P_n^\prime(\cos \Theta) \]

\[ C_n = \frac{-(n+1)}{2n(n+1)} \int_{0}^{\pi} \sin \theta a P_n(\cos \theta) d\theta \]

\[ \Rightarrow A = \frac{\hat{\phi}}{2} \sum \frac{\sin \theta}{n(n+1)} \left( \frac{1}{\mu} \right) P_n(\cos \Theta) P_n^\prime(\cos \Theta) \]

$B_r \neq B_\theta$ follow

7.14 Biot and Savart's Law. Field of straight wire

\[ \mathbf{B} \propto \mathbf{I} \times \mathbf{a} \]

\[ \mathbf{B} = \frac{\mu I}{2\pi a} \]

7.15 Field in cylindrical hole in conducting rod

\[ i_z \text{ is uniform without "hole"} \]

\[ \mu \pi r^2 \frac{1}{2} \times \mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{s} = 2\pi r \mathbf{B}_\theta \]

\[ \Rightarrow \mathbf{B}_\theta = \frac{1}{2} \mu_0 i_z \mathbf{a} \times \mathbf{a} \]

For a current density $\mathbf{a}$, flowing only in inner cylinder:

\[ \mathbf{B} = \frac{1}{2} \mu_0 i_z \mathbf{a} \times (\mathbf{r} - \mathbf{r}') = \frac{1}{2} \mu_0 \mathbf{a} \cdot \mathbf{a} \]

The total current is $I = \pi (b^2 - a^2) i_z$

So, subtracting the $B_\theta$'s gives

\[ B_y = \frac{\mu_0 CI}{2\pi (b^2 - a^2)} \]
7.18 \textcircled{1} \textbf{FORCE ON ELECTRIC CIRCUIT IN MAGNETIC FIELD}

\[ \vec{F} = I \oint \vec{A} \cdot \vec{B} \]

\[ F = \vec{v} \times \vec{B} \text{ \text{LORENTZ FORCE}} \]

7.19 \textcircled{2} \textbf{EXAMPLES OF FORCES TWIXT ELECTRIC CIRCUITS}

1. \textbf{TWO INFINITE PARALLEL WIRES}

\[ \begin{array}{c}
\begin{array}{c}
\uparrow \quad I' \\
\downarrow \\
\rightarrow \quad q \\
\rightarrow \quad \rightarrow \\
\end{array}
\end{array} \]

\[ F = -\frac{\mu_0 I I'}{2\pi q} \]

2. \textbf{TWO LOOPS}

\[ \begin{array}{c}
\begin{array}{c}
\uparrow \quad I' \\
\downarrow \\
\rightarrow \quad q \\
\rightarrow \quad \rightarrow \\
\end{array}
\end{array} \]

\[ F = \pi \mu_0 I I' \sin \alpha \sum \frac{\rho_n^2}{(\rho^2 + c^2)} \rho_n (\cos \theta) \rho_n \phi \]

7.20 \textcircled{3} \textbf{VECTOR POTENTIAL \& MAGNETIZATION}

\[ \frac{1}{M} = \left( \frac{1}{\mu_0} - \frac{1}{\mu} \right) \frac{1}{B} \]

\[ I = \oint B \cdot d\vec{s} / \mu = I \]

7.21 \textcircled{4} \textbf{MAGNETIC BOUNDARY CONDITIONS}

\[ \frac{\partial V_0}{\partial r} - \frac{\partial V_i}{\partial r} = -\sigma \varepsilon \]

\[ V_0 = V_i \]

\[ A' = A'' \]

\[ \frac{\partial A'}{\partial r} - \frac{\partial A''}{\partial r} = \mu_0 \left[ (M' - M'') \times \hat{n} \right] \]

\[ \text{[EQUIVALENT TO $\nabla \times \left[ \frac{B'}{\mu} - \frac{B''}{\mu''} \right] = 0$]} \]

(PLUS OTHER GIVEN CONDITIONS)
7.28 @ Magnetomotive and Magnetic Intensity

Some qualities

\[ \nabla \cdot \mathbf{I} = 0 \]
\[ \oint i \cdot ds = \mathbf{E} \]
\[ \oint B \cdot ds = \mathbf{I} \]

\[ i = \partial \nabla \mathbf{E} \]
\[ B = \mu \nabla \mathbf{\Omega} \]

[\( \Omega \) is magnetomotive = \( \nabla \mathbf{E} \) (in amp-turns)]

\[ \mathbf{D} = \varepsilon \mathbf{E} \]
\[ \mathbf{B} = \mu \mathbf{H} \]

(Boundary conditions given for \( \mathbf{H} \))

\( \mathbf{H} = \text{Magnetic Field Intensity} \)
VIII ELECTROMAGNETIC INDUCTION

8.000 FARADAY'S LAW OF INDUCTION

\[ \nabla \times E = - \frac{dB}{dt} \]

or \[ E = -\frac{dA}{dt} \]

8.010 MUTUAL ENERGY OF TWO CIRCUITS

Bring two loops, carrying currents \( I \) and \( I' \), together. The currents are kept constant by appropriately changing the \( E \) of their batteries. Half of the energy supplied by the batteries is used doing the mechanical work, half goes into the magnetic field.

8.020 ENERGY IN A MAGNETIC FIELD

\[ W_B = \frac{1}{2} \mu \int_V \mathbf{B}^2 \, dv \]

\[ \frac{W_B}{V} = \frac{1}{2} \mu = \text{ENERGY DENSITY} \]

8.030 MUTUAL INDUCTANCE

DEF: Flux, \( N_{12} \) through circuit 1 produced by 2

\[ M_{12} = \int_{S_2} B_2 \cdot \mathbf{n} \, dS_1 = M_{21} \]

\[ = \oint_{S_1} A_2 \cdot dS_1 \left< \text{FOLLOWS FROM STOKES THEOREM} \right> \]

\[ = \frac{\mu}{4\pi} \oint_{S_2} \oint_{S_1} \frac{dS_1 \cdot dS_2}{r} \]

Torque \( \tau = F = I_1 I_2 \delta M_{12} / \delta \theta \)
8.05 MUTUAL INDUCTANCE OF SIMPLE CIRCUITS

\[ M_{12} = \mu_{0} n m \left[ a - (a^2 - b^2)^{\frac{1}{2}} \right] \]

If \( a \gg b \), \( M_{12} = \frac{\mu_{0} n m b^2}{2a} \)

8.06 MUTUAL INDUCTANCE OF TWO COAXIAL LOOPS

8.07 VARIABLE MUTUAL INDUCTANCE

8.08 SELF-INDUCTANCE

\[ L_{II} = \frac{1}{2} \int B^2 \, dv = \frac{1}{2} L_{II} I_1^2 = \frac{1}{2} \int i \cdot A \, dv \]

\[ E_1 = -L_{II} \frac{dI_1}{dt} \]

8.09 SELF-INDUCTANCE OF A THIN WIRE

\[ L_{III} = \text{SELF-INDUCTANCE PER UNIT LENGTH} = \frac{\mu_0}{2\pi} \]

8.10 SELF-INDUCTANCE OF CIRCULAR LOOP

\[ L_{II} \approx b \left[ \mu_0 \varepsilon_{0} \left( \frac{\rho}{a} - 2 \right) + \frac{1}{2} \right] \]
IX. Magnetism

9.00 Paramagnetism and Diamagnetism

$\mu > 1 \Rightarrow$ Paramagnetic
$\mu < 1 \Rightarrow$ Diamagnetic (Temp. Independent)
$\mu >> 1 \Rightarrow$ Ferromagnetic

In strong paramagnetic (but not diamagnetic)

$\mu = \mu_0 + \frac{\mu_c}{1/T + \Theta} \leq$ Curie's Law

9.01 Magnetic Susceptibility

$\vec{M} = K \vec{H} = \left( \frac{1}{\mu_0} - \frac{1}{\mu} \right) \vec{B}$

$K = \text{Magnetic Susceptibility} = \left( \frac{1}{\mu_0} - \frac{1}{\mu} \right) = K_m = \frac{\Theta}{T + \Theta}$

Energy increase (Density)

$\Delta \omega / V = -\frac{\mu_0}{2} K H^2$

9.02 Magnetic Properties of Crystals

$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$

$\mu_{12} = \mu_{21}, \text{ etc.}$

By suitably orienting axes:

$B_x = \mu_1 H_x \quad B_y = \mu_2 H_y \quad B_z = \mu_3 H_z$

On axis, also

$M_x = k_1 H_x \quad M_y = k_2 H_y \quad M_z = k_3 H_z$
9.04 Ferromagnetism

Permeability varies with $B$

9.05 Hysteresis. Permanent Magnetism

9.06 The nature of permanent magnetism

9.07 Uniform magnetism. Equivalent shell current

The vector potential due to $M$ is

$$\vec{A}_M = \frac{\mu_0 \nu}{4\pi} \int \frac{M \times \vec{n}}{r} \, dS \quad \text{(for uniform $M$)}$$

$$= \frac{\mu_0 \nu}{4\pi} \int \iint M d\vec{x} \cdot d\vec{S}$$

$\vec{A} \propto M$ for such a case.
X. EDDY CURRENTS

10.000. INDUCED CURRENTS IN EXTENDED CONDUCTORS

\[ \nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \quad \text{FARADAY'S LAW} \]
\[ \vec{E} = -\frac{d\vec{A}}{dt} \]
\[ \gamma (\nabla \times \vec{A}) = -\frac{d\vec{B}}{dt} \quad \text{OHM'S LAW} \]
\[ \gamma \vec{A} = -\frac{d\vec{A}}{dt} \]
\[ \nabla \times \vec{B} = \mu \vec{J} \quad \text{AMPERE'S LAW} \]
\[ \nabla^2 \vec{A} = -\mu \vec{J} \]

MAY COMBINE THESE TO GIVE:
\[ \frac{d^2}{dt^2} \frac{d\vec{A}}{dt} = \nabla^2 \vec{A} \quad \frac{d^2}{dt^2} \frac{d\vec{B}}{dt} = \nabla^2 \vec{B} \]

AVERAGE POWER OVER A CYCLE \( d \vec{P} = \frac{1}{2} \gamma^2 \nabla \times \vec{A} \times \vec{d} \phi \)

\( \gamma = \hat{\gamma} \text{ CONJUGATED} \)

10.010. SOLUTION FOR VECTOR POTENTIAL FOR EDDY CURRENTS

\[ A = \nabla \times (\vec{U} \nabla w_1 + \vec{U} \times \nabla w_2) \]

GIVES \( \nabla^2 w = \frac{\mu}{\gamma} \frac{d\vec{w}}{dt} \)

\( \phi = \vec{r}, \vec{j}, \vec{k} \rightarrow \vec{B} = -\frac{\mu}{\gamma} \frac{d\vec{w}}{dt} [\vec{r} \times \nabla w_2 + \vec{U} \nabla w_1] + \vec{r} \cdot \nabla (\vec{U} w_1) \)

10.020. STEADY STATE SKIN EFFECT

\( \hat{\gamma}_x \text{ IS UNIFORM} \)

\( d \text{ POWER ABSORBED PER METER} = \frac{1}{2} \omega \mu \delta I_e^2 \)

\( \delta = \left( \frac{1}{2} \omega \mu \gamma \right)^{-\frac{1}{2}} \text{ SKIN DEPTH} \)

\( B_0 = \sqrt{2} \mu I_e \)

\( L_i = \frac{1}{\mu \delta} = \frac{1}{\omega \delta} = \frac{R_i^2}{\omega} \text{ (L \& R PER SQUARE METER)} \)

10.030. SKIN EFFECT ON TUBULAR CONDUCTOR

10.040. SKIN EFFECT ON SOLID CYLINDRICAL CONDUCTOR
10.09 EDDY CURRENTS IN PLANE SHEETS

THIN SHEET, UNIFORM CURRENT DENSITY \( \frac{2}{\mu_v} \)

FLUCTUATING MAGNETIC FIELD \( A'(x,y,z,t) \).

RESULTING MAGNETIC FIELD \( A(x,y,z,t) \)

\[ A_S = \text{TANGENTIAL COMPONENTS OF } A \]

\[ \frac{\partial z}{\mu_v} = \frac{6A_S}{\delta} \]

10.10 EDDY CURRENTS IN PLANE SHEET BY IMAGE METHOD

\( A \) FROM \( \frac{\partial A}{\partial t} = \frac{2\gamma}{\mu_v} \frac{\partial A}{\partial z} \)

\( f_1 = A_1', \Rightarrow t < 0 \quad f_2 = A_2', \Rightarrow t > 0 \)

\( A = f_1(x,y, -|z| - 2\gamma \mu_v^{-1} t) \)

\[ - f_2(x,y, -|z| - 2\gamma \mu_v^{-1} t) \]

\( \uparrow\downarrow \text{VELOCITY} = \frac{2\gamma}{\mu_v} \)

\[ \frac{2\gamma}{\mu_v} \]
10.16 Zonal Eddy Currents in Spherical Shell

All eddy currents flow in coaxial shells

\[ A' + A = \text{total vector potential} \]

\[ A = \text{vector potential due to eddy current} \]

Let \( \varepsilon_n \Rightarrow \tan \varepsilon_n = \frac{(2n+1)}{\mu_0 A_0} \]

For small \( \varepsilon_n \), we have good shielding.

10.18 General Eddy Currents in Spherical Shell

10.19 Torque on Spinning Spherical Shell Twixt Magnetic Poles

10.20 Eddy Currents in Thin Cylindrical Shells
PLANE ELECTROMAGNETIC WAVES

11.00 @ MAXWELL'S FIELD EQUATIONS

AMPERE'S LAW: \( \nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} + \sigma \mathbf{V} \)

FARADAY'S LAW: \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)

\( \nabla \cdot \mathbf{D} = 0 \)

\( \nabla \cdot \mathbf{B} = 0 \)

WE ALSO HAVE:

CHARGE CONSERVATION: \( \nabla \cdot \mathbf{J} = -\frac{\partial \mathbf{D}}{\partial t} \)

OHM'S LAW: \( \mathbf{E} = \gamma \mathbf{J} \quad (\mathbf{J} = \gamma \mathbf{E}) \)

11.01 @ PROPAGATION Eqs., DYNAMIC POTENTIALS.

GAUGES. HERTZ VECTOR.

IN UNIFORM MEDIUM WITH NO CHARGES

\( \nabla^2 \mathbf{B} = \mu \frac{\mathbf{J}}{\mathbf{c}} + \rho \mathbf{c} \frac{\mathbf{B}}{\mathbf{c}^2} \)

\( \nabla^2 \mathbf{E} = \epsilon \frac{\mathbf{J}}{\mathbf{c}} + \rho \mathbf{c} \frac{\mathbf{E}}{\mathbf{c}^2} \)

\( \nabla^2 \mathbf{A} = -\mu \frac{\mathbf{J}}{\mathbf{c}} - \rho \mathbf{c} \frac{\mathbf{A}}{\mathbf{c}^2} = -\mu \mathbf{s} \mathbf{A} \)

-COULOMB GAUGE \( \nabla^2 \mathbf{A} = \frac{\rho}{\mathbf{c}} \mathbf{A} \)

-LORENTZ GAUGE \( \nabla^2 \mathbf{A} = 0 \mathbf{A} / \mathbf{c} \)

-HERTZ VECTOR \( \mathbf{A} = -\nabla \hat{z} \)

11.02 @ PONTING VECTOR

\( \mathbf{T} = \frac{\mathbf{E} \times \mathbf{B}}{\mu} \)
10: For a given set of colinear charges, find the relationship describing the equal lines of force.

We hypothesize the "lines" of force \( C \), all flux passing through surface \( A \) must pass thru \( B \) (since none go through \( C \)). Assume the point charge \( q_i \) will give out \( q_i \) flux lines (as in tubes of force). The number of flux lines, \( N_i \), due to \( q_i \) which pass thru \( A \) are:

\[
N_i = \frac{\Omega_i}{4\pi} q_i
\]

where \( \Omega_i \) is the solid angle which \( A \) subtends at \( x_i \). It follows that:

\[
N = \sum_{i=1}^{n} \frac{1}{4\pi} \Omega_i q_i
\]

\[
= \sum_{i=1}^{n} \frac{p}{4\pi} \left( 2\pi \sin \frac{\alpha_i}{2} \right) q_i
\]

\[
= \frac{p}{2} \left( 1 - \cos \alpha_i \right) q_i
\]

\[
= \frac{1}{2} Q - \frac{1}{2} \sum_{i=1}^{n} \cos \alpha_i q_i
\]

where the constant \( Q \) is

\[
Q = \sum_{i=1}^{n} q_i
\]
WE WISH TO SET \( N = C = \text{constant} \). IT

FOLLOWS THAT

\[
C = \frac{1}{2} Q - N = \frac{1}{2} \sum_{i=1}^{n} q_i \cos \alpha_i
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \frac{q_i (x - x_i)}{\sqrt{y^2 + (x - x_i)^2}}
\]

SINCE \( \Phi \) (tension) AND \( \Psi \) (force)

ARE FUNCTIONS OF \( C \) \& \( E \) ONLY, THE

ORIGIN OR SHAPE OF THE FIELD IS

IMMATERIAL. WE HAVE (IN HOMOISOTROPIC MEDIA)

\[
\Psi(E) = -\frac{EE^2}{2}
\]

\[
\Phi(E) = \frac{EE^2}{2}
\]

FOR OUR PROBLEM:

\[
E = \frac{-1}{4 \mu c^2} \sum_{i=1}^{n} \frac{q_i^2}{r_i^2} \mathbf{r}_i
\]
10.1

First, find the potential at \( P_1 \) by superposition:

\[
V_{P_1} = V_{P_1}^{(1)} + V_{P_1}^{(2)} \quad (V_{P_1} \text{ due to ring } i, \ i = 1, 2)
\]

First, find \( V_{P_1}^{(1)} \):

\[
\rho_i = \frac{Q_1}{2\pi a} \quad \rho_i = \frac{Q_1}{2\pi a}
\]

\[
V_{P_1}^{(1)} = \frac{\rho_1 (a d\phi)}{4\pi \epsilon} = \frac{\rho_1 d\phi}{4\pi \epsilon} = \frac{Q_1}{4\pi \epsilon a}
\]

\[
V_{P_1}^{(1)} = \frac{\rho_1}{4\pi \epsilon} \int_0^{2\pi} d\phi = \frac{\rho_1}{2\epsilon} = \frac{Q_1}{4\pi \epsilon a}
\]

Find, now, \( V_{P_1}^{(2)} \):

\[
\Rightarrow V_{P_1}^{(2)} = \frac{\rho_2 a d\phi}{2\epsilon d} = \frac{Q_2}{4\pi \epsilon d}
\]

WHERE, AS BEFORE, \( \rho_2 = \frac{Q_2}{2\pi a} \)

Also \( d = \sqrt{a^2 + b^2} \)

This gives:

\[
V_{P_1} = \frac{1}{4\pi \epsilon} \left[ \frac{Q_2}{d} + \frac{Q_1}{a} \right]
\]

Due to symmetry, we may also write:

\[
V_{P_2} = \frac{1}{4\pi \epsilon} \left[ \frac{Q_1}{d} + \frac{Q_2}{a} \right]
\]

The work, \( W_i \), required to bring a test charge \( q_i \) (from \( \infty \) where \( V = 0 \)) to \( P_i \) is:

\[
W_i = q_i V_{P_i}
\]
Thus

\[ W_1 = \frac{q}{4\pi\epsilon} \left[ \frac{Q_1}{d} + \frac{Q_2}{a} \right] \Rightarrow Q_2 = \frac{4\pi\epsilon w_1 d}{q} - \frac{Q_1 d}{a} \]
\[ W_2 = \frac{q}{4\pi\epsilon} \left[ \frac{Q_1}{d} + \frac{Q_2}{a} \right] \Rightarrow Q_2 = \frac{4\pi\epsilon w_2 a}{q} - \frac{Q_1 a}{d} \]

EQUATING GIVES:

\[ \frac{4\pi\epsilon w_1 d}{q} - \frac{Q_1 d}{a} = \frac{4\pi\epsilon w_2 a}{q} - \frac{Q_1 a}{d} \]

SOLVING FOR \( Q_1 \):

\[ Q_1 \left[ \frac{a}{d} - \frac{d}{a} \right] = \frac{4\pi\epsilon}{q} \left[ a w_2 - d w_1 \right] \]

\[ Q_1 \frac{d^2 - a^2}{a d} = \frac{4\pi\epsilon}{q} \left[ d w_1 - a w_2 \right] \]

\[ \Rightarrow Q_1 = \frac{4\pi\epsilon a d}{q (d^2 - a^2)} \left[ d w_1 - a w_2 \right] \]

Again:

\[ d = \sqrt{a^2 + b^2} \Rightarrow d^2 - a^2 = b^2 \]

AND

\[ Q_1 = \frac{4\pi\epsilon a \sqrt{a^2 + b^2}}{q b^2} \left[ \sqrt{a^2 + b^2} w_1 - a w_2 \right] \]

Due to the problem's symmetry, we may immediately write:

\[ Q_{1,2} = \frac{4\pi\epsilon a \sqrt{a^2 + b^2}}{q b^2} \left[ \sqrt{a^2 + b^2} w_{1,2} - a w_{2,1} \right] \]
OBVIOUSLY, C MUST LIE TO THE RIGHT
OF B, SINCE AT C (EQUILIBRIUM), WE
REQUIRE \( \vec{E} = -\nabla V = 0 \) TO FIND THIS
POINT EXPLICITLY:
\[
\vec{E} = \frac{1}{4\pi e_0} \left[ \frac{4q}{(a+q)^2} - \frac{q}{a^2} \right] = 0
\]
OR
\[
\frac{4}{(a+q)^2} = \frac{1}{a^2} \Rightarrow a = q, \text{ with } \theta = \frac{\pi}{3}.
\]

Thus, we have

\[
\frac{1}{q} C = \frac{4(x+2a)}{\sqrt{(x+2a)^2 + y^2}} - \frac{(x+q)}{\sqrt{(x+q)^2 + y^2}}, \quad (i)
\]
\[
= 4 \cos \theta_2 - \cos \theta_1
\]

The line of force passing thru \((0,0)\) @ C
has a \( C \) given by
\[
\frac{C_0}{q} = \frac{8q}{2a} - \frac{q}{a} = 3
\]
THE LINE IS THUS DESCRIBED BY

\[ 3 = 4 \cos \theta_2 - \cos \theta_1 \leq \text{pass thru } C \]  \hspace{1cm} (2)

THE CORRESPONDING POTENTIAL TO EQ. 1 IS

\[
\begin{align*}
\frac{1}{\sqrt{(x+a)^2 + y^2}} - \frac{4}{\sqrt{(x+2a)^2 + y^2}} &= C' \\
C' \text{ for } x = y = 0 \text{ is } \frac{1}{a} - \frac{1}{2a} = C_0' = \frac{1}{a}
\end{align*}
\]

ALL LINES OF FORCE INTERSECT POINTS A \( \frac{1}{a} \) \( B \) (as Fig 1.08).

DUE TO THE PROBLEM'S SYMMETRY,
THE LINES OF FORCE WILL BE SYMMETRIC ABOUT THE X AXIS.
THERE WILL BE NO DISCONTINUITY OF THE LINE PASSING THRU C.
IT STANDS TO REASON THAT THE LINE PASSES THRU C AT 90° TO
DETERMINE THIS ANALYTICALLY, ONE MAY SOLVE (OR APPROXIMATE)
EQ. 2 IN THE FORM
\[ y = f(x) \]
(I wasn't able to do this). IF
\[ \frac{dy}{dx} \bigg|_{x = 0} = \infty, \text{ then the } 90° \text{ conjecture is proved. Similarly, } \frac{dy}{dx} \bigg|_{x = 2a} = \tan 60° = \sqrt{3} \]
WOULD SUFFICE FOR SHOWING
EQ. 2 PASSES THRU A AT AN
COULD IN PRINCIPLE BE APPLIED TO
THE SECOND HALF OF THE
PROBLEM. AGAIN, I COULDN'T GET
A HANDLE ON IT.
WE ARE GIVEN THAT THE TOTAL E FIELD (WHICH BY SYMMETRY, WILL POINT "UP") HAS AN INTENSITY = \( \frac{q}{2 \varepsilon} \) (WILL SHOW THIS).

FOR A POINT CHARGE, \( E = \frac{q}{4 \pi \varepsilon_0 r^2} \).

THE Z COMPONENT OF \( E \) DUE TO THE AREA \( dA \) IS

\[
dE_z = \frac{\sigma \, dA}{4 \pi \varepsilon_0 R^2} \cos \theta
\]

BUT \( dA = r \, dr \, d\phi \)

\[\Rightarrow dE_z = \frac{\sigma \, r \, dr \, d\phi}{4 \pi \varepsilon_0 R^2} \cos \theta\]

THE E FIELD INTENSITY DUE TO \( ds \) IS THEN

\[dE_z = \int_0^\theta \int_0^R dE_z \]

\[= \frac{\sigma \, r \, dr}{4 \pi \varepsilon_0 R^2} \int_0^{2\pi} d\phi = \frac{\sigma \, r \, dr}{2 \varepsilon_0 R^2} \cos \theta\]

BUT \( R = \frac{1}{\cos \theta} \)

\( r = h \tan \theta \Rightarrow dr = h \sec^2 \theta \, d\theta \)

THUS

\[dE_z = \frac{\sigma \, h \, \cos^2 \theta \, \sec^2 \theta \, d\theta}{2 \varepsilon_0} = \frac{\sigma \, h \, \cos \theta \, \sin \theta \, d\theta}{2 \varepsilon_0} \]

THE TOTAL INTENSITY, AS PREVIOUSLY MENTIONED, IS

\[E = \int_{\text{plane}} dE_z = \frac{\sigma}{2} \int_0^{\pi/2} \sin \theta \, d\theta\]

\[E_z(\theta) \quad \text{but} \quad E_z = E \quad = \frac{\sigma}{2 \varepsilon_0}\]

Applying Gauss theorem for infinite plane (cont.)

it could be done "in one line."
We wish to consider the case where \( R = 2 \text{h} \) (or more specifically, \( h = \frac{1}{2} \)" and \( R = 1 \)""). The \( \vec{E} \) field resulting from within the corresponding circle (radius \( \frac{\sqrt{3}}{2} \)) is thus

\[
E_z (\text{circle}) = \frac{\sigma}{2\varepsilon} \int_0^{\cos^{-1} \frac{\text{h}}{\text{R}}} \sin \theta d\theta
\]

\[
= \frac{\sigma}{2\varepsilon} \left[ -\cos \theta \right]_0^{\pi/3}
\]

\[
= \frac{\sigma}{4\varepsilon} = \frac{1}{2} E_z
\]

Thus, half the total \( \vec{E} \) field results from the charge distribution within the circle, the other half obviously comes from the charge external to the surface.
2.4 P. FIND \( \hat{D} \) ON SURFACE OF TWO CONDUCTORS

\#1: PLANE CONDUCTOR, THICKNESS \( \theta \)

\#2: SLAB OF THICKNESS \( d \)

BOTH HAVE SAME DIMENSIONS (SAY \( a \times a = A \))

WHERE \( a >> d \),

\((1)\) CASE 1: BOTH CONDUCTORS HAVE SAME \( q \), \( Q \)

\#1 WILL HAVE A SURFACE DENSITY

\[ E' = \frac{q}{A} = \frac{Q}{A} = D_1 \]

(\( \hat{D} \) NORMAL TO PLANE)

\#2 WILL HAVE A SURFACE DENSITY

\[ D_2 \]

OF \( D_2 \), \( O_2 \), \( \frac{Q}{A} \) \[ \frac{2A}{2A + 2Ad} \]

(\( \hat{D} \) NORMAL TO SURFACE EVERYWHERE)

\[ \Rightarrow \frac{1}{2} THE \ DISPLACEMENT \ OF \ #1 \]

\((2)\) CASE \#2: BOTH WILL HAVE SAME \( \vec{0} \) AND TUS HAVE SAME NORMAL \( \hat{D} \)

COMPONENT (IN MAGNITUDE) AT EACH POINT, (EXCEPT @ #2S CORNERS \( \hat{D} = 0 \))

\[ No!! \]

see CHIN S. did it
Are \( \hat{D} \) and \( \hat{E} \) always parallel?

If no, give example.

\( \hat{D} \) and \( \hat{E} \) are not always parallel to each other. (They are in a homogenous isotropic medium, but isotropic materials are not parallel in a crystalline dielectric. Here, the term \( \hat{E} \) in the definition \( \hat{D} = \varepsilon \hat{E} \) takes on a tensor nature:

\[
\begin{bmatrix}
D_x \\
D_y \\
D_z
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\]

(from 1.19)

Here, \( \hat{D} \) is obviously not parallel to \( \hat{E} \).
2.04. WE KNOW THAT A POINT CHARGE $q$ INDUCES
A CHARGE $-\frac{q}{r}$ ON A GROUNDED SPHERE;

\[ q \rightarrow \frac{q}{r} \rightarrow \frac{1}{r} \]

CONSIDER THEN, THE PROBLEM AT HAND.
INITIALLY, WE HAVE THE THREE CHARGED SPHERES

GROUND SPHERE #1, THE INDUCED CHARGE
BY THE UNGROUNDED SPHERES IS

\[ Q_1 = \frac{-2qr}{r} \]

INSULATE SPHERE #1 SO THAT ITS CHARGE
DOESN'T CHANGE. GROUND SPHERE #2,
THE CHARGE INDUCED IS THEN

\[ Q_2 = -q(Q_1 + Q_2) \]

INSULATE UNGROUNDED $Q_2$. GROUND $Q_3$,
THE INDUCED CHARGE IS

\[ Q_3 = -q(Q_2 + Q_1) \]
To find the $Q_i$'s, simply work backwards:

\[
\begin{align*}
Q_1 & = -\frac{2Qa}{r} \\
Q_2 & = -\frac{a}{r} (Q + Q_1) \\
& = -\frac{a}{r} (Q - \frac{2Qa}{r}) \\
& = -\frac{aQ}{r} \left( 1 - \frac{2a}{r} \right) \\
Q_3 & = -\frac{a}{r} \left[ Q_2 + Q_1 \right] \\
& = -\frac{a}{r} \left[ -\frac{aQ}{r} \left( 1 - \frac{2a}{r} \right) + \frac{2Qa}{r} \right] \\
& = \frac{aQ}{r} \left[ \frac{aQ}{r} \left( 1 - \frac{2a}{r} \right) + \frac{2Qa}{r} \right] \\
& = \frac{a^2Q}{r^2} \left[ \left( 1 - \frac{2a}{r} \right) + \frac{2Q}{r} \right] \\
& = \frac{a^2Q}{r^2} \left( 3 - \frac{2a}{r} \right)
\end{align*}
\]
WE MAY LOOK AT THE PROBLEM AS TWO CAPACITORS IN SERIES. \( C_1 \) IS THE CAPACITANCE FROM THE CONDUCTOR TO THE OUTER SURFACE OF THE DIELECTRIC.

\[
\Rightarrow C_1 = \frac{4\pi \varepsilon_0 ab}{b-a} = \frac{4\pi \varepsilon_0 q b}{b-a} \quad \text{(Eq. 2.03(1))}
\]

\( C_2 \) IS THE CAPACITANCE FROM THE OUTER SURFACE OF THE DIELECTRIC TO \( \infty \):

\[
\Rightarrow C_2 = 4\pi \varepsilon_0 b \quad \text{(Eq. 2.03(2))}
\]

THE TOTAL CAPACITANCE, \( C \), IS THEN

\[
\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{b-a}{4\pi \varepsilon_0 ab} + \frac{1}{4\pi \varepsilon_0 b} = \frac{1}{4\pi \varepsilon_0 b} \left[ \frac{b-a}{ka} + 1 \right] = \frac{1}{4\pi \varepsilon_0 b} \frac{b-a+ka}{ka} \Rightarrow C = \frac{4\pi \varepsilon_0 ka b \varepsilon_0}{b-a+ka}
\]
To solve this, use Green's Reciprocation Theorem:

\[ V_{p_1} = \frac{q}{4\pi \varepsilon_0 r} \]

Point
\[ q_{p_1} = 0 \]
\[ V_S = \frac{q}{C} \]

Sphere
\[ q_S = q \]

Here, \( C \) is the capacitance in Prob 2013.

Then, due to Green's Reciprocation Theorem:

\[ V_{p_1} q_{p_1}' + V_S q_S' = q_{p_1} V_{p_1}' + q_S V_S' \]

\[ \frac{q^2}{4\pi \varepsilon_0 r} + \frac{q Q}{C} = 0 \Rightarrow Q = -\frac{q C}{4\pi \varepsilon_0 r} \]

Where \( C = \frac{4\pi k ab \varepsilon_0}{b - a + ka} \)

\[ Q = \frac{-q}{4\pi \varepsilon_0 r} \frac{4\pi k ab \varepsilon_0}{b + (k-1)a} = -q \frac{ab k}{r(b + (k-1)a)} \]
WE REQUIRE \( q_1 = q_2 = q \) = CHARGE 

INDUCED BY THE POINT CHARGE q AT \( P_1 \). TO APPLY GREEN'S 

RECIPIR. THEOREM, WE WISH TO FIND THE POTENTIAL \( V_p' \) TO WHICH 

POINT \( P_1 \) WOULD BE RAISED DUE TO \( V_1' \) AND \( V_2' \) 

PLACED ON INNER \( \frac{1}{2} \) OUTER CONDUCTORS RESPECTIVELY, 

SINCE ALL THE FLUX FROM q 

IN (1) LANDS ON CONDUCTORS; 

\[ q = -2q \] 

APPLYING GREEN'S RECIPROCAL THEOREM: 

\[ qV_1' + qV_p' + qV_2' = 0 \] 

OR \[ qV_1' - 2qV_p' + qV_2' = (V_1' - 2V_p' + V_2')q = 0 \]

THE REQUIREMENT OF EQUAL INDUCED CHARGES 

IS THUS: \[ V_1' - 2V_p' + V_2' = 0 \] 

OR, EQUIVALENTLY: \[ V_1' - V_p' = V_p' - V_2' \] (1)

WE FIND THIS FROM CONFIGURATION (II). PLACE 

A GAUSSIAN SPHERICAL SURFACE AT \( r = \frac{a+b}{2} \). THEN 

\[ q' = \int \text{G}r \cdot \hat{n} \, dS = \frac{4\pi r^2}{2}E_1 \]

\( \Rightarrow E = \frac{q'}{4\pi r^2} \)

\( \Rightarrow V_i - V_p' = \frac{q'}{4\pi \epsilon_1} \int_{r=a+b/2}^{r=1} \frac{dr}{r^2} = \frac{q'}{4\pi \epsilon_1} \frac{1}{r} \mid^a_{b} \)

\[ = \frac{-q'}{4\pi \epsilon_1} \left( \frac{2}{a+b} - \frac{1}{a} \right) = \frac{q'}{4\pi \epsilon_1 a(a+b)} \] (2)

SIMILARLY, WE HAVE 

\[ V_p' - V_2' = \frac{q'}{4\pi \epsilon_2} \int_{b}^{1} \frac{dr}{r^2} = \frac{-q'}{4\pi \epsilon_2} \frac{1}{1} \mid^{b}_{a_2} \]

\[ = \frac{-q'}{4\pi \epsilon_2} \left[ \frac{1}{b} - \frac{2}{a+b} \right] = \frac{-q'}{4\pi \epsilon_2} \left[ \frac{(a+b)-(a+b)/2}{b(a+b)} \right] \]

\[ = \frac{q'}{4\pi \epsilon_2} \left[ \frac{b-a}{b(a+b)} \right] \] (3)
To satisfy Eq. (1) we equate Eqs. (2) \( \frac{b-a}{q'} \) and \( \frac{b-a}{q'} \) of \( \frac{1}{4\pi \epsilon_1} \) and \( \frac{1}{4\pi \epsilon_2} \). Thus,

\[
\frac{1}{\epsilon_1 a} = \frac{1}{\epsilon_2 b}
\]

The necessary ratio of dielectrics is thus

\[
\frac{\epsilon_1}{\epsilon_2} = \frac{b}{a}
\]
WE HAVE FROM 2.28, THE GENERAL EXPRESSION FOR FORCE ON A CONDUCTOR:

\[ F_p = \int_S \frac{\mathbf{D} \cdot \mathbf{E}}{2} \cdot \mathbf{n} d\mathbf{s} \]

CONSIDER THE PART OF THE CAPACITOR CONTAINING \( S_1 \). FOR A LENGTH \( l \), THE SURFACE CHARGE DENSITY ON THE CONDUCTOR IS \( \sigma = \frac{Q}{2\pi a} \). \( \sigma \) IS NORMAL TO THE CYLINDER AND EQUAL IN MAGNITUDE TO \( \sigma \). Thus

\[ F_1 = \int_{S_1} \frac{1}{2} \varepsilon_0 \mathbf{E} \cdot \mathbf{D} d\mathbf{s} \]

WE CAN SPEAK OF THE "PRESSURE" (NORMAL) TO THE TOP HALF OF THE CYLINDER:

\[ \rho_n = \frac{dF_1}{dS} = \frac{1}{2\varepsilon_0} \frac{q^2}{(2\pi a)^2} \]

DUE TO SYMMETRY, THE RESULTING FORCE ON THE UPPER HALF WILL BE IN THE \( y \) DIRECTION:
THE COMPONENT OF PRESSURE IN THIS DIRECTION IS
\[ p_y = \rho \cdot g \cdot \frac{q^2}{2 \varepsilon_1} \]
\[ p_y = \rho \cdot g \cdot \frac{q^2}{2 \varepsilon_1} \]

THE TOTAL FORCE ACTING ON THE TOP CYLINDER IS THEN
\[ F_{y_1} = \int_0^l p_y \, ds = \frac{1}{2} \rho \cdot g \cdot \frac{q^2}{(2\pi)^2 \varepsilon_1} \int_0^l \sin \theta \, d\theta \, d\varepsilon \]

WE TAKE ds TO BE A LONG THIN SLICE OF LENGTH \( l \)

\[ ds = l \cdot d\theta \]

Thus
\[ F_{y_1} = \frac{1}{2} \rho \cdot g \cdot \frac{q^2}{(2\pi)^2 \varepsilon_1} \int_0^l \sin \theta \, d\theta \]

Due to symmetry, the component of force acting on the lower cylinder is (in \(-y\) direction)
\[ F_{y_2} = \frac{q^2}{(2\pi)^2 \varepsilon_2} \]

The total force on the cylinder, \( F \) (per unit length) is thus
\[ F = \frac{1}{2} \left[ F_{y_1} - F_{y_2} \right] \]

\[ = \frac{q^2}{(2\pi)^2 \varepsilon_1} \left[ \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2} \right] \]

\[ = \frac{q^2}{(2\pi)^2 \varepsilon_1} \left( \varepsilon_2 - \varepsilon_1 \right) \]

\[ = \frac{q^2 (\varepsilon_2 - \varepsilon_1)}{(2\pi)^2 \varepsilon_1 \varepsilon_2} \]
401. \[ \text{GIVEN POTENTIAL BOUNDARY CONDITIONS} \]

WILL NOW SHOW THAT THE GIVEN BOUNDARY CONDITIONS ARE SATISFIED BY THE LINE CHARGE DISTRIBUTION:

THE VALUES OF \( q \) WILL BE SHOWN TO BE DETERMINED BY \( V_0 \).

IF \( U \) IS THE POTENTIAL ON THE \( Z \) PLANE:

\[ W = V + \hat{z} U = -\frac{1}{2\pi\epsilon} \sum_{s=1}^{n} q_s \ln(z - z_s) \quad (\text{Eq. 4.12(4)}) \]

THE POTENTIAL \( U \) IS THUS GIVEN BY

\[ U = \frac{1}{2\pi\epsilon} \sum_{s=1}^{n} q_s \arg(z - z_s) = \text{Im}[W] \]

WHERE, FROM FIG 1:

\[ n = 3 \]
\[ z_1 = -a, \quad q_1 = q \]
\[ z_2 = 0, \quad q_2 = -2q \]
\[ z_3 = a, \quad q_3 = q \]
Consider first, the potential on the Y axis:

\[
\begin{align*}
\theta_1 + \theta_3 &= \pi, \quad \text{and} \quad \theta_2 &= \frac{\pi}{2}, \\
U &= -\frac{q}{2\pi\varepsilon} [\theta_1 - 2\theta_2 + \theta_3] = 0
\end{align*}
\]

This boundary condition is satisfied.

Consider next, the x axis for |x| > a:

\[
\begin{align*}
\theta_1 &= \theta_2 = \theta_3 = 0 \Rightarrow U = 0 \\
\text{for} \quad 0 < x < a, \quad \theta_1 = \theta_2 = 0 \quad \text{and} \quad \theta_3 = \pi. \quad \text{Thus}
\end{align*}
\]

\[
U = V_0 = -\frac{q}{2\pi\varepsilon} (\pi) = -\frac{q}{2\varepsilon}
\]

This gives our relationship between the image charge value q, and the stated boundary potential V_0:

\[
q = -2\varepsilon V_0
\]

Due to symmetry, \( U = V_0 \) for \(-a < x < a\).

Due to the uniqueness theorem, evaluation of the field due to the "image" charges is equivalent to analysis of the given boundary value.
PROBLEM. THUS, THE ANALYSIS TO FOLLOW MAY BE SAID TO CONSTITUTE A "PROOF." AGAIN, WE HAVE

\[ U = \frac{1}{2\pi \epsilon} \sum_{n=1}^{\infty} q_n \arg (z - z_n) \]

\[ \begin{array}{c}
\begin{array}{c}
\text{Y} \\
\text{X}
\end{array}
\end{array} \]

\[ q_1 \quad q_2 \quad q_3 \]

\[ -a \quad 2a \quad a, q \quad x, y \]

FOR \( x > 0 \) AND \( y > 0 \), WE HAVE (FROM THE FIGURE)

\[ U = \frac{-q}{2\pi \epsilon} \left[ \theta_1 - 2\theta_2 + \theta_3 \right] \]

\[ = \frac{V_0}{\pi} \left[ \theta_1 - 2\theta_2 + \theta_3 \right] \]

\[ = \frac{V_0}{\pi} \left[ \tan^{-1} \frac{y}{x + a} - 2 \tan^{-1} \frac{y}{x} + \tan^{-1} \frac{y}{x - a} \right] \]

\[ \Rightarrow \pi U = V_0 \left[ \tan^{-1} \frac{y}{x + a} - 2 \tan^{-1} \frac{y}{x} + \tan^{-1} \frac{y}{x - a} \right] \]
403. Without loss of generality, assume that \( \frac{q}{i} \) is on the real axis.

\[ \begin{align*}
\text{Fig 1.}
\end{align*} \]

Now perform an inversion: \( z_1 = \frac{a^2}{z} \). The charge \( \frac{q}{i} \) is now at \( x_1 = \frac{a^2}{c} \).

The image for this case is in 4.05, and is

\[ \begin{align*}
\text{Performing an inverse mapping, } 0 \rightarrow \infty, \quad \frac{1}{i} \rightarrow \frac{a^2}{c}.
\end{align*} \]

Thus, from the first sentence on top of pg. 88, The desired image of Fig. 1 is

\[ \begin{align*}
\text{W within cylinder.}
\end{align*} \]

This is reasonable, it's just the dual of Sec. 4.05, i.e., an external line charge at \( \frac{a^2}{c} \) should give equivalent images.
4.8.

**Perform an Inversion:***

\[ Z_1 = \frac{2a^2}{z + ia} \]

1. \( z = a \Rightarrow Z_1 = \frac{2a^2}{a + ia} \frac{(a - ia)}{(a - ia)} = \frac{2a^2}{2a^2} (0 - ia) = a (1 - i) \)

2. \( z = ia \Rightarrow Z_1 = \frac{2a^2}{z - ia} = \frac{a}{i} = -ia \)

3. \( z = -a \Rightarrow Z_1 = (a - ia) \frac{(a + ia)}{(a + ia)} = \frac{-2a^2}{2a^2} (0 + ia) = -a (1 + i) \)

4. \( z = -ia \Rightarrow Z_1 = \pm i \infty \) \( (z = 0 \Rightarrow Z_1 = \frac{2a^2}{ia} = -2ia) \)

**Perform a Translation:**

\[ Z_2 = Z_1 + iq = \frac{2a^2}{z + ia} + iq \]

\[ = \frac{2a^2 + iq (z + ia)}{z + ia} \]

\[ = \frac{2a^2 - a^2 + ia \bar{z}}{z + ia} \]

\[ = a \left[ \frac{q + \bar{z}}{z + ia} \right] = ia \left[ \frac{z - ia}{z + ia} \right] \]

---

**Diagram:**

[Diagram showing the process of inversion and translation]
THE FOLLOWING POINT CHARGE DISTRIBUTION WILL SATISFY THE GIVEN POTENTIAL BOUNDARY CONDITIONS

\[ W = \frac{V_0}{\pi} \left[ -\ln(z_2 + a) - \ln z_2 + \ln(z_2 - a) \right] \]

BOTTOM HALF OF \( z_2 \) PLANE CORRESPONDS TO INSIDE OF CIRCLE

\[
W = V + \sum_{i=1}^{n} q_i \ln \left( z - z_i \right) \\
U = \sum_{i=1}^{n} q_i \arg \left( z - z_i \right) \\
U = \frac{-q}{2\pi \epsilon} \left[ -\Theta_1 = \Theta_2 + \Theta_3 \right]
\]

ON \( x_2 \) AXIS:

\( x_2 > 0 \Rightarrow \Theta_1 = \Theta_2 = \Theta_3 = 0 \)
\( \Rightarrow U = 0 \)

\( 0 < x_2 < a \Rightarrow \Theta_1 = \Theta_2 = 0 \quad \Theta_3 = -\pi \)
\( \Rightarrow U = \frac{-q}{2\epsilon} = V_0 \quad \Rightarrow V_0 = \frac{q}{2\epsilon} \)

\( -a < x_2 < 0 \Rightarrow \Theta_1 = 0 \quad \Theta_2 = \Theta_3 = -\pi \)
\( \Rightarrow U = 0 \)

\( x_2 < -a \Rightarrow \Theta_1 = \Theta_2 = \Theta_3 = \pi \)
\( U = \frac{-q}{2\pi \epsilon} \left[ \pi + \pi - \pi \right] = \frac{-q}{2\epsilon} = -V_0 \)

THE CORRESPONDING COMPLEX DISTRIBUTION IS THUS

\( W = \frac{V_0}{\pi} \left[ -\ln(z_2 + a) - \ln z_2 + \ln(z_2 - a) \right] \)
Now, go back to the \( z \) plane:

\[
W = \frac{V_0}{\pi} \left[ \ln \frac{Z_2 - a}{Z_2 + a} - \ln Z_2 \right]
\]

\[
= \frac{V_0}{\pi} \left[ \ln \left( \frac{\mathbf{i}a (Z_2 - \mathbf{i}a)}{\mathbf{i}a (Z_2 - \mathbf{i}a) + a} \right) - \ln \mathbf{i}a (Z_2 - \mathbf{i}a) \right]
\]

\[
= \frac{V_0}{\pi} \left[ \ln \left( \frac{i(a(Z_2 - i)a) - a(Z_2 + i)a)}{i(a(Z_2 - i)a) + a(Z_2 + i)a)} \right) + \frac{Z_2 + i}{Z_2 - ia} \right]
\]

\[
= \frac{V_0}{\pi} \left[ \ln \frac{Z(i - 1) - a(i - 1)}{Z(i + 1) + a(i + 1)} + \frac{Z + i}{Z - ia} \right]
\]

\[
= \frac{V_0}{\pi} \left[ \ln \frac{(i - 1) - a}{Z + a} + \frac{Z + i}{Z - ia} \right]
\]

\[
= \frac{V_0}{\pi} \left[ \ln \frac{(i + 1) - a}{Z + a} + \frac{Z + i}{Z - ia} \right]
\]

\[
= \frac{V_0}{\pi} \left[ \ln i + \ln \frac{Z - a}{Z + a} - \ln i + \ln a(Z - ia) \right]
\]

\[
= \frac{V_0}{\pi} \left[ \ln \frac{Z - a}{Z + a} + \ln a(Z - ia) \right]
\]

Now \( \ln \frac{Z - a}{Z + a} = \frac{(x - a)+jy}{(x+a)+jy} \)

\[
= \ln \left[ \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} \right]
\]

\[
\Rightarrow \arg \frac{Z - a}{Z + a} = \arg \left[ \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} \right]
\]

\[
= \arg \left[ (x+a)(x-a) + y^2 + j2ay \right]
\]

\[
= \arg \left[ x^2 - a^2 + y^2 + j2ay \right]
\]

\[
= \arg \left[ r^2 - a^2 + j2ay \right] \quad \exists \quad r^2 = x^2 + y^2
\]

\[
= \tan^{-1} \frac{2ay}{r^2 - a^2}
\]
\[ \frac{z+ia}{z-ia} = \ln a \left( \frac{x+i(y+a)}{x+i(y-a)} \right) \]

\[ = \ln a \left[ \frac{(x+i(y+a))(x-i(y-a))}{x^2 + (y-a)^2} \right] \]

\[ \Rightarrow \text{arg} \frac{z+ia}{z-ia} = \text{arg} \frac{z+ia}{z-ia} \]

\[ = \text{arg} \left[ \frac{x+i(y+a)}{x-i(y-a)} \right] \]

\[ = \text{arg} \left[ \frac{x^2 + (y+a)(y-a) + i(x(y+a) - (y-a)^2)}{x^2 - a^2 + i2ax} \right] \]

\[ = \text{arg} \left[ \frac{r^2 - a^2 + i2ax}{r^2 - a^2} \right] \]

\[ = \tan^{-1} \frac{2ax}{r^2 - a^2} \]

\[ W = V + jU = \frac{V_0}{\pi} \left[ \ln \frac{z-a}{z+a} + \ln \frac{z+ia}{z-ia} \right] \]

Then, since

\[ W = V + jU = \frac{V_0}{\pi} \left[ \text{arg} \left\{ \frac{z-a}{z+a} \right\} + \text{arg} \left\{ \frac{z+ia}{z-ia} \right\} \right] \]

\[ U = \text{Im} W = \frac{V_0}{\pi} \left[ \tan^{-1} \frac{2aY}{r^2 - a^2} + \tan^{-1} \frac{2ax}{r^2 - a^2} \right] \]
FROM THE GEOMETRY AND 4.12 (4):

\[ W = \frac{-q}{2\pi \epsilon} \ln \left[ \left( z - \left( \frac{1}{2} + \frac{i\sqrt{3}}{2} \right) c \right) \left( z - \left( \frac{3}{2} - \frac{i\sqrt{3}}{2} \right) c \right) \left( z - c \right) \right] \]

But

\[ \frac{1}{2} + \frac{i\sqrt{3}}{2} = e^{i2\pi / 3} \]
\[ \frac{3}{2} - \frac{i\sqrt{3}}{2} = e^{i4\pi / 3} \]

Thus

\[ W = \frac{-q}{2\pi \epsilon} \ln \left[ (z - c e^{i2\pi / 3})(z - c e^{i4\pi / 3})(z - c) \right] \]

\[ W = U + iV \]

IN THE CONVENTION WE'VE WRITTEN \( W \), \( U \) CORRESPONDS TO EQUAPOTENTIAL LINES.
Thus

\[ U = Re \mathbf{U} = K = \text{constant} \]

\[ K' = \ln \left[ |z - ce^{i \frac{2\pi}{3}}| |z - ce^{i \frac{4\pi}{3}}| |z - c| \right] \]

\[ K'' = \left| (z - ce^{i \frac{2\pi}{3}})(z - ce^{i \frac{4\pi}{3}})(z - c) \right| \]

where \( K' \) and \( K'' \) are also constants. \((K'' = e^{K'})\)

It remains to expand this relationship using the relationship

\[ z = re^{i \theta} \]

\[ K'' = \left| (re^{i \theta} - ce^{i \frac{2\pi}{3}})(re^{i \theta} - ce^{i \frac{4\pi}{3}})(re^{i \theta} - c) \right| \]

\[ = \left| (r^2e^{i \theta} + c^2e^{i \frac{2\pi}{3}} - c re^{i(\theta + \frac{2\pi}{3})} - c re^{i(\theta + \frac{4\pi}{3})}) - c^2re^{i\theta} + c^2re^{i\theta} - c^2re^{i\theta} + c^2re^{i\theta} \right| \]

\[ = \left| r^3e^{i \theta} + c^2re^{i \theta} - cr^2e^{i \theta} - c re^{i \theta} - cr^2e^{i \theta} - c re^{i \theta} - c^2re^{i \theta} + c^2re^{i \theta} \right| \]

\[ = r^6 + c^6 - r^3c^3 \left[ e^{i \frac{3\pi}{3}} + e^{-i \frac{3\pi}{3}} \right] \]

\[ = r^6 + c^6 - 2r^3c^3 \cos 3\theta \]

This ungodly expression was evaluated on three sheets of computer paper, and gives the desired answer:

\[ K''^2 = \text{const} \]

\[ = r^6 + c^6 - r^3c^3 \left[ e^{i \frac{3\pi}{3}} + e^{-i \frac{3\pi}{3}} \right] \]

\[ = r^6 + c^6 - 2r^3c^3 \cos 3\theta \]
**COMPUTATION OF POTENTIAL FROM 5.103**

Original

\[ b \]
\[ d \]
\[ a \]
\[ q_1 \]
\[ q_2 \]
\[ q_3 \]
\[ (P_1) \]
\[ (P_2) \]
\[ (P_3) \]
\[ \alpha \]
\[ \beta \]
\[ \gamma \]

\[ d = \frac{+a b}{\sqrt{a^2 + b^2}} \]
\[ \alpha + \beta = \pi \]

**WE MUST DETERMINE COORDINATES OF P_1, P_2, P_3.**

- \( P_2 = (x_2, y_2) = (0, -d) = (0, \frac{-a b}{\sqrt{a^2 + b^2}}) \)
- \( P_3 = (x_3, y_3) = (4b \cos \beta, -b \sin \beta) \)

\[ \frac{b}{(b^2 - d^2)} \]
\[ \frac{b^2}{(b^2 - d^2)} \]

\[ \Rightarrow P_3 = \left( \frac{b}{\sqrt{a^2 + b^2}}, \frac{-a b}{\sqrt{a^2 + b^2}} \right) \]

- \( P_1 = (x_1, y_1) = \left( a \cos \alpha, a \sin \alpha \right) \)

\[ \frac{a}{\sqrt{a^2 + b^2}} \]
\[ \frac{a^2}{\sqrt{a^2 + b^2}} \]

\[ \Rightarrow P_1 = (x_1, y_1) = \left( \frac{-a}{\sqrt{a^2 + b^2}}, \frac{-a b}{\sqrt{a^2 + b^2}} \right) \]

**OBSERVERLY,** \( Y_1 = Y_2 = Y_3 = \frac{-a b}{\sqrt{a^2 + b^2}} \)
We may find coordinates of $P_a, P_b, P_c, P_d$

$P_a = (x_a, y_a)$, $P_b = (x_b, y_b)$

$P_c = (x_c, y_c)$, $P_d = (x_d, y_d)$

$P_a = (0,0)$

$P_d = \left( \frac{-2a^2}{b}, \frac{-2a^3}{b} \right)$

$P_b = \left( -4a \cos \alpha, -4a \sin \alpha \right) = \left( \frac{4a^2}{\sqrt{a^2+b^2}}, \frac{-4a b}{\sqrt{a^2+b^2}} \right)$

$P_c = \left( x_a + x_d, y_b + y_d \right) = \left( \frac{-2a^2}{\sqrt{a^2+b^2}}, \frac{-2a}{\sqrt{a^2+b^2}} \right)$
IN MAPPING THE INVERSION BACK TO THE ORIGINAL, THE CHARGE \( q \) AT \( P_4 \) GOES TO INFINITY AND HAS NO EFFECT ON THE ORIGINAL POTENTIAL FIELD. THE CHARGE \(-q\) AT \( P_4 \) MAPS TO \( P_3 \):

\[
\frac{-q}{q_3} = \frac{2a^2}{b(2a)} = \frac{a}{2b} \quad \text{(FROM 5.10 (1))}
\]

\[
\Rightarrow q_3 = -\frac{2b}{a} q = -\frac{2b}{a} (4\pi \epsilon)
\]

SIMILAR MAPPING GIVES (\( P_c \) TO \( P_d \))

\[
q_2 = 4\pi \epsilon \frac{ab}{2a^2 + b^2} = 4\pi \epsilon b \frac{2}{2a^2 + b^2}
\]

\[
q_1 = -\frac{a}{2a} (4\pi \epsilon) = (4\pi \epsilon) \left(-\frac{1}{2}\right)
\]

WE MAY NOW COMPUTE THE POTENTIAL VIA 1.06 Eq. 3:

\[
v(x, y) = \frac{1}{4\pi \epsilon} \sum \frac{q_i}{r_i^2}
\]

\[
= \frac{\left(-\frac{1}{2}\right)}{\left[ (x + \frac{a}{\sqrt{a^2 + b^2})^2} + (y + \frac{ab}{\sqrt{a^2 + b^2}})^2 \right]^{\frac{1}{2}}}
\]

\[
= \frac{b}{2\sqrt{a^2 + b^2}} \left[ x^2 + (y + \frac{ab}{\sqrt{a^2 + b^2}})^2 \right]^{\frac{1}{2}}
\]

\[
+ \left[ (x - \frac{b^2}{\sqrt{a^2 + b^2}})^2 + (y + \frac{ab}{\sqrt{a^2 + b^2}})^2 \right]^{\frac{1}{2}}
\]

I was asking you to find the pot. in the inverse system and then transform it to the original. Obviously result will be the same.
IN MAPPING THE INVERSION BACK TO THE ORIGINAL, THE CHARGE \( q \) AT \( P_4 \) GOES TO INFINITY AND HAS NO EFFECT ON THE ORIGINAL POTENTIAL FIELD. THE CHARGE \(-q\) AT \( P_d \) MAPS TO \( P_3 \):

\[
\frac{-q}{q_3} = \frac{2a^2}{b(2a)} = \frac{a}{2b} \quad \text{(from 5.10 (1))}
\]

\[
\Rightarrow q_3 = -\frac{2b}{a} \cdot \frac{q}{q} = -\frac{2b}{a} (4\pi\varepsilon)
\]

SIMILAR MAPPING GIVES \((P_c \text{ to } P_d)\):

\[
q_2 = 4\pi\varepsilon \frac{ab}{\sqrt{a^2 + b^2}} = 4\pi\varepsilon \frac{b}{2\sqrt{a^2 + b^2}}
\]

\[
q_1 = -\frac{a}{2a} (4\pi\varepsilon) = (4\pi\varepsilon) (-\frac{1}{2})
\]

WE MAY NOW COMPUTE THE POTENTIAL VIA 1.06 Eq. 3:

\[
v(x, y) = \frac{1}{4\pi\varepsilon} \sum \frac{q_i}{r_i^2}
\]

\[
= \left[ (x + \frac{a}{\sqrt{a^2 + b^2}})^2 + (y + \frac{ab}{\sqrt{a^2 + b^2}}) \right] \frac{1}{r_2}
\]

\[
= -\frac{b}{2\sqrt{a^2 + b^2}}
\]

\[
+ \left[ x^2 + (y + \frac{ab}{\sqrt{a^2 + b^2}})^2 \right] \frac{1}{r_2}
\]

\[
= \left[ (x - \frac{b^2}{b} \right) \right] \frac{\left[ (x - \frac{b^2}{b} \right) \right] \frac{1}{r_2}
\]

"I was asking you to find the potential in the inverse system and then transform it to the original. Obviously result will be the same.\]
CONFORMAL MAPPINGS (CHAP. 4)

4.17. \( z_1 = \ln \frac{z + j a}{z - j a} \)

4.18. THE SCHWARZ TRANSFORM (GENERAL)

\[
z = C_1 \int \frac{d z_1}{(z_1 - u_1)^{\frac{1}{\lambda_1}} (z_1 - u_2)^{\frac{1}{\lambda_2}} \ldots} + C_2
\]

- \( C_1 \) scales and rotates
- \( C_2 \) translates
4.19. POLYGONS WITH ONE POSITIVE ANGLE

\[ Z = C_1 Z_1 \frac{\alpha}{\pi} + C_2 \]

\[ Z = C_1 2^{\frac{\alpha}{\pi} - 1} + C_2 = C_1 2^{\frac{\alpha}{\pi}} + C_2 \]

1. \( \alpha = 2\pi \)

\[ W = Z_1 \rightarrow \text{PARABOLIC} \]

2. \( \alpha = \frac{3\pi}{2} \)

3. \( \alpha = \frac{\pi}{2} \)

It is interesting to point out that the same problems have different problems in 2, using two different \( \alpha \) values.
HEXAGON WITH ZERO ANGLE ($\alpha = 0$)

$$Z = C_1 \ell \theta Z_1 + C_2$$  (A SCHWARZ XFORM)

($C_2 = 0$)

$\ell \theta$

UPPER HALF OF $Z_1$ PLANE

$\ell \theta$

$C_1 \ell \theta$

$\ell \theta$

$C_1 \ell \theta$

$\ell \theta$

$\ell \theta$

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4.21 POLYGONS WITH ONE NEGATIVE ANGLE: INVERSION
\[ \alpha = -\pi \Rightarrow z = \frac{a^2}{z_1} \]

4.22 POLYGONS WITH TWO ANGLES
60. It was shown in 5.17 that the potential inside the sphere, for \( a < r \leq b \) is given by

\[
V = \frac{Q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!!} \left[ \left( \frac{a}{r} \right)^{2n+1} - \left( \frac{a}{b} \right)^{2n+1} \right] \frac{r^{2n}}{b^{2n}} P_{2n}(\cos \theta)
\]

On the sphere, the \( \vec{E} \) field is normal (since the sphere is a conductor). Thus, we may obtain the field strength, \( E \), there, merely by differentiating the above expression with respect to \( r \) and evaluate at \( r = b \).

\[
E_b = -\frac{\partial V}{\partial r} \bigg|_{r=b}
\]

And now, the math:

\[
E_b = \frac{Q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!!} \left[ \frac{a^{2n+1}}{r^{2n+2}} - 2n \left( \frac{a}{b} \right)^{2n+1} \frac{r^{2n}}{b^{2n+2}} \right] P_{2n}(\cos \theta)
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!!} \left[ \frac{a^{2n+1}}{r^{2n+2}} - 2n \left( \frac{a}{b} \right)^{2n+1} \frac{r^{2n}}{b^{2n+2}} \right] P_{2n}(\cos \theta)
\]

\[
= \frac{Q}{4\pi \varepsilon} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!!} \left[ \frac{a^{2n+1}}{r^{2n+2}} - 2n \left( \frac{a}{b} \right)^{2n+1} \frac{r^{2n}}{b^{2n+2}} \right] P_{2n}(\cos \theta)
\]

\[
= \frac{Q}{4\pi \varepsilon} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!!} \left[ \frac{a^{2n+1}}{r^{2n+2}} - 2n \left( \frac{a}{b} \right)^{2n+1} \frac{r^{2n}}{b^{2n+2}} \right] P_{2n}(\cos \theta)
\]

\[
= \frac{Q}{4\pi \varepsilon} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!!} \left[ \frac{a^{2n+1}}{r^{2n+2}} - 2n \left( \frac{a}{b} \right)^{2n+1} \frac{r^{2n}}{b^{2n+2}} \right] P_{2n}(\cos \theta)
\]
5.61.

[ATTENTION HEREOF RESTRICTED TO $a \leq r \leq b$ UNLESS STATED OTHERWISE]

$\mu = \cos \Theta \ ; \ \sum_{n=0}^{\infty}$

1. Let the potential due to the inner sphere be

$$V_{in} = \sum_{n=0}^{\infty} \left( A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\mu)$$

Since it's grounded, $V_{in}(a) = 0$

$$\Rightarrow A_n a^n = -\frac{B_n}{a^{n+1}} \Rightarrow B_n = -a^{2n+1} A_n$$

Thus

$$V_{in} = \sum_{n=0}^{\infty} A_n \left[ r^n - \frac{a^{2n+1}}{r^{n+1}} \right] P_n(\mu) \quad (1)$$

2. Let the potential due to the outer sphere be

$$V_{out} = \sum_{n=0}^{\infty} \left( D_n r^n + \frac{E_n}{r^{n+1}} \right) P_n(\mu)$$

Since it too is grounded, $V_{out}(c) = 0$

$$\Rightarrow D_n c^n = -\frac{E_n}{c^{n+1}} \Rightarrow E_n = -D_n c^{2n+1}$$

Thus

$$V_{out} = \sum_{n=0}^{\infty} D_n \left[ r^n - \frac{c^{2n+1}}{r^{n+1}} \right] P_n(\mu) \quad (2)$$

3. The potential due to the charge $q$ is

$$V_c = \frac{q}{4\pi \varepsilon r}$$

For $a \leq r < b$

$$V_c = \frac{q}{4\pi \varepsilon r} \sum_{n=0}^{\infty} \left( \frac{b}{r} \right)^n P_n(\mu) \quad (3)$$

For $b \leq r \leq c$

$$V_c = \frac{q}{4\pi \varepsilon b} \sum_{n=0}^{\infty} \left( \frac{r}{b} \right)^n P_n(\mu) \quad (4)$$
MEETING BOUNDARY CONDITIONS

THE TOTAL POTENTIAL BETWEEN INNER AND OUTER SPHERES IS \( V = V_{\text{in}} + V_{\text{out}} + V_0 \).

\( V(a) = 0 \). EQUATING COEFFICIENTS OF \( P_n (\mu) \)

Using Eqs. 1, 2, and 3:

\[
\frac{-q}{4\pi \varepsilon_0} \left( \frac{b}{a} \right)^n = A_n \left[ a^n - \frac{a^{2n+1}}{2n+1} \right] + B_n \left[ a^n - \frac{b^{2n+1}}{2n+1} \right]
\]

\[= D_n \left[ a^n - \frac{a^{2n+1}}{2n+1} \right]
\]

\[= D_n \left[ \frac{a^{2n+1}}{2n+1} - \frac{b^{2n+1}}{2n+1} \right]
\]

\[\Rightarrow D_n = \frac{-q}{4\pi \varepsilon_0} \left( \frac{b}{a} \right)^n \frac{a^{2n+1}}{2n+1} - \frac{b^{2n+1}}{2n+1}
\]

\[= \frac{-q}{4\pi \varepsilon_0} \frac{b^{2n+1}}{2n+1} \frac{a^{2n+1}}{2n+1}
\]

\( V(c) = 0 \). EQUATING COEFFICIENTS OF \( P_n (\mu) \)

Using Eqs. 1, 2, and 4:

\[
\frac{-q}{4\pi \varepsilon_0} \left( \frac{c}{b} \right)^n = A_n \left[ c^n - \frac{a^{2n+1}}{2n+1} \right] + D_n \left[ c^n - \frac{b^{2n+1}}{2n+1} \right]
\]

\[= A_n \left[ c^n - \frac{a^{2n+1}}{2n+1} \right]
\]

\[= A_n \left[ \frac{c^{2n+1}}{2n+1} - \frac{a^{2n+1}}{2n+1} \right]
\]

\[\Rightarrow A_n = \frac{-q}{4\pi \varepsilon_0} b^{2n+1} \frac{c^{2n+1}}{2n+1} - \frac{a^{2n+1}}{2n+1}
\]

(CONT)
WE WISH TO FIND \( V \) FOR \( a < r < b \). FROM Eqs. 5,2,1 \( (2) \)

\[
V = V_e + V_h + V_{\text{out}} = \frac{q}{4\pi \varepsilon r} \left( \frac{b}{R} \right)^n + A_n \left( \frac{R^n - \frac{a^{2n+1}}{r^{n+1}}}{} \right) + p_n \left( \frac{R^n - \frac{c^{2n+1}}{r^{n+1}}}{} \right) P_n (\mu)
\]

SUBSTITUTING Eqs. 5 AND (6):

\[
V = \frac{q}{4\pi \varepsilon} \left[ \sum \frac{b^n}{r^{n+1}} + \frac{C^{2n+1}}{b^{n+1}} \left( \frac{a^{2n+1} - c^{2n+1}}{r^{n+1}} \right) \left( \frac{r^n - \frac{a^{2n+1}}{r^{n+1}}}{} \right)
- \left( \frac{a^{2n+1} - c^{2n+1}}{r^{n+1}} \right) \left( \frac{r^n - \frac{c^{2n+1}}{r^{n+1}}}{} \right) \right] P_n (\mu)
\]

\[
= \frac{q}{4\pi \varepsilon} \sum \left( \frac{b^{n+1} (c^{2n+1} - a^{2n+1})}{r^{n+1} (a^{2n+1} - c^{2n+1}) b^{n+1}} \right)
\]

\[
= \frac{q}{4\pi \varepsilon} \sum \left[ b^{n+1} \left( \frac{c^{2n+1} - a^{2n+1}}{a^{2n+1} - c^{2n+1}} \right) \left( r^{2n+1} - a^{2n+1} \right)
+ b^{n+1} \left( \frac{c^{2n+1} - r^{2n+1}}{c^{2n+1} - a^{2n+1}} \right) \right] P_n (\mu)
\]

\[
= \frac{q}{4\pi \varepsilon} \sum \left[ b^{n+1} \left( \frac{a^{2n+1} - a^{2n+1}}{a^{2n+1} - c^{2n+1}} \right) \left( r^{2n+1} - a^{2n+1} \right)
- b^{n+1} \left( \frac{c^{2n+1} - r^{2n+1}}{r^{2n+1} - a^{2n+1}} \right) \right] P_n (\mu)
\]

\[
= \frac{q}{4\pi \varepsilon} \sum \left[ b^{n+1} \left( \frac{a^{2n+1} - c^{2n+1}}{a^{2n+1} - c^{2n+1}} \right) \left( r^{2n+1} - a^{2n+1} \right)
- b^{n+1} \left( \frac{c^{2n+1} - r^{2n+1}}{r^{2n+1} - a^{2n+1}} \right) \right] P_n (\mu)
\]

\[
= \frac{q}{4\pi \varepsilon} \sum_{n=0}^{\infty} b^{n+1} \left( \frac{a^{2n+1} - c^{2n+1}}{a^{2n+1} - c^{2n+1}} \right) \left( r^{n} - \frac{a^{2n+1}}{r^{n+1}} \right) P_n (\cos \theta)
\]
602C. Consider an arbitrarily shaped sheathed conductor with given cross-sectional area.

To minimize the leakage current, consider the following (two-dimensional) Gaussian "surface." The (two-dimensional) leakage current is given by

\[ I_L = -\int \frac{1}{q} \frac{\delta V}{\delta n} \, d\ell \]

For the inner conductor at a given potential, the dotted surface is at equipotential and \( I_L \) is clearly minimized by minimizing the length of the path of integration, i.e., by minimizing \( \ell \). Under the constraint \( A_c \) remains constant, it is clear that the optimal shape of the inner curve is circular. \( \Rightarrow \)
We are here employing the fact that minimization of leakage current corresponds to maximizing lateral resistance. It remains to find the optimal sheath shape. Consider the following geometry.

Suppose we now divide the sheath into a number of small resistances with lateral resistances $\Delta R_n$. These resistances are in parallel. To maximize the total resistance, it follows from elementary circuit theory that we wish to make all $\Delta R_n$'s the same. Thus, due to symmetry, we require that the outer surface of the sheath also be circular.

This configuration maximizes resistance to current leakage.
We know from 4.14(2) that the capacitance between two similar cylinders each of radius $a$ and separated by a distance $d$ is

$$C = \pi \varepsilon \left[ \cosh^{-1} \frac{d}{2a} \right]^{-1}$$

The corresponding (two-dimensional) resistance is then given from 6.06(7) as

$$R = \frac{\varepsilon}{\pi \varepsilon} \cosh^{-1} \frac{d}{2a}$$

but in addition, you have a dish to this you have a dish.
WE HAVE FROM ART. 7.10
\[ A_\phi = \frac{\mu_0}{4\pi} \int \frac{ds_4}{r} \]
WHERE
\[ ds_4 = a \cos \phi \, dd \]
\[ r = \sqrt{a^2 + \beta^2 + z^2 - 2az \cos \phi} \]

BUT, FROM 5.197, WE CAN WRITE
\[ \frac{1}{r} = \sum_{s=0}^{\infty} (2s+1) e^{-k|z|} J_s(ka) J_s(k\beta) \, dk \]

BUT, DUE TO SYMMETRY, THE RESULTING VECTOR POTENTIAL, \( A_\phi \), WILL BE INDEPENDENT OF THE VARIABLE \( \phi \).

ALSO, WE HAVE THE CLEARLY OBVIOUS SITUATION OF \( z_0 = 0 \), THE EQUATION WRITTEN IMMEDIATELY ABOVE THUS BECOMES
\[ \frac{1}{r} = \sum_{s=0}^{\infty} (2s+1) \int_0^\infty e^{-k|z|} J_s(ka) J_s(k\beta) \, dk \]

THUS, THE VECTOR POTENTIAL BECOMES
\[ A_\phi = \frac{\mu_0 I_0}{4\pi} \int_0^{2\pi} \sum_{s=0}^{\infty} (2s+1) e^{-k|z|} J_s(ka) J_s(k\beta) \, dk \]

\[ = \frac{\mu_0 I_0}{2} \sum_{s=0}^{\infty} (2s+1) \int_0^\infty e^{-k|z|} J_s(ka) J_s(k\beta) \, dk \]

BUT, CLEARLY, DUE TO THE PROPOSED SOLUTION (i.e., DUE TO B.C.), THE ONLY CONTRIBUTION TO THIS WILL BE \( s = 1 \), CHOOSING \( J_0(ka) = 0 \), THIS IS
\[ A_\phi = \frac{\mu_0 I_0}{2} \int_0^\infty e^{-k|z|} J_1(ka) J_1(k\beta) \, dk \]
We address the problem of finding $B$ in the square loop shown. The contribution $B$, to the flux density at a point $P$ from the side of the conducting loop located at $x = \frac{a}{2}$ is directed perpendicular to the plane formed by the loop and the point $P$. Its magnitude is

$$B = \frac{I}{2\pi sd} \sin \alpha$$

where $s$ is the distance to the point, and $\alpha$ the angle between

$\phi$. What problem from book is it?
Thus \[ B = \frac{\mu I}{2\pi s} \frac{1}{\sqrt{(a/2)^2 + s^2}} \]

Taking the contribution from two sides:

\[ B \cos^2 \phi = B \frac{a}{2} = \frac{\mu I a d}{8\pi s^2 d} \frac{1}{\sqrt{(a/2)^2 + s^2}} \]

But \[ s^2 = h^2 + (a/2)^2 \]

\[ \Rightarrow B \cos^2 \phi = \frac{\mu ad I}{8\pi d \sqrt{h^2 + a^2/4 + d^2/4}} \frac{1}{h^2 + a^2/4} \]

We may immediately write the contribution from the other sides as

\[ \frac{\mu ad I}{8\pi d \sqrt{h^2 + a^2 + d^2/4}} \frac{1}{h^2 + a^2/4} \]

The total flux is the sum of the above relationships

\[ B = 4\pi \sqrt{h^2 + a^2 + d^2/4} \left[ \frac{1}{h^2 + a^2/4} + \frac{1}{h^2 + d^2/4} \right] \]

What happened to an additional wire with current \( I \)?
WE KNOW THAT $2\theta = 2\pi \Rightarrow \theta = \frac{n}{m} \Rightarrow \theta = \frac{\pi}{m}$

FROM BIOT AND SAVART'S LAW (ART. 7.14) WE KNOW THAT THE MAGNETIC INDUCTION DUE TO AN INFINITE WIRE AT A POINT $P$ A DISTANCE $a$ FROM IT IS

$$B = \frac{\mu I}{2\pi a}$$

IT FOLLOWS FROM ART (7.14) THAT THE MAGNETIC INDUCTION FROM A WIRE OF FINITE LENGTH AT A POINT LYING ON ITS PERPENDICULAR BISECTOR IS

$$B = \frac{\mu I}{2\pi a} \sin \frac{\Theta}{2}$$

WHERE $\Theta$ IS THE ANGLE SUBTENDED BY THE LINE. IT FOLLOW THAT, FOR THE POLYGON CONSIDERED, THAT $B$ DUE A SINGLE SEGMENT IS

$$B_s = \frac{\mu I}{2\pi a} \sin \left( \frac{n\pi}{2m} \right)$$

SUPERIMPOSING ALL $2n$ SEGMENTS GIVES THE DESIRED ANSWER

$$B = 2\pi B_s = \mu I \pi a \sin \left( \frac{n\pi}{2m} \right)$$

because at the center all $B_s$ are parallel.
To find this relationship for \( a = b \), we need only to evaluate \( \lim_{b \to a} M_{12} \).

The term of interest is

\[
T_n = \lim_{b \to a} ab (a^2 - b^2)^{n+1} p_{n+1}^{\prime} \left( \frac{a^2 + b^2}{a^2 - b^2} \right)
\]

In this limit, the argument of \( p_{n+1}^{\prime} \) will tend to \( \infty \). From eq.23(12):

\[
p_n (v) \xrightarrow{\mu \to a} \frac{2^n}{n!} n! (n-m)! \mu^n
\]

It follows that

\[
p_{n+1}^{\prime} \left( \frac{a^2 + b^2}{a^2 - b^2} \right) \xrightarrow{b \to a} 2^{n+1} (n+1)! (n_1!)
\]

Thus

\[
T_n = ab (a^2 - b^2)^{n+1} 2^{n+1} (n+1)! n!
\]

\[
= ab \frac{(a^2 + b^2)^{n+1}}{(2a^2)^{n+1}} n!
\]

But, since \( a = k \),

\[
T_n = a^{2n+3} \frac{2^{n+1}(n+1)! n!}{(n+1)! \cdot n!}
\]

\[
= a \cdot (n+1)! \cdot n!
\]
SUBSTITUTING THIS VALUE FOR $ab (a^2 - b^2)^{n+1} P_{n+1} \left( \frac{a^2 + b^2}{a^2 - b^2} \right)$ IN EQUATION 1 GIVES THE DESIRED ANSWER:

$$M_{12} = 2\pi \mu a \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!! (a^2 + b^2)^{n+1}}{n! (n+1)! (2n+4)!!} \left( \frac{a}{r} \right)^{2n+3} P_{2n+2} (\cos \alpha)$$
8.34. This is essentially equivalent to Prob. 32. Our mutual inductance is now
\[ M_{12} = 2\pi \mu ab \sum_{q=0}^{\infty} \frac{(-1)^{p+q+1} (2q+1)!!}{(a^2-b^2)^{q+1}} P_{q+1}(\frac{a^2-b^2}{2q+1}) P_{q+1}(\frac{a^2-b^2}{2q}) \]

From the previous problem, we may write
\[ P_{q+1}(\frac{a^2+b^2}{a^2-b^2}) \frac{b}{b+a} = \frac{2^{q+1}}{(q+1)!} \]

Substituting into \( M_{12} \):
\[
\lim_{b \to a} M_{12} = 2\pi \mu ab \sum_{q=0}^{\infty} \frac{(-1)^{p+q+1} (2q+1)!!}{(a^2-b^2)^{q+1}} \frac{(a^2+b^2)^{q+1}}{(2q+1)(2q+3)!! q!(q+1)! 2^{q+1} P_{q+1}(\alpha \alpha_p)}
\]

But \( b = a \). This gives the desired answer:
\[
\lim_{b \to a} M_{12} = 2\pi \mu a^2 b \sum_{q=0}^{\infty} \frac{(-1)^{p+q+1} (2q+1)!!}{(2q+1)(2q+3)!! q!(q+1)!} \frac{(a^2+b^2)^{q+1}}{(2q+1) P_{q+1}(\alpha \alpha_p)}
\]
4. Find potential outside grounded conducting sheet, interior of cavity.

9. Due to line charge $q$ placed $E_i = b \neq 0 \neq 0$.

10. Find the image law for this case. (15)

2. Find $V_0$ outside & potential $V_i$ inside using mutual

My answer for charge placed half way between plane & center of ellipse with relative capacitance $k$ and radius $a$. The distance of sphere is distance 49 away from plane.

\[
\frac{\alpha}{\theta} = \frac{1}{2}
\]

(15)

3. Field lines inside conductors $90^\circ$ unstable. Find field $V_0$ whose sides approach with $1 \times \frac{1}{2} = V_0$

\[\text{ref. pg. 22, 229, 232, 235 (25)}\]
By pg. 70, the image is

\[ \begin{array}{c}
\text{charge here. It's grounded!} \\
\text{Potential is}
\end{array} \]

\[ V = \frac{1}{2\pi \epsilon} \left[ \frac{2}{3} q_s \operatorname{Re} \ln (z - z_s) \right] \]

\[ = -\frac{1}{2\pi \epsilon} \left[ -q_1 \operatorname{Re} \ln (z - \frac{a^2}{b}) + q \operatorname{Re} \ln (z - b) \right] \]

\[ = \frac{q}{2\pi \epsilon} \ln \sqrt{(x - \frac{a^2}{b})^2 + y^2} - \ln \sqrt{(x - b)^2 + y^2} \]

\[ V \neq 0 \text{ at the cylinder} \]
Inside, Image is. From pg 69

\[ q'' = \frac{2}{1 + k} q \]

Image

\[ \begin{array}{ccc}
-2a & b + \frac{q^2}{b} & 2a \\
q' & q'' & -q'' & -q'
\end{array} \]

Use Eq. 1 with

\[ \begin{align*}
Z_1 &= -b + \frac{q^2}{b} \\
Z_2 &= -2a \\
Z_3 &= 2a \\
Z_4 &= b - \frac{q^2}{b}
\end{align*} \]

\[ \begin{align*}
q_1 &= q' \\
q_2 &= q'' \\
q_3 &= \frac{1}{k} q'' \\
q_4 &= -q'
\end{align*} \]
This form opens the problem to a single sheet:

\[ v, \]

\[ V_0 \text{ and ?...} \]
Let total flux leaving a line charge $= \Phi$.

From $\frac{1}{2}$, the total flux entering prism is (for both) = $\Phi/4$? Why?

(This is due to symmetry.)

Let $\rho = \Phi/4 = \text{linear charge density}$. If we take a Gaussian surface enclosing box, we get zero. Is this the

What is the conclusion?
For a sphere: \( Q = \frac{q}{f} \left( 1 + \pi s_n \right) \quad q = 1 \quad \text{(unit charge)} \)

\[ Q = \frac{-v}{v'} \cdot q \]

\[ r = q \left( 1 + \pi s_n \right) \]
Equal to (pg. 69)

\[ q' = \frac{1-k}{1+k} q \]

\[ V_0 = \frac{1}{2\pi} \sum q_s \Re \ln (z - z_s) \]

\[ z_1 = -b \]
\[ z_2 = -b + \frac{q^2}{b} \]
\[ z_3 = -2q \]
\[ z_4 = 2q \]
\[ z_5 = b - \frac{q^2}{2} \]
\[ z_6 = b \]
$P_{ij} = \nabla_1, \# = 2$

$S_{11} = S_{22} = S_{33} = S_{44} = S_7$
$S_{12} = S_{13} = S_{24} = S_{42} = S_5$
$S_{14} = S_{23} = S_2$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} S_7 & S_5 & S_5 & S_5 \\ S_5 & S_7 & S_5 & S_5 \\ S_5 & S_5 & S_7 & S_5 \\ S_5 & S_5 & S_5 & S_7 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$V_1 = V_4 \quad V_2$

ARC!
102.

**Suppose we place a Gaussian surface immediately on the prism, such that all the line charges are enclosed by incremental circles:**

\[ q = \oint \vec{E} \cdot \hat{n} \, ds \]

\[ \text{Total charge enclosed} = 0 \]

Gauss' flux theorem states that the total flux normal to the surface is equivalent to the total charge enclosed. For the surface above, this is clearly zero.

It follows from symmetry that the total flux that is leaving the surface is the same as that entering, that is, \( \frac{1}{2} \) (50%) of the total flux enters the box.
IT FOLLOWS BY SYMMETRY, THAT
\[ S_{12} = S_{21} = S_{14} = S_{41} = S_{23} = S_{32} = S_{34} = S_{43} \]
\[ S_{11} = S_{22} = S_{33} \]
\[ S_{13} = S_{31} = S_{24} = S_{42} \]
WE NOW GO THROUGH THE PERScribed STEPS:
(a) PLACE CHARGE \( Q \) ON \( #1 \)
(b) TOUCH \( #1 \) TO \( #2 \)
\[ \Rightarrow V_1 = V_2 \]
\[ V_1 = S_{11} Q_1 + S_{21} Q_2 \quad V_2 = S_{12} Q_1 + S_{22} Q_2 \]
THUS \[ S_{11} Q_1 + S_{21} Q_2 = S_{12} Q_1 + S_{22} Q_2 \]
\[ (S_{11} - S_{12}) Q_1 = (S_{22} - S_{21}) Q_2 \]
BUT \[ S_{11} - S_{12} = S_{22} - S_{21} \]
\[ \therefore Q_1 = Q_2 \]
WE KNOW THAT, DUE TO CHARGE
CONSERVATION: \[ Q_1 + Q_2 = Q \]
IT FOLLOWS THAT
\[ Q_1 = Q_2 = \frac{Q}{2} \]
(c) TOUCH # 1 TO # 3

⇒ \( V_1 = V_3 \)

\[
V_1 = S_{11} Q_1' + S_{21} Q_2' + S_{31} Q_3'
\]

\[
V_3 = S_{13} Q_1' + S_{23} Q_2' + S_{33} Q_3'
\]

Thus \((S_{11} - S_{13}) Q_1' + (S_{21} - S_{23}) Q_2' + (S_{31} - S_{33}) Q_3' = 0\)

But \( S_{21} - S_{23} = 0 \) and \( S_{11} - S_{13} = - (S_{31} - S_{33}) \)

⇒ \( Q_1' = Q_3' \)

We know that \( Q_1' + Q_3' = Q_1'' = \frac{a}{2} \)

It follows that

\[
Q_1' = Q_3' = \frac{a}{2} \quad \forall y \Rightarrow Q_2'' = \frac{3a}{4}
\]

(d) TOUCH # 1 TO # 4

⇒ \( V_1 = V_4 \)

\[
V_1 = S_{11} Q_1'' + S_{21} Q_2'' + S_{31} Q_3'' + S_{41} Q_4''
\]

\[
V_4 = S_{14} Q_1'' + S_{24} Q_2'' + S_{34} Q_3'' + S_{44} Q_4''
\]

Or

\[
(S_{11} - S_{14}) Q_1'' + (S_{21} - S_{24}) Q_2'' = (S_{34} - S_{31}) Q_3'' + (S_{44} - S_{41}) Q_4''
\]

But \( S_{11} - S_{14} = S_{44} - S_{41} \) and \( S_{21} - S_{24} = (S_{34} - S_{31}) \)

Also we know that

\[
Q_2'' - Q_3'' = \frac{a}{2} \quad \text{(2)}
\]

\[
Q_1'' + Q_4'' = Q_1'' = \frac{a}{4} \quad \text{(4)}
\]
REWITING (2) USING (4) GIVES

\[ Q_4'' = Q_1'' + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} \left( Q_2'' - Q_3'' \right) \]

SUBSTITUTING (3)

\[ Q_4'' = Q_1'' + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} \frac{Q}{2} \]

SUBSTITUTING (4)

\[ Q_4'' = \frac{Q}{4} - \frac{Q}{4} + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} \frac{Q}{2} \]

\[ 2Q_4'' = \frac{Q}{2} + \frac{S_{21} - S_{24}}{S_{11} - S_{14}} \frac{Q}{2} \]

\[ = \frac{S_{11} - (S_{11} - S_{14}) + (S_{21} - S_{24})}{4(S_{11} - S_{14})} Q \]

\[ = \frac{S_{11} - (S_{14} - S_{21}) - S_{24}}{4(S_{11} - S_{14})} Q \]

BUT \[ S_{14} = S_{21} \]

IT FOLLOWS THAT

\[ Q_4'' = \frac{Q}{8} \frac{S_{11} - S_{24}}{S_{11} - S_{14}} \]

FROM (4)

\[ Q_1'' = \frac{Q}{4} - \frac{Q_4''}{4} \]

\[ = \frac{Q}{4} - \frac{Q}{4} \left[ \frac{S_{11} - S_{24}}{S_{11} - S_{14}} \right] \]

\[ = \frac{Q}{4} \left[ \frac{(S_{11} - S_{14})^2 - (S_{11} - S_{24})}{2(S_{11} - S_{14})} \right] \]

\[ \therefore Q_1'' = \frac{Q}{8} \frac{S_{11} - S_{14}}{(S_{11} - S_{14})} \]
FOR $\alpha = 0$, AND A COMPLETE SPHERE, WE KNOW THAT THE EXTERNAL POTENTIAL IS OF THE FORM OF A POINT CHARGE AT THE ORIGIN:

$$V = \frac{q}{4\pi \epsilon r} = \left( \frac{q}{4\pi \epsilon a^2} \right) \frac{a^2}{r^2}$$

(1)

THIS IS EXACTLY WHAT THE GIVEN ANSWER BECOMES FOR $\alpha = 0$:

$$V = \frac{a^2 \alpha}{\epsilon} \left[ \frac{a}{r} \cos \alpha + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} \left( P_{2n+1} (\cos \alpha) - P_{2n} (\cos \alpha) \right) \right]$$

(2)

FOR $\alpha = 0$, $\cos \alpha = 1$

AND, FROM ART. 1.157, $P_n(1) = 1$ FOR ANY $n$.

THUS (2) BECOMES

$$V = \frac{a^2 \alpha}{\epsilon} \left[ \frac{a}{r} \right] = \frac{a^2 \alpha}{\epsilon r}$$

(3)

WHICH IS THE SAME AS IS GIVEN IN (1)
Now, it would seem that if the potential \( V(r, \theta, \phi) \) in (3) were valid, then the corresponding potential for the hemisphere in the figure could be given simply by a coordinate rotation about the origin, since \( V(r, \theta, \phi) \) is a function only of \( r \), and since \( r \) would suffer no identity change upon such a rotation, the resulting potential would be the same. That is, Eq. 3 is symmetric about the origin, and a rotation about the origin would leave it unaltered. Obviously, Eq. 3 is not the given solution. What is the flaw in logic here?
5.71c. First off, we know from 2.14(1) that the induced charge on a grounded conductor due to a point charge \( q \) is
\[
Q = -\frac{V_p}{V} q
\]
A first order approximation for the problem at hand is
\[
Q = -\frac{a q}{f}
\]
This first order approximation is valid for a sphere of radius \( a \) and a point charge \( q \) a distance \( f \) from the sphere's center, and is a valid solution for the problem at hand for \( n = 0 \).

To find a better statement, we call on Art. 5.131, Eqs. 6 and 7. Approximations made at the beginning of this article make these relationships approximations. Thus, the analysis to follow should be viewed as a (second order) approximation.

In line with 5.131(6), the surface for the problem at hand may be considered as two surfaces superimposed:
\[
\Sigma_1 : a
\]
\[
\Sigma_2 : a \Lambda, S_0 \quad \text{4 so, how why * ?}
\]
That is, taking \( \sigma = \frac{q}{4\pi a^2} \), we may write
\[
S_n' = \frac{q}{4\pi a^2}
\]
\[
S_n = \frac{q}{4\pi a^2} N_S_n
\]  \( \blacksquare \)

We write (13) in closed form:
\[
V_0 = \frac{q}{\epsilon} \sum_{n=0}^{\infty} \left( \frac{q}{a} \right)^n \frac{1}{2n+1} S_n
\]

Thus, the potential distribution external to our given surface is
\[
V_0 = \left( \frac{q}{\epsilon} \right) \left( \frac{q}{4\pi a^2} \right) \left[ \left( \frac{a}{r} \right) + \frac{n}{2n+1} \left( \frac{a}{r} \right)^{2n+1} S_n \right]
\]
\[
= \frac{q}{4\pi \epsilon a} \left( \frac{q}{a} \right) \left[ 1 + \frac{n}{2n+1} \left( \frac{a}{r} \right)^{2n} S_n \right]
\]

The \( V_p' \) in (1) is simply obtained by evaluating this expression at
\[
(r, \Theta, \Phi) = \left( \frac{a}{r}, \Theta_o, \Phi_o \right)
\]
\[
V_p' = \frac{q}{4\pi \epsilon a} \left( \frac{q}{a} \right) \left[ 1 + \frac{n}{2n+1} \left( \frac{a}{r} \right)^{2n} S_n(\Theta_o, \Phi_o) \right]
\]  \( \blacksquare \)

We now need to compute \( V' \) as given in (1), approximating our surface as a sphere of radius \( q' \), we have
\[
V' \approx \frac{q'}{4\pi \epsilon a}
\]  \( \blacksquare \)

For a unit charge, \( q = 1 \). Thus, substituting (3) and (4) into (1) gives:
\[
Q = -\frac{a}{q} \left[ 1 + \frac{n}{2n+1} \left( \frac{a}{r} \right)^{2n} S_n(\Theta_o, \Phi_o) \right]
\]
WE FIRST LOOK AT THE CASE WHERE NO DIELECTRIC IS PRESENT IN THE LOWER HALF OF THE CYLINDER. THIS PROBLEM IS ADDRESSED IN ART. 5.298. WITH REFERENCE TO 5.298 (4), WE NEED ONLY TO MAKE THE FOLLOWING OBSERVATIONS

1. DUE TO SYMMETRY, THE POTENTIAL WILL NOT BE A FUNCTION OF $\phi$.

2. ALSO DUE TO SYMMETRY, ONLY ZEROTH ORDER BESSEL FUNCTIONS OF THE FIRST KIND ($\tilde{J}_0$) WILL APPEAR IN THE POTENTIAL EXPANSION. (SINCE $J_0$ IS THE ONLY INTEGRAL ORDER EVEN BESSEL FUNCTION OF THE FIRST KIND, THAT IS $J_0(x) = J_0(-x)$)

3. THE POINT CHARGE $q$ IS PLACED AT $b$ ON THE $z$ AXI S, THUS, WE MUST ACCORDINGLY TRANSLATE THE POTENTIAL BY REPLACING $z$ WITH $z - b$.
FROM THESE OBSERVATIONS, WE REWRITE 5.298(4) AS

\[ V_1 = \frac{q}{2\pi\varepsilon_0} \sum_{r=1}^{\infty} e^{-\mu_r I z-bI} \frac{J_0(\mu_r a)}{\mu_r J_1(\mu_r a)} \]

WE HAVE HERE USED THE FACT THAT

\( J_0(0) = 1 \) TO ELIMINATE THE \( J_0(\mu_r b) \) TERM

IN 5.298(4) (NOTE:THE b IN 5.298(4) IS NOT

THE SAME AS HERE, BUT IS THE DISTANCE

\( \rho \) OF THE POINT CHARGE FROM THE

Z AXIS, WHICH, HERE, IS ZERO).

SINCE WE ARE CONCERNED WITH

THE POTENTIAL ABOVE THE DIELECTRIC

AND WITHIN THE CYLINDER, WE MAY

APPLY RESULTS GIVEN IN ART. 5.05.

BY INTRODUCING THE DIELECTRIC,

WE CHANGE THE POTENTIAL ACCORDING

TO 5.05(3), SETTING THEIR \( K_2 \) TO

OUR \( K_1 \), THEIR \( K_1 \) TO 1, AND

THEIR \( f(x,y,z) \) TO OUR \( V_1(\rho,\phi,z) \)

(GIVEN IN (1)) YIELDS

\[ V(\rho,\phi,z) = V_1(\rho,\phi,z) - \frac{K-1}{K+1} V_1(\rho,\phi,-z) \]

\[ = \frac{q}{2\pi\varepsilon_0} \sum_{r=1}^{\infty} e^{-\mu_r I z-bI} \frac{J_0(\mu_r a)}{\mu_r J_1(\mu_r a)} \]

\[ = \frac{q}{2\pi\varepsilon_0} \sum_{r=1}^{\infty} \left[ e^{-\mu_r I z-bI} - \frac{K-1}{K+1} e^{-\mu_r I (z+b)} \right] \frac{J_0(\mu_r a)}{\mu_r J_1(\mu_r a)} \]

WHERE, OF COURSE, \( z \geq 0, \rho \leq a, \frac{1}{r} J_0(\mu_r a) = 0 \text{ \& } r \)
1. List all properties that the magnetic induction and the electric current have in common.

2. Explain the sense of polarization of the electromagnetic waves. List all types of polarization you know and the consequences it have on the wave propagation.

3. 30 p 278

4. 7 p 320

5. 15 p 410

\[
\begin{align*}
90^\circ & \leq \varphi \leq 180^\circ \\
80^\circ & \leq \varphi < 90^\circ \\
70^\circ & \leq \varphi \leq 80^\circ 
\end{align*}
\]
1. By the definition, magnetic induction as a vector $\mathbf{B}$ in the
  magnetic field, whose magnitude (in webers per
  square meter) is the torque in newtons meters on a hoop
  of moment one whose axis is normal to this direction.
  The magnetic field is determined by a small
  exploring loop of unit
  area and placed grinding
  on unit moment. If
  the torque and therefore
  $p$, depends on the permeability
  $\mu$ of the medium. In a
  uniform medium, $\mu$ is
  constant. In a crystal, $\mu
  \neq \mu_0$ is a tensor. On
  paramagnetic materials, $B
  \neq \mu_0 H$ and related (hysteresis).
  The properties of $B$
  can be summarized
  by the relationship
  
  $B = \mu H$
and by this Maxwell's Eqn:

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = \frac{\varepsilon_0}{\varepsilon} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \mathbf{B} \cdot \mathbf{A} = \nabla \times \mathbf{A} \]

B is produced times expressed in the vector potential A such that

A is unique to within a gradient of a scalar,
(\(J_\mathbf{0}\) is always a gauge)

For computational purposes, A is many times written:

\[ A = \nabla \times \mathbf{W} \]

where \( \mathbf{W} \) has approximately chosen (usually 2) components...

\( \mathbf{W} \) is determined by current density or by \( \mathbf{B} \).

It is not an answer for question (problem).
2. The sense of polarization of an electromagnetic wave is a statement concerning the plane in which the wave propagates, specifically addressing the manner in which the fields change with respect to the direction of propagation. The plane of polarization is defined by the plane of the polarized wave as found by the component of the wave and the direction of the wave normal. By linearly polarization, it is meant that the electric field has only a single wave in the direction and varies sinusoidally in
In this manner of reducing to and to rectangular an elliptically (or spherically
unstable) wave in a general case, the poloidal shape of the
project is again oscillatory in time. This is thought to
be settled in many elliptical medias for increasing z. I refer to
Fig. 11.09 on pg. 429 for a pictorial representation.

In anisotropic media, such as crystals, wave velocity will vary
with propagation velocity. Thus, for this case, polarization is a function
of direction (cosines).

(Ref 11-05)
PROB. 3

20. \[ \mathbf{v} = \int_0^1 \left( \frac{r}{n} \right)^{2n+1} (2n+1)P_{2n+1}(y) \, dy \]

\[ p = \cos \theta \]

Find the potential inside of the sphere.

From symmetry, we know that the result will be independent of \( \phi \). This is reflected in the answer.

Also, we know that

We know the general potential (inside the sphere) is

\[ V = \frac{1}{2\pi} \left[ A_0 r^n + B_n r^{-n-1} \right] P_n (\cos \theta) \]

The field inside must be finite. Thus we must set \( B_n = 0 \). Thus

\[ V = \frac{1}{2\pi} A_0 r^n P_n (\cos \theta) \]
We also see, by inspection, that \( V \) is an odd function of \( \theta \),
that \( \theta V(\theta) = -V(\theta) \).
As such, due to the "oddness" of \( V \), the only the odd and odd trigonometric polynomials will contribute to the potential relationship. As such, we may write \((\theta)\) as

\[
V_\theta = \frac{3T}{\mu} A_{2n+1} (r)^{2n+1} P_{2n+1} (\cos \theta)
\]

We should proceed from here to find boundary conditions; next, by finding the external potential and supply the current at the pole poles. By using these boundary conditions would give

\[
A_{2n+1} \frac{4n+3}{(2n+1)(2n+2)} \frac{1}{d\theta}
\]
Substituting this into (2) would give the desired result:

\[ V_4 = 2 \pi \left[ \frac{41 + 3}{(2 \pi + 1)(2 \pi + 2)} \right] \left( \frac{r}{2} \right)^{2n+1} \times P_{2n+1}(\cos \theta) \]

Are you sure that you will solve the problem if you don't have the answer given?
The force toward center is \( \frac{\mu I^2}{16\pi C} \).

Up to point \( 219 \), that the force between two circles separated by a distance \( a \) is \( F = \frac{\mu I^2}{2\pi a} \).

Thus, the force \( F_1 \) acting between the two wires (of separation \( \alpha \) separation)

\[
F_1 = -I_1(\pi\alpha)(\frac{1}{2C})/2\pi(2C)
\]

\[
= -\mu I^2/16\pi C
\]

The wires are attracted to each other. Thus, the forces toward the center acting on (one of) the inner cylinders is simply

\[
F = -\frac{\mu I^2}{16\pi C}
\]

\( \square \)

What with outer cylinder?
\[ \tan \theta = \frac{-29}{\mu \omega a} \]

The vector potential inside is given in Eq. 10.20:

\[ \mathbf{A} = \frac{1}{\mu} \sum_{n=1}^{\infty} \sin \alpha \sin \left( \omega t - \phi_n \right) \]

In our case, from Eq. 2, only one of these terms is applicable.

\[ \mathbf{A} = \frac{1}{\mu} \sin \alpha \sin \left( \omega t - \phi_n \right) \]

Now the vector potential must be finite inside (which it is). We gotta plug around and figure out what \( \phi_n \) is away, and find out what \( \alpha \) is.
First \[ \sqrt{4q^2 + \mu^2} \, \omega = a \]

we have \( a = \sqrt{4q^2 + \mu^2} \omega \)

it follows that

\[ \sin \theta = \frac{a}{\sqrt{4q^2 + \mu^2}} \omega \]

\( \tilde{J} \) becomes

\[ \tilde{J} = \mathbf{k} \cdot \mathbf{c} \cdot \mathbf{a} \cdot \alpha \cdot \sin(\alpha \theta + \phi) \]

\[ \frac{1}{\sqrt{4q^2 + \mu^2}} \omega \cdot \sin \theta \] (\text{similar to } \tilde{e})

But, we are given the external B field

\[ \mathbf{B} = \mathbf{k} \cdot \cos \theta \cdot \omega \]

The first terms in the displaced portion of (3) are recognized as the negative shifted argument for this relationship, whose, well, if it were

\[ 1 + \omega \cos \alpha \theta \]

for all intents and purposes placing the integral dependence (\text{WKB})
As such, we may replace the displaced position by the negative of \( C \) to give:

\[
N = ( - p_i ) ( - 2\phi ) \left[ 4\phi^2 + \mu \nu^2 w \phi^2 a^2 \right]^{-\frac{1}{2}} \sin \omega t - \epsilon
\]

\[
= 2\phi B \left[ 4\phi^2 + \mu \nu^2 w \phi^2 a^2 \right]^{-\frac{1}{2}} \times \sin \omega t - \epsilon
\]
Find the field for the following configuration of two planes, one earthed, infinite and one at potential \( U_0 \), semi-infinite.

We will apply Schwarz transformation with one positive angle \( \alpha = \frac{\pi}{2} \) and \( U_1 = a \), which will give us field between parallel plates.

\[
\frac{dZ}{dZ_1} = C_1 \left( Z_1 - U_1 \right) \frac{\alpha - \pi}{\pi} \\
U_1 = a, \quad \alpha = \frac{\pi}{2}
\]

\[
z = 2C_1 \left( Z_1 - a \right)^{\frac{1}{2}} + C_2
\]

Now we should find \( C_1 \) and \( C_2 \):

1) For \( Z_1 = a \), we wish to have \( z = 0 \).
2) For \( Z_1 = 0 \), we wish to have \( z = 0 \).

From 1) if apply to \( (\circ) \) we have

\[
C_2 = a
\]

From 2) if apply to \( (\circ) \) we have

\[
0 = 2C_1 \sqrt{a_1} + a \Rightarrow C_1 = -\frac{a}{2\sqrt{a_1}} = \frac{a}{2\sqrt{a_1}}
\]

Thus we have our transformation:

\[
z = j \frac{a}{\sqrt{a_1}} \left( Z_1 - a \right)^{\frac{1}{2}} + a
\]
Now our problem is reduced to following:

This transformation as you see did not fold the part of real axis $x_1 : 0 \leq x_1 \leq a_1$. That line before transformation was a line of force, then after transformation this will be also a line of force and all other for $y > 0$ will be parallel (plates for $y > 0$ are infinite). Originally problem was symmetric with regard to axis $x$ and we will solve the problem only for $y > 0$, however after transformation back to $z$ plane we could extend solution for $y, < 0$.

Because all line of forces are parallel and equipotential line perpendicular to them and parallel each to other and to plates, therefore instead talking about $z$-plane with $z = x + jy$ we could understand it as a $z$-plane with $W = U + jV$. Because $U = \text{const}$ are parallel to $x = \text{const}$ and $V = \text{const}$ to $y = \text{const}$.

We have:

$$\begin{align*}
V \\
\uparrow
\end{align*}$$

in our transformation we should now put $a = U_0$. 

\[ \text{U} \]
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{1}{\partial y} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{1}{\partial y} \right) f = \gamma
\]

and lines of equipotential are given by:

\[
\frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial y} \right) = \gamma
\]

Now we should have \( \gamma < 0 \) from (**) and the lines are straight lines. The solution function are given by

\[
\phi = \gamma \frac{y}{y_0} \quad x = \gamma \frac{x}{x_0}
\]

and \( x_0 = y_0 = 1 \) for the point source. \( \gamma \) is a constant.
in original system.

Similarly for stream functions:

\[(U - U_0) = \frac{U_0^2}{2a_1V} y_1\]

\[\frac{U_0^4}{4a_1^2V^2} y_1^2 - V^2 = \frac{U_0^2}{a_1} (a_1 - x_1)\]

rearranging

\[y_1^2 = -4a_1 \left( \frac{V}{U_0} \right)^2 (x_1 - a_1) + 4a_1 \left( \frac{V}{U_0} \right)^4\]

stream lines

Taking \(0 \leq V \leq \infty\) for each \(V = \text{const.}\) we have

we get:

In every point in real system we have now \(U\) and \(V\).

For instance we could determine induced charge on grounded plate and distribution on plate charged to \(U_0\). (How?)
THIS OUTLINE TAKEN FROM
"FUNDEMENTALS OF ELECTRIC WAVES" 2nd Ed
by H.H. Skilling
(WILEY, NEW YORK, 1948)
I. Experiments on the Electrostatic Fields

A. Fields

1. Exp. 1:
\[ \vec{F} = q \vec{E}; \vec{E} : \text{electric field strength} \]

2. Exp. 2:
\[ \oint \vec{F} \cdot d\vec{s} = \oint \vec{E} \cdot d\vec{s} = 0 \]

3. Exp. 3:
\[ C_0 \oint \vec{E} \cdot d\vec{a} = Q \quad (\text{in vacuum}) \]

4. Exp. 4:
\[ KE_0 = C \]

\( K \): relative dielectric constant
\( \varepsilon \): permittivity
\( C_0 \): "of free space"

\[ \oint \varepsilon_0 \vec{E} \cdot d\vec{a} = Q \quad (\text{in } \varepsilon \text{ material}) \]
B. Electrostatic Flux

\[ D = \varepsilon E \Rightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q \]

**Electrostatic Flux Density**

**Electrostatic Flux**

\[ \mathbf{D} \cdot d\mathbf{a} = q \]

**Lines of Flux**

C. Units

- \( E = \text{Electric Field Strength} \sim \frac{\text{V}}{\text{m}} \)
- \( \mathbf{D} = \text{Flux Density} \sim \frac{\text{Coulombs}}{\text{m}^2} \)
- \( \varepsilon = \text{Permittivity} \)
- \( \varepsilon_0 = \text{permittivity of free space} \)
- \( k = \frac{\varepsilon}{\varepsilon_0} = \text{Relative Dielectric Constant} \)
PROB

1 (i) A body carrying a positive electric charge of 1000 \( \mu \text{C} \) is in an electric field of 5000 \( \text{V/cm} \). What is the electric force of the body in newtons?

ANS.

\[ F = QE \]

\[ = (1000 \times 10^{-12} \text{ coul}) (5000 \frac{\text{V}}{\text{cm}} \times 100 \text{ cm/m}) \]

\[ = 5 \times 10^{-4} \text{ newtons} \]
The electric field strength is measured at all points of a sphere of 10 cm radius in air. It is found to be everywhere normal to the surface, 10,000 V/m in magnitude and directed outward. How much charge is contained within the sphere?

Ans:

\[ Q = \int \oint E \cdot dS \]
\[ = \varepsilon_0 \left( \frac{4}{3} \pi R^3 \right) \left( 1000 \text{ V/m} \right) \]
\[ = \left( 8.855 \times 10^{-12} \right) \frac{4}{3} \pi (0.1)^3 \left( 1000 \text{ V/m} \right) \]
\[ = 3.709 \times 10^{-10} \text{ COULOMBS} \]
\[ = 0.0371 \mu \text{ COULOMBS} \]
\[ = 37100 \mu \text{ COULOMBS} \]
I (3). How much flux comes out of the spherical surface in the previous problem? What quantity would come out of the same charge in petroleum oil? ($\varepsilon = 2\varepsilon_0$) What would be the value of $E$ at the surface in petroleum oil?

Ans.

$D = \varepsilon_0 E$

Flux = $\Phi D \cdot ds = 37100 \mu$ coulombs

(same as in 2)

For petroleum oil

$D = 2\varepsilon_0 E$

We must still have

$Q = \Phi D \cdot ds \Rightarrow$ electrostatic

Flux is still the same, but the $E$ field is now half of what it was in air, since

$Q = 2\varepsilon_0 \Phi E \cdot ds$
D. VECTOR FIELDS

E. GRADIENT

PROPERTIES: 1. GRADIENT LINE IS ALWAYS PERPENDICULAR TO CONTOUR LINES, (IT IS STEEPEST SLOPE) 2. THE CLOSER THE CONTOUR LINES, THE STEEPER THE GRADIENT.

\[ \vec{\nabla} \text{p} = \vec{\text{A}} \frac{\delta \text{p}}{\delta x} + \vec{\text{B}} \frac{\delta \text{p}}{\delta y} + \vec{\text{K}} \frac{\delta \text{p}}{\delta z} \]

\[ (\vec{\nabla} \text{p}) \cdot ds = \frac{\delta \text{p}}{\delta x} dx + \frac{\delta \text{p}}{\delta y} dy + \frac{\delta \text{p}}{\delta z} dz \]

F. DIVERGENCE

\[ \vec{\nabla} \cdot \vec{\text{A}} = \frac{\delta \text{A}_x}{\delta x} + \frac{\delta \text{A}_y}{\delta y} + \frac{\delta \text{A}_z}{\delta z} \]

(AN INCOMPRESSIBLE FLOW HAS ZERO DIVERGENCE)

G. CURL

\[ \vec{\nabla} \times \vec{\text{A}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \text{A}_x & \text{A}_y & \text{A}_z \end{vmatrix} \]

H. DEL (NABLA): \[ \vec{\nabla} = \frac{\delta}{\delta x} \hat{i} + \frac{\delta}{\delta y} \hat{j} + \frac{\delta}{\delta z} \hat{k} \]

GRADIENT: \[ \vec{\nabla} \text{p} \rightarrow (\text{VECTOR}) \]

DIVERGENCE: \[ \vec{\nabla} \cdot \vec{\text{A}} \rightarrow (\text{SCALAR}) \]

CURL: \[ \vec{\nabla} \times \vec{\text{A}} \rightarrow (\text{VECTOR}) \]

* DIVERGENCE IS A RATE OF CHANGE OF A FIELD STRENGTH IN THE DIRECTION OF THE FIELD
* CURL IS A RATE OF CHANGE OF THE FIELD STRENGTH IN A DIRECTION AT RIGHT ANGLES TO THE FIELD.

I. \[ \vec{\nabla} \times \vec{\nabla} \text{F} = \vec{0} \] ALWAYS;
\[ \vec{\nabla} \cdot \vec{\nabla} \text{A} = \vec{0} \]

\[ \vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla} \]

SCALAR: \[ \vec{\nabla}^2 \text{F} = (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}) \text{F} \]

VECTOR: \[ \vec{\nabla}^2 \vec{\text{A}} = \vec{\nabla}^2 (\vec{\text{A}}_x + \vec{\text{A}}_y + \vec{\text{A}}_z) \]
II. VECTOR ANALYSIS

A. VECTOR MULTIPLICATION

- SCALAR OR DOT PRODUCT:
  \[ \mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \mathbf{A} \mathbf{B} \cos \theta \]

- VECTOR OR CROSS PRODUCT:
  \[ \mathbf{C} = \mathbf{A} \times \mathbf{B} \]

EXAMPLE: FORCE ON A WIRE IN A \( \mathbf{B} \) FIELD

\[ \mathbf{F} = I \mathbf{L} \times \mathbf{B} \]

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \]

B. UNIT VECTORS

\[ \mathbf{i} \cdot \mathbf{i} = 1 \quad \mathbf{i} \times \mathbf{i} = 0 \]
\[ \mathbf{j} \cdot \mathbf{j} = 0 \quad \mathbf{j} \times \mathbf{j} = \mathbf{k} \]
\[ \mathbf{k} \cdot \mathbf{k} = 0 \quad \mathbf{k} \times \mathbf{k} = -\mathbf{j} \]

\[ \mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \]
\[ \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \]
\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \]

C. TRIPLE PRODUCTS

\[ [\mathbf{ABC}] = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = +(\mathbf{B} \times \mathbf{C} \cdot \mathbf{A}) = -[\mathbf{CBA}] \]
J. Polar Coordinates

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ r = \sqrt{x^2 + y^2} \]
\[ \theta = \tan^{-1} \left( \frac{y}{x} \right) \]

\[ A_r = A_x \cos \theta + A_y \sin \theta \]
\[ A_\theta = A_y \cos \theta - A_x \sin \theta \]
\[ A_x = A_r \cos \theta - A_\theta \sin \theta \]
\[ A_y = A_r \sin \theta + A_\theta \cos \theta \]

\[ \nabla p = i \frac{\partial p}{\partial x} + j \frac{\partial p}{\partial y} \]

\[ \text{NOW} \]
\[ \frac{\partial p}{\partial x} = \frac{\partial p}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial p}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial p}{\partial r} \frac{x}{\sqrt{x^2 + y^2}} + \frac{\partial p}{\partial \theta} \left( \frac{y}{x} \right)^2 + 1 \]
\[ = \frac{\partial p}{\partial r} \cos \theta - \frac{\partial p}{\partial \theta} \frac{\sin \theta}{r} \]
II. PROB.

\[ V_x = \sin Y, \quad V_y = 0 \quad \text{sketch the field of } V \]

Find its divergence

\[ \nabla \cdot \vec{V} = \frac{\partial}{\partial x} \sin Y = 0 \]

\[ \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \sin Y & 0 \end{vmatrix} = \sin Y \hat{\imath} - \cos Y \hat{\jmath} \]

\[ = -K \frac{\sin Y}{Y} \sin Y = -K \cos^2 Y \]
III. Certain Theorems Relating to Fields

A. Divergence

\[ \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \]

Amount of change of \( D_x \) in \( dx = \frac{6D_x}{\varepsilon_x} dx \)

\[ \text{\# of flux lines leaving right hand side is} \]
\[ (D_x + \frac{6D_x}{\varepsilon_x} dx) dy dz \]

Difference between \# of lines on left and right is
\[ \frac{6D_x}{\varepsilon_x} dx dy dz \]

Generalizing, the \# of flux lines leaving which do not enter is \( (6D_x + \frac{6D_y}{\varepsilon_y} + \frac{6D_z}{\varepsilon_z}) dx dy dz \)

\[ \therefore \text{Divergence is \# of flux lines originating per unit area} \]
\[ \nabla \cdot \mathbf{D} = \frac{6x \varepsilon_y \varepsilon_z^{-1}}{6x \varepsilon_y \varepsilon_z^{-1}} \]

B. Gauss's Theorem

\[ \oint \mathbf{D} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{D} \, dv \]

In a given volume, the \# of flux lines originating within can be computed by

1. Integrating the divergence over the volume
2. Seeing how much flux comes out
3. That doesn't go in
B. Curl

\[ C = \text{Circulation} = \oint \vec{E} \cdot d\vec{S} \]

**Curl is a microscopic circulation/area and has a direction normal to the plane in which it is maximum.**

Using counterclockwise sense:

\[ C = -E_{y_1} \, dy + E_{y_2} \, dy - E_{x_2} \, dx + E_{x_1} \, dx \]

\[ = \left( E_{y_2} - E_{y_1} \right) dy - \left( E_{x_2} - E_{x_1} \right) dx \]

\[ = \frac{\delta E_y}{\delta x} \, dx \, dy - \frac{\delta E_x}{\delta y} \, dy \, dx \]

\[ \Rightarrow \text{Curl} = \left( \frac{\delta E_y}{\delta x} - \frac{\delta E_x}{\delta y} \right) \mathbf{k} \]

C. Stokes's Theorem

\[ \text{Circ} = \oint \vec{E} \cdot d\vec{S} = \iint_A \nabla \times \vec{E} \cdot d\vec{A} \]

The circulation can be computed by:

1. The line integral definition
2. Summing the itty bitty curls within the area

Note: Surface need not necessarily be planar.

D. Comparison of Theorems

Both Gauss' and Stokes' theorem relate the microscopic to the macroscopic aspect.
E. SCALAR POTENTIAL

\( \rho = \text{an equapotential surface} \)

\( \vec{E} = \text{corresponding field} \)

\( \vec{E} = -\nabla \rho \) (assumed if \( \nabla \times \vec{E} = 0 \))

\( \{ \forall \rho \in \vec{E} \}

\{ 3 \vec{E} \in \nabla \times \vec{E} = 0 \Rightarrow \exists \rho \in -\nabla \rho = \vec{E} \}

- AN ELECTROSTATIC FIELD HAS NO CURL
- A STATIC MAGNETIC FIELD HAS NO CURL IN
  REGIONS THAT ARE NOT CARRYING CURRENT.
- CURLESS FIELDS HAVE EQUIPOTENTIAL LINES
  DIVIDING IT INTO "LAMELLERS"

\( \Rightarrow \nabla \times \vec{E} = 0 \Rightarrow \vec{E} \) IS "LAMELLAR" OR "IRROTATIONAL"

F. SOLENOIDAL FIELDS AND VECTOR POTENTIALS

IF \( \vec{B} = \nabla \times \vec{A} \), THEN \( \nabla \cdot \vec{B} = 0 \)

AND IF \( \nabla \times \vec{B} = 0 \), THEN \( 3 \vec{A} \Rightarrow \vec{B} = \nabla \times \vec{A} \)

\( \vec{A} = \text{vector potential} \)

\( \vec{B} = \text{"Solenoidal" or "Sourceless"} \)

- ALL MAGNETIC FIELDS ARE SOLENOIDAL
IV. THE ELECTROSTATIC FIELD

**REVIEW:**
\[ \vec{F} = q \vec{E} \]
\[ \oint \vec{E} \cdot d\vec{s} = 0 \]
\[ \oint \vec{D} \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{a} = Q \]

**FUNDAMENTAL LAWS OF ELECTROSTATICS**

- **USING STOKE'S THEOREM**
  \[ \oint \vec{E} \cdot d\vec{s} = \int (\nabla \times \vec{E}) \cdot d\vec{a} = 0 \Rightarrow \nabla \times \vec{E} = 0 \]

  \[ V = \text{ELECTROSTATIC POTENTIAL} \]
  \[ \vec{E} = -\nabla V \]

- **USING GAUSS' THEOREM**
  \[ \oint \vec{D} \cdot d\vec{a} = \int \nabla \cdot \vec{D} \, dV = Q \]

  \[ \therefore \text{WHERE} \quad Q = 0 \quad \Rightarrow \quad \nabla \cdot \vec{D} = 0 \]
  \[ \nabla \cdot \vec{D} = \rho = \text{CHARGE DENSITY} \]
  \[ \nabla \cdot \vec{E} = \rho / \varepsilon = -\nabla^2 V \Rightarrow \text{POISSON'S EQ} \]

  \[ \therefore \text{WHEN THERE'S NO CHARGE, WE HAVE LAPLACE EQ} \]
  \[ \nabla^2 V = 0 \]

  \[ \begin{bmatrix} \vec{E} = -\nabla V \\ \nabla^2 V = \rho / \varepsilon \end{bmatrix} \text{SUMMARY} \]
A. CONDUCTORS

FOR THE ELECTROSTATIC CASE

\( E = 0, \quad V = \text{const.} \quad (\text{in conductor}) \)

Q WILL ALWAYS GO TO SURFACE

\( \sigma = \text{surface charge density} = D_n = \varepsilon E_n \)

B. A CHARGED SPHERE

\[ \nabla^2 V = 0 \]

(1) \( \nabla^2 V = 0 \)

(2) SURFACE EQUIPOTENTIAL

(3) TOTAL CHARGE = Q

IN SPHERICAL COORDINATES

\[ \nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2} \frac{\partial V}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial V}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} \]

DUE TO SYMMETRY: \( \frac{\partial^2 V}{\partial \theta^2} = \frac{\partial^2 V}{\partial r^2} + \frac{\partial}{\partial r} \frac{\partial V}{\partial r} \)

SOLUTION IS

\[ V = \frac{Q}{r} + b \]

\[ V(\infty) = 0 \Rightarrow b = 0 \Rightarrow V = \frac{Q}{r} \]

\( \sigma = \frac{4\pi r^2}{\varepsilon} \Rightarrow E_n = \frac{\sigma}{\varepsilon} \frac{4\pi}{r^2} \)

\[ E = \frac{\partial V}{\partial r} = \frac{Q}{r^2}, \quad E(r_0) = \frac{Q}{r_0^2} \Rightarrow q = \varepsilon \frac{Q}{4\pi} \]

OR

\[ V = \frac{4\pi \varepsilon Q}{r}, \quad E = \frac{Q}{4\pi \varepsilon r^2} \]
C. SPHERICAL CONDENSER

\[ E = \frac{Q}{4\pi \kappa r^2} \]

D. VOLTAGE

\[ V_{12} = \int_1^2 E \, dS = \frac{W_{12}}{Q} \]

For sphere:

\[ V_{12} = \int_a^b \frac{Q}{4\pi \epsilon_0 r^2} \, dr = \frac{Q}{4\pi \epsilon_0 b} - \frac{Q}{4\pi \epsilon_0 a} \]

E. CAPACITANCE

\[ C = \tau \sqrt{\frac{Q}{V}} \]

For double sphere:

\[ C = \tau \sqrt{\frac{4\pi \epsilon_0 ab}{b-a}} \]

\[ \kappa \epsilon = \text{ALWAYS} \]

F. POLARIZATION

(How come \( E \) is less in \( \varepsilon \) than in \( \epsilon_0 \)?)

\[ +Q \quad -Q \]

\[ \text{inelastic} \]

\[ \text{in E field, the atoms develop an equivalent surface charge thru polarization} \]

G. INVERSE SQUARE LAW

\[ F = \frac{Q_1 Q_2}{4\pi \epsilon_0 r^2} \quad \text{COULOMB'S LAW} \]

H. FIELD IN A HOLLOW SPHERE

\[ \mu = 0 \]
II. THE POTENTIAL INTEGRAL

Potential from a number of charges: \( V = -\frac{1}{4\pi\varepsilon_0} \int \frac{P}{r} \, dV \)

In limit, we have

\[ V = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma \, d\mathbf{a}}{r} \]

\[ = \frac{1}{4\pi\varepsilon_0} \left[ \int_{\sigma_1}^{\sigma_2} \int_{y_1}^{y_2} \frac{1}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} \, dx \, dy \right] \]

\[ = -\frac{\sigma}{\varepsilon_0} \left( y, \frac{y_2}{2} \right) \]

\[ \int_{\sigma_1}^{\sigma_2} \int_{y_1}^{y_2} \frac{1}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} \, dx \, dy \]

II. ELECTROSTATIC ENERGY

\[ dW = V \, dQ = \frac{Q}{C} \, dQ \Rightarrow W = \int_{0}^{Q} \frac{Q^2}{2C} \, dQ = \frac{Q^2}{2C} = \frac{1}{2} Q V \]

for energy distribution

\[ W = \frac{1}{2} \int \mathbf{P} \cdot \mathbf{E} \, dV \]
V. **ELECTRIC CURRENT**

\[ I = \frac{\text{d}q}{\text{d}t} \]

**OHM'S LAW:** \[ V = RI \]

**FOR CYLINDER:** \[ \frac{1}{\ell R} = \frac{\text{AREA}}{\text{LENGTH}} \]

\( \gamma = \text{CONDUCTIVITY} \)

\[ \Rightarrow \text{VOLTAGE: } V = \int E \cdot \text{d}s \]

\[ \text{CURRENT: } I = \int \mathbf{J} \cdot \text{d}a \; ; \; \mathbf{J} = \text{CURRENT DENSITY} \]

![Diagram of current flow through a cylinder](image)

\[ \frac{1}{\ell R} = \gamma \frac{\text{d}q}{\text{d}s} \]

\[ \Rightarrow \mathbf{J} \cdot \text{d}a = \gamma \frac{\text{d}q}{\text{d}t} E \cdot \text{d}s \]

**THUS:** \[ \mathbf{J} = \gamma E \leq \text{MICROSCOPIC OHM'S LAW} \]

**WHEN FIELD IS DC:** \[ \nabla \cdot \mathbf{J} = 0 \leq \text{KIRCHHOFF'S CURRENT LAW} \]

**MORE GENERAL CASE:**

**FLUX FROM SURFACE**

\[ \frac{\text{d}}{\text{d}t} (\text{FLUX}) = \frac{\text{d}}{\text{d}t} Q = I \]

\[ \text{FLUX} = \oint D \cdot \text{d}a \]

\[ \Rightarrow I = \frac{\text{d}}{\text{d}t} \oint D \cdot \text{d}a = \oint \frac{\text{d}D}{\text{d}t} \cdot \text{d}a \]

**CURRENT INTO VOLUME IS**

\[ \oint (\mathbf{J} + \frac{\text{d}D}{\text{d}t}) \cdot \text{d}a = 0 \]

**BY GAUSS' THEOREM:** \[ \int \nabla \cdot (\mathbf{J} + \frac{\text{d}D}{\text{d}t}) \text{d}V = 0 \]

**OR** \[ \nabla \cdot (\mathbf{J} + \frac{\text{d}D}{\text{d}t}) = 0 \]

**OR** \[ \nabla \cdot (\gamma \mathbf{E} + \mathbf{e} \frac{\text{d}E}{\text{d}t}) = 0 \]

\[ \mathbf{J} = \frac{\text{d}D}{\text{d}t} = \mathbf{e} \frac{\text{d}E}{\text{d}t} \leq \text{DISPLACEMENT CURRENT} \]

\[ \mathbf{J}_C = \gamma \mathbf{E} \leq \text{CONDUCTION CURRENT} \]

\[ \mathbf{J}_T = \mathbf{J}_d + \mathbf{J}_c \; ; \; \nabla \cdot \mathbf{J}_T = 0 \]

\[ \mathbf{J}_T \leq \text{SOLENOIDAL} \]
A. Electromotive Force

\[ E_s = E \text{ due to } Q \]
\[ E_m = E \text{ due to changing magnetic field} \]
\[ E = E_s + E_m \]

\[ V = \int E_s \cdot dS \]
\[ e_m f = \int E_m \cdot dS \]

\( E_m \) from chemical or mech. energy, xferred to \( E_s \) energy

\[ V + e_m f = R I \]
\[ I = \int E_s \cdot dS \]
\[ = \int (E_{m1} + E_s) \cdot dS \]
VI. THE MAGNETIC FIELD

A. MAGNETIC FORCE
\[ \vec{F} = I \vec{L} \times \vec{B} \]
\[ \vec{L} = \text{LENGTH OF WIRE} \]
\[ I = \text{CURRENT THRU WIRE} \]
\[ \vec{B} = \text{MAGNETIC INDUCTION (FLUX DENSITY)} \]
\[ \sim \left( \frac{\text{weber}}{\text{m}^2} \right) \]

B. MAGNETIC FLUX \[ \Rightarrow \phi = \int \vec{B} \cdot d\vec{a} \]

A. \vec{B} FIELD CAN BE PRODUCED BY AN \vec{E} FIELD
\[ \text{EMF} = \oint \vec{E} \cdot d\vec{s} = -\frac{d\phi}{dt} \leq \text{FARADAY'S LAW} \]

APPLYING STOKES THEOREM
\[ \oint \vec{E} \cdot d\vec{s} = \int \nabla \times \vec{E} \cdot d\vec{a} = \int \frac{d}{dt} \vec{B} \cdot d\vec{a} \]
\[ \Rightarrow \nabla \times \vec{E} = -\frac{d}{dt} \vec{B} \]

(NOTE: IF \vec{E} IS ELECTROSTATIC, \( \nabla \times \vec{E} = 0 \))

C. VOLTAGE INDUCED BY MOTION

LET LOOP BE TRAVELING WITH SPEED \( \vec{v} \)
\[ \Rightarrow \nabla \times \vec{E} = -\frac{\vec{v}}{\mu t} \vec{B} + \vec{v} \times (\nabla \times \vec{B}) \]

IN ALL CASES
\[ \oint \vec{B} \cdot d\vec{a} = \int \nabla \cdot \vec{B} \, dv = 0 \Rightarrow \nabla \cdot \vec{B} = 0 \text{ (A MAXWELL)} \]

ALSO
\[ \frac{1}{\mu} \oint \vec{B} \cdot d\vec{s} = I \]

\( \mu = \text{PERMEABILITY} \), \( \mu_0 = 4\pi \times 10^{-7} \)

FERROMAGNETIC \( \Rightarrow \mu >> \mu_0 \)

FOR NON-HOMOGENEOUS MATERIAL
\[ \oint \frac{\vec{B}}{\mu} \cdot d\vec{s} = I \]
\[ \vec{B} = \mu \vec{H} \quad \Rightarrow \quad \vec{H} = \text{MAGNETIC INTENSITY} \]
\[ \Rightarrow \oint \vec{H} \cdot d\vec{s} = I \text{ (A MAXWELL)} \]
Now \[ I = \oint \mathbf{J} \cdot d\mathbf{a} = \oint \mathbf{H} \cdot d\mathbf{s} = \oint (\nabla \times \mathbf{H}) \cdot d\mathbf{a} \]
\[ \implies \nabla \times \mathbf{H} = \mathbf{J} \]

**D. Convention Regarding Sign**

- **THUMB IS "BOREAL" TO FINGERS**
- **BOREAL, BY CONVENTION, IS POSITIVE**

**EXAMPLE:**
- **PATH OF INTEGRATION**
- **SURFACE OF INTEGRATION, ON THIS PATH, H IS CONSTANT**

Now
\[ \oint H_0 \, ds = I = H_0 \oint ds = H_0 \, 2\pi r \]

This gives "BIOT-SAVART LAW": \[ H_0 = \frac{I}{2\pi r} \]

\[ H_r = H_z = 0 \] (why?)

\[ H_r = 0 \] since, first, it must be the same at a distance \( r \) from the wire. Top and bottom of pill box would cancel. Must be zero also around surface, since \[ \oint \mathbf{B} \cdot d\mathbf{s} = 0 \] there.

\[ H_z = 0 \] since integration about any line external to wire must be zero.
E. Force Between Currents

\[ F = I_1 I_2 \left( \frac{\mu_0 I_1}{2\pi d} \right) = \frac{\mu_0 I_1 I_2}{2\pi d} \]

Solving for force per unit length:

\[ F/L_2 = \frac{\mu_0 I_1 I_2}{2\pi d} \]

This is a form of Ampere's Law.

Note: "Like currents attract."

F. Magnetic Flux Linkages

Each surface penetration is a "flux linkage."

G. Magnetic Potential

In electrostatics, \( \nabla \times E = 0 \) \( \Rightarrow \) \( E = -\nabla \phi \)

\( \nabla \times H = 0 \) only when there is no current

But, \( \nabla \cdot H = 0 \) \( \Rightarrow \) \( H = \nabla \times A \)

\( A = \text{magnetic vector potential} \)

\( \nabla \times H = 0 \) \( \Rightarrow \) \( \nabla \times \nabla \times A = 0 \)

\( = \nabla (\nabla \cdot A) - \nabla^2 A \)

For magnetostatic field, let \( \nabla \cdot A = 0 \)

\( \Rightarrow \nabla^2 A = -\nabla \cdot J \)

(analogous to Poisson's Eq: \( \nabla^2 V = -\rho/\varepsilon \) )
IN COMPONENT FORM
\[ \nabla^2 A_U = -j_U \quad (U = x, y, z) \]
\[ \Rightarrow A_U = \frac{1}{4\pi} \int \frac{j_U \, dv}{r} \]

(ANALOGOUS TO \[ V = \frac{1}{4\pi\epsilon} \int \frac{\rho \, dv}{r} \])

IN VECTOR FORM:
\[ \vec{A} = \frac{1}{4\pi} \int \frac{\vec{j} \, dV}{r} \Rightarrow \text{NOTE} \ \hat{A} \ \text{AND} \ \hat{j} \ \text{HAVE SAME DIRECTION} \]

("\hat{A} \ \text{IS LIKE} \ \hat{j}, \ \text{BUT} \ \text{"FUZZY" AROUND THE EDGES}"")

EXAMPLE: IN A SHORT WIRE \( \vec{j} = Ia \sin \omega t \)

ASSUME \( \omega \) \( \text{IS SO SMALL, MAGNETOSTATIC CONDITION PREVAILS} \) (\( \vec{A} \), "QUASI-STATIONARY" STATE)

\[ \vec{A} = \frac{1}{4\pi} \int \frac{\vec{j} \, dV}{r} \]
\[ A_x = \frac{1}{4\pi} \int \frac{j_x \, dV}{r} = \frac{1}{4\pi} \int \frac{i \, dx}{r} = \frac{1}{4\pi} \int \frac{Ia \sin \omega t \, dx}{r} \]

\[ \Rightarrow A_x = \frac{Ia \sin \omega t}{4\pi r} \int_{-l/2}^{l/2} dx = \frac{Ia l}{4\pi r} \]

H. MAGNETIC ENERGY

MAGNETIC ENERGY DENSITY: \[ \frac{1}{2} \vec{B} \cdot \vec{H} \]
\[ \Rightarrow \text{MAG. EN.} = \frac{1}{2} \int \vec{B} \cdot \vec{H} \, dV \]

(WE ASSUME HERE THAT ENERGY IS STORED IN THE MAGNETIC FIELD)
I. Theories

\[ \text{Ferromagnetic Polarization} \]

II. Diamagnetic Materials

Explains reluctance of some materials to magnetic fields.
VIII. EXAMPLES AND INTERPRETATION

A. EXAMPLE

\[ \Phi = \frac{N}{I} \int H \cdot d\mathbf{s} \]

\( N \) turns

\( I \) current

\[ \Phi = \frac{N}{I} \int H \cdot d\mathbf{s} = \frac{NI}{I} = N \]

\( H \) is the magnetic field here, which will only have a component in the \( \theta \) direction (since \( j = \vec{\nabla} \times \mathbf{H} \)).

\[ \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{a} = I \text{out} \]

\( r < r_1 \) or \( r > r_2 \), no current passes within the circle and \( H = 0 \).

\( \Phi \) if the surface lies within the surface:

\[ \Phi = \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{a} = NI \]

\[ = N \int_\Phi \mathbf{H} \cdot d\mathbf{s} = 2\pi r H_0 \Rightarrow H_0 = \frac{NI}{2\pi r} \text{for } r_1 \leq r \leq r_2 \]

\[ \left\{ \begin{array}{l}
\int \mathbf{H} \cdot d\mathbf{s} = \text{MAGNETIC POTENTIAL DIFFERENCE} \\
\oint \mathbf{H} \cdot d\mathbf{s} = \text{MAGNETOMOTIVE FORCE}
\end{array} \right. \]

COMPUTATION OF FLUX:

\[ \Phi = \oint \mathbf{B} \cdot d\mathbf{a} = \int \mu_0 H_0 d\mathbf{a} = \int_{r_1}^{r_2} \frac{\mu_0 NI}{2\pi r} z_1 dr \]

\[ = \frac{\mu_0 NI z_1}{2\pi} \ln \frac{r_2}{r_1} \]

WHERE \( z_1 \) is the thickness of the core.

COMPUTATION OF INDUCTANCE:

\[ L = \frac{\Phi}{I} \]

\[ = \frac{\mu_0 N^2 z_1}{2\pi} \ln \frac{r_2}{r_1} \]
B. **EXAMPLE**

We wish to find the electric field associated with this old-fashioned carbon filament. If \( \mathbf{J} \parallel \) to wire, \( \mathbf{J} = \hat{x} E \). For 110V and 10 inches of wire, \( E = 11 \text{ V/inch} \), in a conductor not carrying current, \( E = 0 \). Also, in a (complicated) electric field outside the conductor.

C. **EXAMPLE:** Consider a changing current in the toroid of Example A. The magnetic field changes, and the changing magnetic field introduces an electric field.

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\Rightarrow \int \nabla \times \mathbf{E} \cdot d\mathbf{A} = \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{1}{\mu_0} \oint \mathbf{B} \cdot d\mathbf{a} = -\frac{1}{\mu_0} \oint \mathbf{F}
\]

↑ SURFACE ↑ LINE (FROM STOKE'S THEOREM)

Choose line integral around one wrapping. Summing up all N windings gives

\[
\text{EMF} = -N \frac{d\Phi}{dt}
\]

Or, since \( L = \frac{N^2 \Phi}{I} \)

\[
\text{EMF} = -L \frac{dI}{dt}
\]

**NOTE:** EMF opposes current flow.
D. EXAMPLE: "INDUCTION ACCELERATOR"

Increasing I will induce increasing magnetic field (out of page) which will induce an E field. Electrons will be increasing E field. Increasing current I will cause the E field curl will be out of page.

Assume $E_\theta = -\alpha r$

$\nabla \times E = \frac{1}{\kappa} \left( A + \frac{4\pi}{c} \right) = -\alpha 2\alpha$

$\frac{\delta A}{\delta t} = -\alpha 2\alpha$ or $\frac{\delta B}{\delta t} = 2\alpha$

$\Rightarrow E_\theta = -\frac{\alpha}{2} \frac{\delta A}{\delta t}$ INSIDE

TURNS OUT THAT $E_\theta = -\frac{\alpha}{2} \frac{\delta B}{\delta t}$ OUTSIDE
XIII. MAXWELL'S HYPOTHESIS

ASSUMPTIONS AND EXPERIMENTS

E1: \( E \) FIELD DEFINED

E2: ELECTROSTATIC FIELD IS WITHOUT CURL \((\nabla \times \vec{E} = 0)\)

E3: \( \nabla \cdot \vec{E} = \rho / \varepsilon \) WHEN \( \vec{E} \) IS ELECTROSTATIC

E4: \( \vec{E} \) ESTABLISHED

E5: OHM'S LAW: \( V = IR \)

E6: \( \vec{B} \), THE MAGNETIC FIELD, DEFINED

E7: A CHANGING \( \vec{B} \) FIELD INDUCES AN \( \vec{E} \) FIELD.

E8: A MAGNETOSTATIC FIELD IS WITHOUT DIV \((\nabla \cdot \vec{B} = 0)\)

E9: \( \nabla \times \vec{H} = \vec{J} \) WHEN \( \vec{B} \) IS MAGNETOSTATIC

A1: ASSUME \( \nabla \cdot \vec{E} \) IS PROPORTIONAL TO \( \vec{J} \)

WHEN \( \vec{E} \) IS DYNAMIC

A2: DYNAMIC \( \vec{B} \) FIELD HAS NO DIVERGENCE \((\nabla \cdot \vec{B} = 0)\)

MAXWELL'S ASSUMPTION: WE KNOW

TOTAL CURRENT = CONDUCTION CURRENT

+ DISPLACEMENT CURRENT

CONDUCTION CURRENT \( \Rightarrow \) MAGNETIC FIELD

\[ \nabla \times \vec{H} = \vec{J} \Rightarrow \text{MAG Field (MAXWELL'S HYPOTH)} \]

\[ \nabla \times \vec{H} = \vec{J} \Rightarrow \text{CONDUCTION CURRENT} \]

WITH DISPLACEMENT CURRENT: \( \nabla \times \vec{H} = \vec{J} + \frac{\varepsilon \partial \vec{E}}{\partial t} \)

MAXWELL'S EQUATIONS:

\[ \nabla \times \vec{H} = \vec{J} + \frac{\varepsilon \partial \vec{E}}{\partial t} \]

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

\[ \vec{D} = \varepsilon \vec{E} \]

\[ \vec{B} = \mu \hat{H} \]

\[ \nabla \cdot \vec{B} = 0 \]

\[ \vec{J} = \sigma \vec{D} \]
IN HOMOGENEOUS MEDIA WITH NO CHARGE OR CONDUCTIVITY
(SUCH AS FREE SPACE)
\[ \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \]
\[ \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \]
\[ \nabla \cdot \mathbf{H} = 0 \]
\[ \nabla \cdot \mathbf{E} = 0 \]

**DERIVATION OF WAVE EQUATIONS**
\[ \nabla \cdot (\nabla \times \mathbf{E}) = -\mu \nabla \cdot \left( \frac{\partial \mathbf{H}}{\partial t} \right) \]
\[ = -\mu \frac{\partial}{\partial t} \nabla \cdot \mathbf{H} = -\mu \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = -\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

A VECTOR IDENTITY: \[ \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

But \[ \nabla \cdot \mathbf{E} = 0 \] (since there is no charge)
\[ \Rightarrow \nabla^2 \mathbf{E} = +\mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

3 DIFF. EQUATIONS FOR \( E_1 = E_2 = 0 \), \( E_3 = f(x - ct) \) \( \implies C = \sqrt{\frac{\mu \epsilon}{\mu \epsilon}} \)

IN FREE SPACE \( \Rightarrow C = \text{SPEED OF LIGHT} = \frac{1}{\sqrt{\mu \epsilon}} \)

WE CAN ALSO SHOW: \[ \nabla^2 \mathbf{H} = \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \]

(Note: \( \mathbf{E} \) \& \( \mathbf{H} \) are physically inseparable quantities)
IX. PLANE WAVES

A. ELECTRIC FIELD

$$\nabla^2 E = \mu \epsilon \frac{\partial^2 E}{\partial t^2}$$

Assume $$E_x = E_z = 0 \Rightarrow \nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = \mu \epsilon \frac{\partial^2 E_y}{\partial t^2}$$

Also assume wave doesn't change in $$y$$ or $$z$$:

$$\Rightarrow \frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial z} = 0$$

$$\Rightarrow \frac{\partial^2 E_y}{\partial x^2} = \mu \epsilon \frac{\partial^2 E_y}{\partial t^2} \Rightarrow \text{assume } E_y = f_1(x - vt) + f_2(x + vt)$$

B. MAGNETIC FIELD

From A, assume $$E_y = E_m \cos \omega t - B_x$$

$$\omega = \frac{2\pi}{\beta}, \; v = \frac{\omega}{c}, \; \lambda = \frac{2\pi}{\beta} = \frac{\lambda}{v}$$

Now $$\frac{\partial B_z}{\partial t} = -\nabla \times E$$

$$\Rightarrow \frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} = -\beta E_m \sin(\omega t - \beta x)$$

$$\Rightarrow B_z = \frac{\beta}{\omega} E_m \cos(\omega t - \beta x) = \frac{\beta}{\omega} E_y$$

Thus $$E_y = \sqrt{\mu \epsilon} B_z$$

$$Z = \sqrt{\frac{\mu}{\epsilon}} \Rightarrow \text{INTRANSCUDE IMPEDANCE (377 ohms FREE SPACE)}$$

Thus, $$E$$ and $$H$$ are in phase and \perp
c. Polarization

If E vector oscillates, but maintains the same direction, the wave is polarized (for example, $E_x = E_y = 0$)

Plane of Polarization
- In optics, the plane $\parallel$ H-field
- In radio, $\perp$ E-field

D. Complex Notation

$E_x = E_m \cos(\omega t - \beta x)$

Physically, we want $R_0 e^{j\theta} \leq \phi_0 \cos \theta$

E. Propagation in a Conducting Media

Until now, we have assumed $\mu = 0$

(re a perfect dielectric).

Restrict attention to sinusoids:

$\vec{E} = E_0 e^{j\omega t}$

$\vec{H} = H_0 e^{j\omega t}$

Still assume $\rho = 0$

Maxwell:

$\nabla \times \vec{E} = -\mu \frac{\delta \vec{H}}{\delta t}$

$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\delta \vec{E}}{\delta t}$

or

$\nabla \times E_0 e^{j\omega t} = -j \omega \mu H_0 e^{j\omega t}$

$\nabla \times H_0 e^{j\omega t} = \sigma E_0 e^{j\omega t} + j \omega \varepsilon E_0 e^{j\omega t}$

or

$\nabla \times E_0 = -j \omega \mu H_0$

$\nabla \times H_0 = (\sigma + j \omega \varepsilon) E_0$

We now wish to derive a wave equation $\Rightarrow$
\[ \vec{\nabla} \times \vec{\nabla} \times \vec{E}_0 = -j \omega \mu (\vec{\nabla} \times \vec{H}_0) \]
\[ -\nabla^2 \vec{E}_0 = -j \omega \mu (\delta + j \omega \varepsilon) \vec{E}_0 \]

**Define**

\[ \Gamma^2 = j \omega \mu (\delta + j \omega \varepsilon) \]

\[ \Rightarrow \nabla^2 \vec{E}_0 = \Gamma^2 \vec{E}_0 \iff \text{WAVE EQUATION WITH CONDUCTIVITY} \]

**Assume, as before:**

\[ E_x = E_z = 0 = \frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial z} \]

\[ \Rightarrow \frac{\partial^2 E_y}{\partial x^2} = \Gamma^2 E_y \]

**Solution is**

\[ E_y = E_m e^{\pm \Gamma x} \]

\[ \Rightarrow E_y = E_m e^{j \omega t - \Gamma x} \]

**Note:** \( \delta = 0 \Rightarrow \Gamma = j \beta \) and we have previous result.

Let

\[ \Gamma = j \omega \sqrt{\mu \varepsilon (1 + \frac{\delta}{\omega \varepsilon})} = \alpha + j \beta \]

\[ \Rightarrow E_y = E_m e^{-\alpha x} e^{j (\omega t - \beta x)} \]

- **Magnetic Field (not in phase with \( \vec{E} \) for \( \delta \neq 0 \))**

\[ \vec{\nabla} \times \vec{E}_0 = -j \omega \mu \vec{H}_0 \Rightarrow H_z = \frac{\mu}{\varepsilon} \frac{\partial E_y}{\partial y} \]

\[ \eta = \text{INTRINSIC IMPEDANCE} = \frac{E_y}{H_z} \]

\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} \text{ for } \delta = 0 \]

\[ \eta \text{ is complex in first quadrant} \]

\[ \Rightarrow \vec{E} \text{ field leads } \vec{H} \text{ field in phase } \]

- **Dielectric Loss**

Due to "Dielectric Hysteresis"

\[ \text{proportional to } \omega \]
G. Power and the Poynting Vector

\( \vec{P} = \text{power/m}^2 \); \( \vec{P} \cdot \vec{A} = \text{power thru area } \vec{A} \)

For a given surface:

Outward flow of power = \( \oint \vec{P} \cdot d\vec{A} \)

Recall:

\( (\text{energy from } \vec{E}) = \frac{1}{2} \int \vec{P} \cdot \vec{E} \, dV \)

\( (\text{" \ " \ " } \vec{H}) = \frac{1}{2} \int \vec{B} \cdot \vec{H} \, dV \)

Power decrease = \(- \frac{1}{2} \frac{\delta E}{\delta t} \int (\vec{E} \cdot \vec{H} + \vec{D} \cdot \vec{E}) \, dV \)

Power decrease = outward flow of power

\[ \Rightarrow \oint \vec{P} \cdot d\vec{A} = -\frac{1}{2} \int \frac{\delta E}{\delta t} \left( \vec{H} \cdot \vec{H} + \vec{E} \cdot \vec{E} \right) \, dV \]

\[ = -\int \left( \vec{H} \cdot \frac{\delta \vec{H}}{\delta t} + \vec{E} \cdot \frac{\delta \vec{E}}{\delta t} \right) \, dV \]

\[ = \int \left[ \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) \right] \, dV \]

\[ = \int \nabla \cdot (\vec{E} \times \vec{H}) \, dV \Leftarrow \text{from vector identity} \]

Using Gauss' Theorem

\[ \oint \vec{P} \cdot d\vec{A} = \oiint \vec{E} \times \vec{H} \cdot d\vec{A} \]

\[ \Rightarrow \vec{P} = \vec{E} \times \vec{H} \quad \text{(Note: we assumed } \gamma = 0 \text{ but it is also true for } \gamma > 0) \]

Example:

[Diagram of a cylinder with vectors and field lines]
X. REFLECTION
A. BOUNDARY SURFACES

NORMAL $\mathbf{B}$ FIELD MUST BE CONTINUOUS
TANGENTIAL $\mathbf{E}$ FIELD THE SAME
NORMAL $\mathbf{B}$ FIELD THE SAME
TANGENTIAL $\mathbf{H}$ FIELD THE SAME

B. CONDUCTOR AS BOUNDARY

CAN BE NO $\mathbf{E}$ FIELD IN A PERFECT CONDUCTOR

SKIN EFFECT (INTRO) (SEE NEXT PAGE ⇒ INCREASING $\mathbf{B}$ FIELD (OUT OF PAGE)

$\Rightarrow$ CURRENT $\mathbf{J}$ IS PRODUCED

$\Rightarrow$ $\mathbf{H}$ FIELD WILL HAVE CURL

($\mathbf{J} \times \nabla \times \mathbf{H}$ FOR CONDUCTOR)

$\Rightarrow$ $\mathbf{B}$ WILL DIMINISH AT THE CONDUCTOR'S INARDS

$\Rightarrow$ MOST CURRENT WILL BE AT SURFACE

CURRENT PROVIDES BOUNDARY FOR $\mathbf{B}$ FIELD

$$\oint \mathbf{H} \cdot d\mathbf{s} = I \quad (\text{SINCE } \phi_{\mathbf{B}} = 0 \text{ IN COND})$$

$$= \int \mathbf{H} \cdot d\mathbf{l} \quad \Rightarrow \mathbf{H} = \frac{I}{2}$$
C. Skin Effect

Recall: \[ E_{oy} = E_m e^{j \frac{px}{\lambda}} \]

Thus, in a conductor: \[ J_{oy} = J_m e^{j \frac{px}{\lambda}} \]

\[ \Gamma = j \omega \mu (1 + j \omega \epsilon) \]

For \( \omega \ll \text{Freq of Vis. Light} \):

\[ \Gamma = \frac{l + j}{b} \Rightarrow b = \frac{1}{\sqrt{\mu \epsilon}} \]

\[ \Rightarrow J_{oy} = i_m e^{-\frac{px}{\lambda}} \]

\[ \Rightarrow I_d = \int_0^a J_{oy} \, dx = \frac{i_m}{\lambda} = \frac{I}{\lambda} \]

Power loss per volume:

\[ \frac{1}{\lambda^2} \gamma = \frac{i_m^2 \delta}{4 \sigma} \]

\( \delta = \text{"Skin Depth"} \)

\( \frac{1}{\lambda^2 \gamma} = \text{"Surface Resistivity"} \)

D. Reflection from a Conductor

\[ E_x = E_m \cos \beta (vt - z) + \frac{1}{2} B_z \]

\[ E_x(0) = 0 \Rightarrow \frac{1}{2} B_z = -E_m \cos \beta (vt + z) ; \quad w = BV \]

\[ E_x = E_m \left[ e^{-j \beta z} - e^{j \beta z} \right] e^{j \omega t} \]

\[ B_z = \frac{\rho}{\omega} E_y \Rightarrow H_y = \frac{E_m}{\lambda} \left( e^{-j \beta z} + e^{j \beta z} \right) e^{j \omega t} \]

\[ E_x \perp B_z \text{ are "Standing Waves"} \]

\[ E_x = 2E_m \sin \beta z \sin \omega t \]

\[ H_y = \frac{2E_m}{\lambda} \cos \beta z \cos \omega t \]

Nodes 90° out of phase.
**E. Dielectric Diffraction**

- \( E_x(\text{air}) = E_{m_1} e^{j(\omega t - \beta_1 z)} + E_{m_2} e^{j(\omega t + \beta_1 z)} \)
- \( E_x(d) = E_{m_3} e^{j(\omega t - \beta_3 z)} \)

@ \( z = 0 \), (surface) \( \Rightarrow \) \( E_{m_3} = E_{m_1} + E_{m_2} \)

- \( H_y(\text{air}) = H_{m_1} e^{j(\omega t - \beta_1 z)} + H_{m_2} e^{j(\omega t + \beta_1 z)} \)
- \( H_y(d) = H_{m_3} e^{j(\omega t - \beta_3 z)} \Rightarrow H_{m_3} = H_{m_1} + H_{m_2} \)

\( H \neq E \) are related by intrinsic impedance:

\[ E_{m_1} = \eta_1 H_{m_1}, \quad E_{m_2} = -\eta_1 H_{m_2}, \quad E_{m_3} = \eta_3 H_{m_3} \]

(-) sign from \( E > 0 \Rightarrow H < 0 \)

This gives:

\[ E_{m_2} = \frac{\eta_3 - \eta_1}{\eta_3 + \eta_1} E_{m_1} \]

\( \rho = \frac{\eta_3 - \eta_1}{\eta_3 + \eta_1} \); reflection coefficient

Power flow: \( P_1 = E_1 \times H_1 = E_{m_1} H_{m_1} \cos^2 \omega t \)

We have a standing wave

**SWR = Standing Wave Ratio**

\[ \text{SWR} = \frac{\text{Field strength at Loop}}{\text{Field strength at Node}} \]

- Big SWR
- Small SWR

(Big Ref) (Small SWR)
F. REFLECTION FROM A POOR CONDUCTOR

\[ \Gamma = (\text{COMPLEX}) \text{ PROPAGATION FACTOR} \]

\[ Z = (\text{COMPLEX}) \text{ INTRINSIC IMPEDANCE} \]

\[ E_x(t) = E_m 3 \cos(\omega t - \Gamma z) \]

\[ H_y(t) = H_m 3 \cos(\omega t - \Gamma z) \]

RECALL

\[ n_3 = \left( \frac{\mu_3}{\epsilon_3 (1 + \frac{\epsilon_3}{\omega \mu_3})} \right) \]

XMITTED + REFL WAVES HAVE PHASE SHIFT

G. OBLIQUE REFLECTION

BOUNDARY CONDITIONS: \( E_1 + E_2 = E_3 \), \( H_{t1} + H_{t2} = H_{t3} \)

\[ b - a = \frac{\lambda^2}{\cos \gamma_3} = \frac{\lambda^1}{\cos \gamma_3} = \frac{\lambda^2}{\cos \gamma_2} \]

GIVES "SNELL'S LAW"

\[ \frac{\cos \gamma_1}{\cos \gamma_3} = \frac{\lambda^1}{\lambda_3} = \frac{v_1}{v_3} \]

NOW

\[ E_1 = n_1 H_1 \]

\[ E_2 = n_1 H_2 \]

\[ E_3 = n_1 H_3 \]

\[ H_{t1} = \frac{E_1}{n_1} \sin \gamma_1 \]

\[ H_{t2} = \frac{E_2}{n_1} \sin \gamma_1 \]

\[ H_{t3} = \frac{E_3}{n_3} \sin \gamma_3 \]

OR

\[ \frac{E_2}{E_1} = \frac{n_3 \sin \gamma_1 - n_1 \sin \gamma_3}{n_3 \sin \gamma_1 + n_1 \sin \gamma_3} \]
XI. Radiation

A. Electromagnetic Potentials

For Electrostatic Case (with no $J$ or $Q$)
\[ \nabla^2 V = 0 \quad , \quad \nabla^2 A = 0 \]

For Electromagnetic Case (with no $J$ or $Q$)
\[ \nabla^2 V - \mu \varepsilon \frac{\partial^2 V}{\partial t^2} = 0 \]
\[ \nabla^2 A - \mu \varepsilon \frac{\partial^2 A}{\partial t^2} = 0 \]
\[ \frac{\partial V}{\partial n} = \frac{1}{V} \]

Quasi-Stationary Relations
\[ \vec{A} = \frac{1}{4\pi} \int \frac{\vec{E}}{r} \, dv \quad V = \frac{1}{4\pi \varepsilon} \int \frac{\rho}{r} \, dv \]

Retarded Potentials
\[ V = \frac{1}{4\pi \varepsilon} \int \frac{\rho(t - \frac{r}{v})}{r} \, dv \]
\[ \vec{A} = \frac{1}{4\pi} \int \frac{\vec{E}(t - \frac{r}{v})}{r} \, dv \]

B. Radiation

Short Wire of Length $l$ and Current $i = I \sin \omega t$

Sine $\vec{I} = \vec{I}_x$, we only have $x$ component of a field:
\[ A_x = \frac{1}{4\pi} \int \frac{\rho(t - \frac{r}{v})}{r} I \sin \omega (t - \frac{r}{v}) \, dx \]

For $r \gg l$, we have
\[ A_x = \frac{I}{4\pi r} l \sin \omega (t - \frac{r}{v}) \]

Or, in spherical coordinates:
\[ A_r = \frac{I}{4\pi r} l \cos \theta \sin \omega (t - \frac{r}{v}) \]
\[ A_\theta = \frac{I}{4\pi r} l \sin \theta \sin \omega (t - \frac{r}{v}) \]
\[ A_\phi = 0 \]

Taking curl gives $\vec{H}$:
\[ H_r = H_\theta = 0 \]
\[ H_\phi = \frac{I}{4\pi r} \sin \theta \left[ \frac{\omega}{v} \cos \omega (t - \frac{r}{v}) + \frac{1}{r} \sin \omega (t - \frac{r}{v}) \right] \]
Now \( \nabla \cdot A = -\varepsilon \frac{\partial E}{\partial t} \)

\[ \Rightarrow \nabla \cdot V = -\varepsilon \varepsilon_0 \int A \, dt \]

\[ E = -\nabla \cdot V \]

GIVES

\[ E_r = \frac{I \delta_e}{2\pi \varepsilon_0 r} \cos \theta \left[ \frac{1}{v} \sin \omega (t - \frac{r}{v}) - \frac{1}{v^2} \omega^2 \cos \omega (t - \frac{r}{v}) \right] \]

\[ E_\theta = \frac{I \delta_e}{4\pi \varepsilon_0 r^2} \sin \theta \left[ \frac{\omega^2}{\varepsilon_0} \cos \omega (t - \frac{r}{v}) + \frac{1}{\varepsilon_0} \sin \omega (t - \frac{r}{v}) \right. \]

\[ \left. - \frac{i}{\omega r^2} \cos \omega (t - \frac{r}{v}) \right] \]

\[ E_\phi = 0 \]

or, with \( \lambda = \frac{v}{\omega} = \frac{2\pi \lambda}{r^2} \) AND \( \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \)

\[ E_r = -n \frac{I \delta_e \cos \theta}{\lambda r} \left[ \frac{\lambda^2}{4\pi^2 \eta^2} \cos \left( \frac{2\pi r}{\lambda} - \omega t \right) + \frac{\lambda}{2\pi \eta} \sin \left( \frac{2\pi r}{\lambda} - \omega t \right) \right] \]

\[ E_\theta = \eta \frac{I \delta_e \sin \theta}{2r \lambda} \left[ -\frac{\lambda^2}{4\pi^2 \eta^2} \cos \left( \frac{2\pi r}{\lambda} - \omega t \right) - \frac{\lambda}{2\pi \eta} \sin \left( \frac{2\pi r}{\lambda} - \omega t \right) \right. \]

\[ \left. + \cos \left( \frac{2\pi r}{\lambda} - \omega t \right) \right] \]

\[ E_\phi = 0 \]

\[ H_r = 0 \]

\[ H_\theta = 0 \]

\[ H_\phi = \frac{I \delta_e \sin \theta}{2r \lambda} \left[ -\frac{\lambda}{2\pi \eta} \sin \left( \frac{2\pi r}{\lambda} - \omega t \right) + \cos \left( \frac{2\pi r}{\lambda} - \omega t \right) \right] \]

For \( r \) SMALL: Induction Field:

\[ E_r = -\eta \frac{I \delta_e \cos \theta}{r \lambda} \left[ \frac{\lambda^2}{4\pi^2} \cos \left( \frac{2\pi r}{\lambda} - \omega t \right) \right] \]

\[ E_\theta = -\eta \frac{I \delta_e \sin \theta}{2r \lambda} \left[ \frac{\lambda^2}{4\pi^2} \cos \left( \frac{2\pi r}{\lambda} - \omega t \right) \right] \]

\[ E_\phi = 0 \]

\[ H_r = H_\theta = 0 \]

\[ H_\phi = -\frac{I \delta_e \sin \theta}{2r \lambda} \left[ \frac{\lambda}{2\pi \eta} \sin \left( \frac{2\pi r}{\lambda} - \omega t \right) \right] \]
FOR $\gamma$ LARGE: RADIATION FIELD:

\[
\begin{align*}
E_r &= E\phi = 0 \\
E_\theta &= 2\pi I \frac{\cos}{2\pi} \cos \frac{2\pi r}{\lambda} - \omega t \\
H_r &= H_\phi = 0 \\
H_\phi &= \frac{I \pm \epsilon A \theta}{2\pi} \cos \left( \frac{2\pi r}{\lambda} - \omega t \right)
\end{align*}
\]

**NOTE:** $E_0 = \pi H_\phi$

C. THE SPHERICAL WAVE

RADIATED WAVE (NO DIVERGENCE)

E FIELD

H OUT OF PAGE
XII. ANTENNAS

A. SHORT ANTENNAS

LENGTH OF ANTENNA LESS THAN A $\lambda$

CURRENT FLOWS DUE TO DISTRIBUTED CAPACITANCE

EFFECTIVE LENGTH $l_e = \frac{L}{2}$, W CURRENT $I_0$

EFFECTIVE (RMS) FIELD STR $(\text{m}) = \frac{n I_0 l_e}{2 r \lambda} \sin \theta$

$\lambda = 377 \Omega$ (FOR FREE SPACE)

$I_0$ = EFFECTIVE MIDPOINT CURRENT (AMPS)

$l$ = ACTUAL ANTENNA LENGTH

$l_e$ = EFF. ANT. LENGTH (IF $l \ll \lambda$, $l_e = \frac{1}{2} l$

$\theta$ = ANGLE TWIXT XMITTING & REC. ANTENNA

$c$ = DISTANCE TWIXT ANTENNAS

$\lambda$ = WAVELENGTH.
B. HALF WAVE LENGTH ANTENNAS

CURRENT IS NO LONGER LINEAR, BUT

\[ I = I_0 \cos \frac{2\pi x}{\lambda} \]

OBLIQUITY IS ANOTHER COMPLICATION

INTEGRATING EACH LITTLE \text{d}x GIVES

\[ E_\theta = \int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \frac{n I_0 \sin \theta e}{2\pi \lambda} \cos \left( \frac{2\pi x}{\lambda} \right) \cos \omega (t - \frac{\tau e}{c}) \text{d}x \]

\[ H_\phi = \frac{1}{\mu} E_\theta \]

COMPARISON TO SHORT ANTENNA:

FOR \( \frac{1}{2} \) WAVE (DIPOLE): RMS FIELD = \( \frac{n I_0}{2\pi \tau} \)

FOR SHORT: RMS FIELD = \( \frac{n I_0}{2\pi \lambda} \)

THUS \( L_e = \frac{n I_0}{\tau} \) FOR \( \frac{1}{2} \) WAVE

ACTUAL LENGTH = \( \frac{\lambda}{2} \)

\[ \Rightarrow L_e = \frac{2}{\pi} \]
C. RADIATION PATTERN

Plot is field strength \( \left( \frac{\text{W}}{\text{m}} \right) \) at one mile.

\[
\begin{align*}
\text{TURNs OUT } \frac{3}{4} \text{ WAVE IS MORE EFFICIENT THAN SHORTY}
\end{align*}
\]

D. RADIATED POWER

\[
\mathbf{P} = \mathbf{E} \times \mathbf{H}
\]

\[
\mathcal{P} = \int \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = \int \mathbf{E} \cdot d\mathbf{a} \times \mathbf{H}
\]

\[
= \int 2\pi I^2 \left( \frac{\sin \theta}{2 \lambda} \right) \cos \theta \left( 1 - \frac{r}{c} \right) \left( 2 \pi r^2 \sin \theta \right) d\theta
\]

\[
= \frac{2\pi I^2 \lambda^2}{3 \lambda^2} \cos^2 \theta \left( 1 - \frac{r}{c} \right)
\]

Thus, for a short uniform \( \lambda \) antenna power (average)

\[
= \frac{\frac{2\pi I^2 \lambda^2}{3 \lambda^2}}{\frac{2\pi I^2 \lambda^2}{3 \lambda^2}} = \frac{2\pi I^2 \lambda^2}{3 \lambda^2} \]

E. RADIATION RESISTANCE

\[
= \frac{\text{AVERAGE POWER}}{\text{EFF}}
\]

For short antenna

\[
\frac{2\pi I^2 \lambda^2}{3 \lambda^2} = \frac{789 \left( \frac{\lambda}{\lambda} \right)^2 \Omega}{789}
\]

For half-wave dipole, turns out

\[
= 73.1 \Omega \text{ (INO. OF } \omega)\]
F. ANTENNAS ABOVE GROUND

\[ \text{Reflection gives } 180^\circ \text{ phase shift} \]

\[ \text{We want } \frac{1}{2} \lambda \text{ difference in path for constructive interference} \]

(Note antenna is essentially doubled)

G. GROUNDED ANTENNAS

\[ \text{Radiation from } \]

\[ \text{Direct} \]

\[ \text{Reflected} \]

VT: \[ \theta \]

\[ \text{Vertical antennas} \]

\[ \eta_3 \text{ will be vertically polarized.} \]

\[ \text{For perfect reflection } (-1), \]

\[ \text{Horizontal component would be reversed} \]

\[ \frac{1}{2} \text{ vertical component the same.} \]

\[ \text{If earth were dielectric, refl. would be smaller than incident wave, and} \]

\[ \text{is independent of angular incidence.} \]

BREWSTER ANGLE: ALL INCIDENT ENERGY ABSORBED (AS "CRITICAL ANGLE")

Recall reflection coefficient

\[ \frac{E_{t2}}{E_{t1}} = \frac{\eta_2 \sin \theta_3 - \eta_1 \sin \theta_3}{\eta_2 \sin \theta_3 + \eta_1 \sin \theta_3} \]
IF ANTENNA IS SHORT WIRE, WE TAKE INTO ACCOUNT REFLECTION:
\[ E_{\text{rms field str}} = \frac{\pi I_0 h_0}{\lambda} \text{ Air } \]
MAY INCREASE \( I_0 \) BY PLACING CAPACITANCE NEAR THE TOP TO FURTHER MAKE CURRENT UNIFORM.

H. ANTENNA ARRAYS

\[ \text{LEAD} \quad \text{LAG} \]
\[ \text{CANCEL} \quad \text{REINFORCED} \]

RADIATION PATTERN

I. RECEIVING
E-M FIELD CAUSES I IN A CONDUCTOR
XII. WAVEGUIDES

A. Guided Waves

1. Fields can terminate only on electric charge.
2. For unpolarized plane wave propagation:
   - Current must provide NEC, core for H field boundary.
   - Provide proper charge for E field boundary.

   Boundary
   - \( \nabla \times H = \frac{\delta D_y}{\delta t} \Rightarrow -\frac{\delta H_z}{\delta x} = \frac{\delta D_y}{\delta t} \) \( \text{Current required} \)
   - \( \nabla \cdot D_y \Rightarrow \frac{\delta E_y}{\delta t} = \frac{\delta D_y}{\delta t} \) \( \text{To terminate E field} \)
   - \( H_z = I_x \Rightarrow \frac{\delta H_z}{\delta x} = \frac{\delta I_x}{\delta t} \) \( \text{Conclusion; special case of a Maxwell} \)

A wave can be limited to the space between two conductors since the current that flows on these surfaces is proper to terminate both E and H fields.
B. Finite Waves

BENDING TOP UP INTO A CYLINDER

BENDING BOTTOM DOWN INTO A CYLINDER.

THIS GIVES PARALLEL-WIRE

TRANSMISSION LINE.
C. Hollow Wave Guides

There can be no tangential \( \mathbf{E} \) field at surface since in conductors, but two sinuosity of same amplitude \( \frac{1}{2} \text{ freq.} \), when superimposed at an angle, will add to zero.

A depends on freq. \( \frac{1}{2} \text{ wave guide.} \)

TE \( \text{ mode:} \)

\[ \int \text{ flux} \]

\[ \text{ TE field} \]
D. GROUP VELOCITY

\[ v_g = \text{GROUP VELOCITY} \approx c_0 \text{ (due to zig-zag path)} \]

\[ \lambda = \text{LONGEST WAVELENGTH: } \lambda_0 = 2b \]

\[ f_0 = \text{CUT OFF FREQUENCY: } f_0 = \frac{v_c}{2b} = \frac{1}{2b \sqrt{\mu_0}} \]

E. PHASE VELOCITY:
**F. Derivation**

From Maxwell:

\[ \nabla \times E = \mu_0 \epsilon_0 \frac{\partial B}{\partial t}, \quad \nabla \cdot E = 0 \]

A solution is

\[ E_x = A \sin \left( \frac{2\pi}{L} x \right) \sin (\beta x - \omega t), \quad \beta = \frac{\omega}{\sqrt{\mu \epsilon}}, \quad \omega^2 = \frac{\mu_0 \epsilon_0 \omega}{(\mu \epsilon)} \]

**H Field from Maxwell equation:**

\[ \nabla \times E = \frac{\partial B}{\partial t} \]

\[ \Rightarrow \quad H_x = \frac{\mu_0}{\epsilon_0} \frac{\partial E_y}{\partial t}, \quad H_y = \frac{\mu_0}{\epsilon_0} \frac{\partial E_x}{\partial t} \]

**G. General Solution for Rectangular Guide**

**TM** - Transverse Magnetic Field

**TE** - Transverse Electric Field

**TEM** - No Axial Comp. of Either Field

**Diff. Eq.:**

\[ \nabla^2 E = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0 \]

**B.C.:**

\[ E_x = E_y = 0 \quad \text{at} \quad z = 0, b; \quad E_y = 0 \quad \text{at} \quad y = 0, a \]

**Assume Sinusoidal Form:**

\[ E_x = E_{mx} e^{j \omega t}, \quad E_y = E_{my} e^{j \omega t}, \quad E_z = E_{mz} e^{j \omega t} \]

**Then:**

\[ \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{\epsilon_0 \mu_0}{\epsilon_0 \mu_0} \frac{\partial^2 E_x}{\partial t^2} \]

**Becomes:**

\[ \frac{\partial^2 E_{mx}}{\partial y^2} + \frac{\partial^2 E_{my}}{\partial z^2} = -(\omega^2 \epsilon_0 \mu_0 \eta^2) E_{mx} \]
LET $E_{my} = \frac{\partial^2 E}{\partial y \partial z}$

GIVES

$$\frac{1}{\gamma} \frac{d^2 y}{d z^2} + \frac{1}{\gamma} \frac{d^2 z}{d y^2} = -(\omega^2 \mu + \Gamma^2)$$

LET

$$\frac{1}{\gamma} \frac{d^2 y}{d z^2} = -M^2 - \frac{1}{\gamma} \frac{d^2 z}{d y^2} = -N^2$$

$\Rightarrow M^2 + N^2 = \omega^2 \mu + \Gamma^2$

$\therefore y = A_1 \cos N \nu + B_1 \sin N \nu$

$\therefore z = C_1 \cos M \mu + D_1 \sin M \mu$

$E_{my} = (A_1 \cos N \nu + B_1 \sin N \nu) (C_1 \cos M \mu + D_1 \sin M \mu)$

SIMILAR EXPRESSIONS FOR $E_{my}, \frac{\partial}{\partial z} E_{nz}$

USING BOUNDARY CONDITIONS:

$E_x \big|_{y=0} = 0 \Rightarrow C_1 = 0$

$E_y \big|_{y=0} = 0 \Rightarrow A_1 = 0$

$E_x \big|_{z=0} = b \Rightarrow N = \frac{\mu \Gamma}{b}, \ n = 0, 1, 2, \ldots$

$E_x \big|_{z=a} = 0 \Rightarrow M = \frac{\nu \Gamma}{a}, \ m = 0, 2, \ldots$

$\therefore E_{my} = B_1 D_1 \sin M \mu \cos N \nu \sin \frac{\mu \Gamma}{a}$

THUS

$E_x = K_1 \sin \nu \mu \cos N \nu \sin \frac{\mu \Gamma}{a}$

MAY OBTAIN $E_y \# E_x$ FROM

$\forall \nu, \mu, E = 0$

GIVES:

$$\Gamma^2 = \left(\frac{\mu \Gamma}{a}\right)^2 - \omega^2 \mu$$

MAY GET H FROM

$$\frac{\partial H}{\partial \mu} = \frac{1}{\mu} \nabla \times E$$
H. TRANSVERSE ELECTRIC WAVES: $E_x = 0$

I. TRANSVERSE MAGNETIC WAVE: $H_x = 0$

J. MODES: EACH $n, m_l$ GIVE A MODE
Electrostatic and Magnetostatic Fields

\[ E \cdot \mathbf{F} = \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{E} = 0; \quad \int \mathbf{E} \cdot d\mathbf{s} = \int \mathbf{D} \cdot d\mathbf{a} = \mathbf{Q} \]

\[ \mathbf{D} = \varepsilon \mathbf{E} = \text{ELECTROSTATIC FLUX DENSITY} \]

- Vectors: Divergence
  \[ \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]

- Gradient
  \[ \nabla \mathbf{A} = \left[ \frac{\partial A_x}{\partial x}, \frac{\partial A_y}{\partial y}, \frac{\partial A_z}{\partial z} \right] \]

- Curl:
  \[ \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \]

- Laplacian:
  \[ \nabla^2 P = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{P} \]

- Theorems
  - Gauss:
    \[ \oint \mathbf{D} \cdot d\mathbf{s} = \int \nabla \cdot \mathbf{D} \, dv \]
  - Stokes:
    \[ \iint \mathbf{E} \cdot d\mathbf{S} = \int \nabla \times \mathbf{E} \cdot d\mathbf{a} \]

- Potentials
  - Scalar (Lamella) if \( \nabla \times \mathbf{E} = 0 \) \( \Rightarrow \mathbf{E} = -\nabla \phi \)
  - Vector (Solenoidal Sourceless) if \( \nabla \cdot \mathbf{B} = 0 \) \( \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \)

Fundamental Laws of Electrostatics

\[ \begin{cases} \mathbf{F} = \mathbf{Q} \mathbf{E} \\ \oint \mathbf{E} \cdot d\mathbf{s} = 0 \\ \int \mathbf{D} \cdot d\mathbf{a} = \mathbf{Q} \end{cases} \]

OR

\[ \begin{cases} \mathbf{E} = -\nabla \mathbf{V} \\ \nabla^2 \mathbf{V} = -\rho/\varepsilon \leftrightarrow \text{POISSON'S EQ.} \]

Conductors: \( \mathbf{E} = \mathbf{0}, \mathbf{V} = \text{const}, \sigma = D_n = \text{SURFACE CHARGE} \)

A Charged Sphere: \( \nabla^2 \mathbf{V} = 0 \) gives \( \mathbf{V} = \frac{Q}{4\pi \varepsilon r} \)

\[ C = \frac{Q}{V}, \quad \text{Double Sphere: } C = \frac{4\pi Q \varepsilon abc}{b-q} \quad (\text{over}) \]

Coulomb's Law:

\[ \mathbf{F} = \frac{Qq}{4\pi \varepsilon r^2} \]

\[ V_{1,2} = \int \frac{1}{2} \mathbf{E} \cdot d\mathbf{s}; \quad \text{Electrostatic } \mathbf{E} = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, dv \]

\[ V = \frac{1}{4\pi \varepsilon} \int \rho \, dv \leftrightarrow \text{POSSIBLE INTEGRAL} \]
**Example Capacitance of Double Sphere**

\[ V_{12} = \int_1^2 E \cdot d\vec{S} \]

For inner sphere: \[ E = \frac{Q}{4\pi \varepsilon_0 r^2} \]

\[ V_{ab} = \int_a^b \frac{Q}{4\pi \varepsilon_0 r^2} dr = \frac{Q(b-a)}{4\pi \varepsilon_0 ab} \]

\[ C = \frac{Q}{V_{ab}} = \frac{Q}{\frac{Q(b-a)}{4\pi \varepsilon_0 ab}} \]

\[ \Rightarrow C = \frac{4\pi \varepsilon_0 ab}{b-a} \]

\[ b \to \infty \Rightarrow C = 4\pi \varepsilon_0 a \]

**Electric Current:** \[ I = \frac{dQ}{dt} \]

\[ \gamma = \text{conductivity} = \frac{U}{I \cdot m} \quad \Rightarrow \quad \frac{1}{\gamma} = \frac{U}{\text{area}} \]

**Analogy:** \[ V = \int \vec{E} \cdot d\vec{S} \]

\[ I = \int \vec{J} \cdot d\vec{a} \]

\[ \vec{I} = \gamma \vec{E} \leftarrow \text{Ohm's Law} \]

\[ \nabla \cdot \vec{J} = 0 \leftarrow \text{Kirchhoff's Current Law (static \( \vec{E} \))} \]

**Example:**

\[ \int \rightarrow \int \rightarrow \int \]

\[ I = \oint \vec{E} \cdot d\vec{a} = \oint \vec{D} \cdot d\vec{a} \]

\[ I = \oint \vec{J} \cdot d\vec{a} \]

\[ \Rightarrow \oint (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot d\vec{a} = 0 \]

\[ = \int \nabla \cdot (\vec{J} + \frac{\partial \vec{D}}{\partial t}) dV = 0 \]

\[ \Rightarrow \nabla \cdot (\vec{J} + \frac{\partial \vec{D}}{\partial t}) = 0 \]

**EMF**

\[ \vec{E}_m \rightarrow \text{due to } \vec{J} \quad \vec{E}_m \rightarrow \text{due to changing } \vec{H} \text{ field} \]

\[ V = \int \vec{E}_m \cdot d\vec{S} \quad \text{emf} = \int \vec{E}_m \cdot d\vec{S} \]
THE MAGNETIC FIELD

- MAGNETIC FORCE ON A WIRE: \[ \mathbf{F} = I \mathbf{E} \times \mathbf{B} \]

- MAGNETIC FLUX \[ \Phi = \oint \mathbf{E} \cdot d\mathbf{a} \]

- EMF \[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi}{dt} \]

- \[ \oint \mathbf{E} \cdot d\mathbf{a} = \int \mathbf{dE} \cdot \mathbf{B} \cdot d\mathbf{a} \]

\[ \nabla \times \mathbf{E} = 0 \]

\[ \mathbf{E} = -\nabla \Phi \]

\[ \mathbf{H} = \nabla \times \mathbf{A} \]

\[ \mathbf{J} = \nabla \times \mathbf{H} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

\[ \nabla \times \mathbf{E} = 0 \]

\[ \mathbf{E} = -\nabla \Phi \]

\[ \mathbf{H} = \nabla \times \mathbf{A} \]

\[ \mathbf{J} = \nabla \times \mathbf{H} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

\[ \nabla^2 \mathbf{V} = \frac{-\mathbf{J}}{\varepsilon_0} \]

\[ \mathbf{V} = \frac{1}{4\pi\varepsilon_0} \int \frac{d\mathbf{V}}{r} \]

- MAGNETIC

\[ \mathbf{J} = \frac{1}{4\pi} \int \frac{d^3\mathbf{V}}{r} \]

- ELECTRIC

\[ \mathbf{E} = -\nabla \Phi \]

\[ \mathbf{H} = \nabla \times \mathbf{A} \]

\[ \mathbf{J} = \nabla \times \mathbf{H} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

\[ \nabla^2 \mathbf{V} = \frac{-\mathbf{J}}{\varepsilon_0} \]

\[ \mathbf{V} = \frac{1}{4\pi\varepsilon_0} \int \frac{d\mathbf{V}}{r} \]

- MAGNETIC POTENTIAL

\[ \mathbf{J} = \frac{1}{4\pi} \int \frac{d^3\mathbf{V}}{r} \]
- QUASI STATIONARY MAG. FIELD

\[ \mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{j}}{r} \, d\mathbf{v} \]

\[ A_x = \frac{1}{4\pi} \int \frac{j_x}{r} \, d\mathbf{v} \]

\[ = \frac{1}{4\pi} \int \frac{1}{r} \, d\mathbf{v} \]

\[ = \frac{1}{4\pi} \int \frac{i \, dx}{r} \]

\[ = \frac{1}{4\pi} \int \frac{I \Delta m \omega t}{r} \, dx \]

\[ = \frac{1}{4\pi} \int \frac{I \Delta m \omega t}{r} e^{-\frac{r^2}{4\sigma^2}} \, dr \quad ; \quad r \gg \sigma \]

\[ = \frac{i \Phi}{4\pi r} \]

- MAGNETIC ENERGY \[ = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} \, d\mathbf{v} \]
E \& M FIELDS

- MAXWELL'S EQUATIONS:
  \[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]
  \[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \mathbf{D} = \varepsilon \mathbf{E} \]
  \[ \nabla \cdot \mathbf{B} = 0 \quad \mathbf{B} = \mu \mathbf{H} \]
  \[ \nabla \cdot \mathbf{D} = \rho \quad \mathbf{J} = \sigma \mathbf{E} \]

IN HOMOGENEOUS MATERIAL WITH NO \( \varepsilon \) OR \( \sigma \) (CONDUCTIVITY)

\[ \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \]
\[ \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \]
\[ \nabla \cdot \mathbf{H} = 0 \]
\[ \nabla \cdot \mathbf{E} = 0 \]

\( \) DERIVATION OF WAVE EQUATION

\[ \nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \Leftrightarrow \text{VECTOR IDENTITY} \]

\[ \nabla \times (\nabla \times \mathbf{E}) = \mu \nabla \times \left( \frac{\mathbf{H}}{\varepsilon} \right) = -\mu \frac{\varepsilon}{\varepsilon} \nabla \times \mathbf{H} = -\mu \frac{\varepsilon}{\varepsilon} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\varepsilon}{\varepsilon} \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

\[ \Rightarrow \nabla^2 \mathbf{E} = \frac{\mu \varepsilon}{\varepsilon} \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

FOR \( E_y = E_z = 0 \), \( E_x = \hat{f}(x-ct) \Rightarrow c = \frac{1}{\sqrt{\mu \varepsilon}} \)

\( \) PLANE WAVES

\[ E_y = E_z = 0 \Rightarrow E_x = E_0 \cos (\omega t - Bx) \]
\[ B_2 = \frac{\mu}{\omega} \cos (\omega t - Bx) \]
\[ \Rightarrow Z^2 = \frac{\mu}{\varepsilon} = \text{INTRINSIC Z} \]
PROPAGATION IN A CONDUCTING MEDIA
\[
\begin{align*}
\nabla \times E &= -\mu \frac{\partial H}{\partial t} \\
\nabla \times H &= \nabla E + \frac{\partial E}{\partial t} \\
E &= E_0 e^{j\omega t} \\
H &= H_0 e^{j\omega t} \\
\Rightarrow \nabla^2 E_0 &= \mu \frac{\partial^2 E_0}{\partial t^2} = j\omega \mu (\alpha + j\beta) \frac{\partial E_0}{\partial t} \\
E_y &= E_m e^{-\alpha x} e^{j(\omega t - \beta x)} \\
H_z &= \frac{n}{j\omega \mu} E_y \\
\rho &= E \times H \\
\text{BOUNDARY CONDITIONS} & \\
\text{NORMAL E FIELD CONTIN.} \text{ } & \text{NORMAL B FIELD SAME} \\
\text{TAN E FIELD SAME} \text{ } & \text{TAN H FIELD SAME} \\
\text{SKIN EFFECT} & \\
\Gamma^2 &= j\omega \mu (\alpha + j\omega \epsilon) \times j\omega \mu \delta \\
|\Gamma| &= \delta_m e^{-\alpha \delta} \\
\zeta &= \sqrt{j \rho \mu} = \text{SKIN DEPTH} \\
\text{DIELECTRIC DIFFRACTION} & \\
\rho &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \text{ REFLEC. COEFF,} \\
\text{SWR} &= \text{AVE} \\
\text{SNELL'S LAW:} & \quad \frac{c_1}{n_2} \frac{\sin \theta_1}{\sin \theta_2} = \frac{A_1}{A_2} = \frac{V_1}{V_2}
\end{align*}
\]
ADVANCED FIELDS

GREEN'S FUNCTIONS: \( \nabla^2 \psi (x, t) - \frac{\delta^2}{\delta t^2} \psi = -4\pi \delta (x, t) \)

\[ \nabla^2 - \frac{\delta^2}{\delta t^2} \Rightarrow \Box x \text{ } G = -4\pi \delta (x - x') \delta (t - t') \]

\[ G = \text{GREEN'S FUNCTION} \]

\[ \mathbf{E} = -\frac{1}{4\pi \varepsilon_0} \int \frac{\mathbf{q} \cdot \mathbf{r}}{r^3} \text{ } \mathbf{r} \text{ } \mathbf{E} \text{ FIELD INTENSITY} \]

DIPole MOMENT: \[ \mathbf{m} = q \mathbf{b} \]

\[ Q = \oint S \text{ } \mathbf{E} \cdot \mathbf{n} \text{ } dS \text{ } \Rightarrow \text{GAUSS' FLUX THEM} \]

\[ C = \frac{Q}{V}; \quad \frac{q}{b - q} = 2 \text{ CONC. SPHERES} \]

\[ \frac{2\pi e_0}{b - q} = \text{2 CONC. CYL.} \]

\[ \frac{6a^4}{a} = \text{2 /// PLATES} \]

\[ W = \frac{1}{2} \sum q_i \mathbf{V}_i \mathbf{E} \text{ IN CHARGED C} \]

\[ \sum q_i \mathbf{V}_i = \mathbf{E} \mathbf{Q}_v \text{ } \Rightarrow \text{GREEN'S REG. 'THEM} \]

*GENERAL THEOREMS*

GAUSS: \[ \oint \mathbf{A} \cdot \mathbf{n} \text{ } dS = \int \mathbf{A} \cdot \mathbf{n} \text{ } dV \]

STOKES: \[ \oint \mathbf{F} \cdot d\mathbf{L} = \int \mathbf{F} \cdot \nabla \mathbf{F} \text{ } dA \]

POISSON: \[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

*2-D DISTRIBUTIONS: \( \mathbf{E} = \frac{q}{4\pi \varepsilon_0} \)*

2-D IMAGES

\[ \begin{array}{c}
\begin{array}{c}
\mathbf{O}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\mathbf{b}
\end{array}
\end{array} \Rightarrow \begin{array}{c}
\begin{array}{c}
\mathbf{q}'
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\mathbf{q}
\end{array}
\end{array} \]

\[ \kappa = \frac{1}{k}; \quad \frac{n}{b} \Rightarrow \frac{q}{b} \]

3-D

SPHERE \[ \begin{array}{c}
\begin{array}{c}
\mathbf{O}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\mathbf{q}
\end{array}
\end{array} \Rightarrow \begin{array}{c}
\begin{array}{c}
\mathbf{q}'
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\mathbf{q}
\end{array}
\end{array} \]

\[ \kappa \cdot \mathbf{E} = \kappa^2 \mathbf{E} \quad \mathbf{K}^2 = \text{RADIUS OF INVERSION,} \]

\[ \mathbf{k} \cdot \mathbf{R} = \frac{1}{b} \sum_{n=0}^{\infty} \left( \frac{b}{a} \right)^n \mathbf{P}_n (\mathbf{a} \cdot \mathbf{e}) \]
MUTUAL INDUCTANCE: \[ M_{12} = \int_{\Gamma_1} \mathbf{B}_2 \cdot \mathbf{n} \, ds \]
\[ = \frac{\mu}{4\pi} \int_{\overline{\Gamma_1}} \mathbf{B}_2 \cdot d\mathbf{s}_1 \wedge d\mathbf{s}_2 \]
TUES

DUE MONDAY 2-8, 2-10, 2-16, 3-16

EPR

MAGNETICALLY
SOFT
IRON

PERMANENT MAGNET

LARGER AREA, LARGER HYSTERESIS (LOSS OF ENERGY)

MON

DIFFERENT B-H LOOPS

\[ i = \frac{V}{R} \quad \text{\(V\) input} \]

\[ B = iH \]

\[ W_l = \frac{1}{2} \mu l^2 i^2 \quad \text{\(W_l\) stored energy} \]

\[ \frac{i^2}{l^2} \propto B \quad \text{(USIA: \(\frac{W}{l^2}\))} \]
WEDNESDAY

(Sketch 9-2)

\[ \frac{2300}{115} \]

**No Load = Open Secondary (Test)** - To measure core losses.

- **I In No Load = Exciting Current**
- **Heat from Copper Losses & Hysteresis**

**Short Circuit (Test)**

- **Rated Current = \( \frac{V}{A} \) (Can deliver power all day)**
- **To measure copper loss**

Transformers rated in \( V \cdot A \) (not watts)
8.2) \(|Z| = \frac{60}{2.17} = 27.6 \)

P.F. = \(\frac{60}{27.6 \times 2.17} = 0.576 \) \(\Rightarrow \phi = \cos^{-1}(0.576) = 54.8^\circ \)

\(\Rightarrow Z = 27.6 < 54.8^\circ \)

\(R_{eq} = 27.6 \times 0.576 = 15.9 \Omega \)

\(X_{eq} = 27.6 \times 0.576 = 22.6 \Omega \)

b) \(R_{eq} = 2.17 \)

\(R_{eq} = 8.5 \Omega ; \) \(R_{eq} = 7.9 \Omega \) (to big)

\(R_L = 7.9 \div 400 = 0.0198 \Omega \)
COMPUTING CORE PARAMETERS

\[ R_{h+e} = \frac{V^2}{P_{h+e}} = \frac{(115)^2}{60} = 220 \Omega \]

60 WATTS FROM OTHER SIDE

\[ R_{h+e} = 0.20 \times (20)^2 \]

P.F. = \frac{60}{(115)(1)} = 0.522

\[ I_{h+e} = (1)(0.522) \]

\[ \frac{L_{\text{line}}}{L_2} = \frac{N_1^2}{N_2^2} \Rightarrow L_{\text{ref}} = L_2 \]

\[ I_0 = \text{MAGNETIZING} \]

\[ V = (1)(0.885) = 0.885 \text{ A} \]

\[ X_\phi = \frac{115}{0.885} = 135 \Omega \]

COMPUTE VOLTAGE REGULATION

P.F. = 0.85
THURS

ASSIGNMENT (FOR GIGGLES)

4.1) AUS E = 2000 J COE = 200

4.2) E = 38.4 COE = 57.6

4.3) E = COE = 250 J

4.10) EQ = 0.0123 M

ASSIGNMENT (FOR REAL)

4.11) 13.6 A

4.14) (a) \( W = \lambda f^3 + \frac{\lambda^2}{4} \left( \frac{s}{2} - 0.01 x \right)^2 \) Joules

4.15) (b) \( W' = 2 \lambda f^3 + \frac{\lambda^2}{2} \left( \frac{s}{2} - 0.01 x \right)^2 \) Joules

4.16) (c) \( f = -0.01 \lambda f^2 \left( \frac{s - 0.01 x}{2} \right) \) Newtons

4.17) (a) \( W = 10 x^2 + 29.425 x + 311.25 J \)

(b) \( 20 x + 294.25 \) Newtons

(c) 147.15 Newton Meters

(d) 289.5 Joules; 5.5 Joules

4.21) (a) 9000 Joules

(b) \( 1.125 \times 10^6 \) Newtons

(c) \( 4.5 \times 10^6 \) Newtons

(d) 9000 Joules

(23, 24, 25, 26, 27 due in 10th week)
\[ e(t) = \sqrt{2} \times 2300 \sin(2\pi \cdot 60 \cdot t) = 1150 \frac{d\phi}{dt} \]

\[ \phi(t) = \frac{\sqrt{2} \times 2300}{2\pi \cdot 60 \cdot 1150} \cos(2\pi \cdot 60 \cdot t) \]

\[ \phi_{\text{max}} = \frac{\sqrt{2} \times 2300}{2\pi \cdot 60 \cdot 1150} = 0.0075 \, \text{W} \]

\[ 0.075 \, \text{W} \times 10^8 \frac{\phi}{W} = 7.5 \times 10^5 \, \text{LINES} \]

\[ A_{\text{area}} = \frac{\phi}{10} = \frac{7.5 \times 10^5}{0.8 \times 10^5} = 9.375 \, \text{m}^2 \]

\[ = 0.00605 \, \text{m}^2 \]

\[ \text{MON} \]

\[ f = -\frac{\delta W_m}{\delta x} = \frac{\delta W_m}{\delta x} \]

\[ W_m' = \frac{1}{2} L i^2 \]

\[ W_m = \frac{\lambda^2}{2L} \]

TDLS

\[ \lambda \text{- FLUX LINKAGE} \]

\[ \text{FOR LINEAR SYSTEM} \]

\[ W_{m} = W_{m}' \text{ FOR LINEAR SYSTEM} \]

\[ \text{SLOPE: } W_m(0) - W_m(\lambda) \]

\[ \lambda \]

\[ W_m \]

\[ W_m' \]
\[ f = \frac{\delta \mathcal{W}_m}{\delta x} = \frac{\delta \mathcal{W}_m'}{\delta x} \]

\[ \mathcal{W}_m (A_x x) \quad \mathcal{W}_m' (A_x' x) \quad (A = L_i) \]

\[ x_1 \rightarrow \mathcal{W}_m \text{ increases} \]

\[ E = \frac{dA}{dt} \text{ (Faraday's Law)} \]

**THURS**

11.1 READ 11-9, 11-10, 11-11, 11-14

\[ f = \frac{\delta \mathcal{W}_m}{\delta x} = \frac{1}{2} \frac{\delta L}{\delta x} \Rightarrow \text{FORCE OF ELECTROMECHANICAL ENERGY} \]

**Figure 11-13**

\[ f = \frac{N^2 \omega_0 A}{(d-x)} \frac{d(L_i)}{dx} \]

\[ V(t) = i R + \frac{d}{dt} \]

\[ = i R + L \frac{di}{dt} + i \frac{d^2 x}{dt^2} \]

\[ (m_k) \quad \text{SPRING CONSTANT SPEED VELOCITY} \]

\[ f_0(t) = M \frac{d^2 x}{dt^2} + D \frac{dx}{dt} + K x = \frac{d^2 x}{dt^2} \]

\[ \frac{dL}{dx} \text{ common to 2, 3. Without } \frac{dL}{dx}, \text{ NO EQUATION COUPLING WANT TO FIND } x \text{ AND } i \]

(Highly non-linear differential)

For linearization: assume a.d.c. component and "small" a.c. component signal (cont)
\[ f_0(t) = f_0 + f_1(t) \quad \text{NEGLIGENCE PRODUCTS} \]
\[ V(t) = V_0 + V_1(t) \quad \text{OF SMALL SIGNALS} \]
\[ i(t) = i_0 + i_1(t) \quad \text{WITH EACH OTHER} \]
\[ x(t) = x_0 + x_1(t) \]

\[
\begin{align*}
\frac{d}{dt} i(t) &= \frac{d}{dt} i_1(t) \quad \text{ETC.} \\
\frac{d^2}{dt^2} x(t) &= \frac{d^2}{dt^2} x_1(t) \quad \text{ETC.}
\end{align*}
\]

Now \( x_1 < d - x_0 \)

\[ V_0 + V_1 = i_0 R + i_1 R \quad \text{L_0} \quad \frac{d}{dt} i_1 + \frac{i_0 L_0}{d - x_0} \frac{d}{dt} x_1 \]
\[ f_1 = M \frac{d^2 x_1}{dt^2} + \frac{Q}{d} \frac{d}{dt} x_1 + K x_1 + k x_0 - \frac{i_0^2}{2} \frac{L_0}{d - x_0} \frac{d}{dt} x_1 \\
- \frac{i_0^2}{2} \frac{L_0}{d - x_0} \frac{d}{dt} x_1 - \frac{L_0}{d - x_0} L_1 i_1 \]

SEPARATING INTO A.C. AND D.C.:

\[ V_0 = i_0 R \]
\[ 0 = k x_0 - \frac{1}{2} i_0 \frac{L_0}{d - x_0} + L_0 \quad \text{EVALUATED @ x_0} \]
\[ \Rightarrow L_0 = \frac{N^2}{d - x_0} \]

**11.1 DUE MON**

[Diagram of an electrical circuit with labeled parts: E.M. SYSTEM, R, W, E_0 + E_s(t).]

WRITE D.C. + A.C. EQUATIONS

SOLVE FOR QUIESCENCE

\[ E_0 + E_s(t) \]
\[ L_0 = L(1 + q x^2) \]
\[ a > 0; \quad q < 0 \quad |q| << 1 \]
SYSTEM ALWAYS GOES TO STATE OF MAXIMUM INDUCTANCE

ALL ELECTRICAL CIRCUIT IN E-M SYSTEM

\( v_1(t) = i_1 R + L_1 \ddot{x}_1 + L_0 \dot{x}_2 + \frac{d^2 x_1}{dt^2} \)

\( f_1(t) = M \dddot{x}_1 + D \ddot{x}_1 + (k - \frac{d L_0}{d x_0}) x_1 - \frac{d L_0}{d x_0} \)

\[ F_0 = k_0 i_1 = -k_0 \dot{i}_1 \]

\( v_0(t) = k_0 x_2 + \frac{d L_0}{d x_0} \)

\( f_0 = \frac{d^2 x_0}{dt^2} \)

\( \frac{R}{1000} \)
GYRATOR

\[ V_I = -\alpha \frac{d}{d\theta} \]

\[ V_B = -\alpha \frac{d}{d\theta} \]

\[ V_{II} = \frac{I_1 Z_{11}}{Z_B} + I_2 Z_{12} \]

\[ V_{IV} = \frac{I_1 Z_{21} + I_2 Z_{22}}{Z_B} \]

\[ V_A = \frac{-\alpha^2}{Z_B} = \frac{-\alpha^2}{Z_A} \]

\[ \frac{V_A}{V_B} = \frac{Z_B}{Z_A} \]

\[ F_0 \]

\[ \frac{R/\sqrt{L}}{L/\sqrt{L}} \]

\[ \frac{\text{consider even harmonics}}{f(\theta) = -f(\pi + \theta)} \]

\[ \text{To compute co-efficients:} \]

\[ \int_0^{2\pi} f(\theta) (\sin n\theta) \, d\theta \]

\[ \int_0^{2\pi} f(\theta) \cos 2n\theta \, d\theta \]

\[ \int_0^{2\pi} f(\theta) \cos 2n\theta \, d\theta \]

\[ \int_0^{\pi} f(\theta + \pi) \cos 2n\theta \, d\theta \]

\[ \int_0^{\pi} f(\theta) \cos 2n\theta \, d\theta \]

\[ \int_0^{\pi} f(\theta + \pi) \cos 2n\theta \, d\theta \]

\[ \Rightarrow \text{no co-efficients for odd harmonics} \]
**SALIENT POLES**

**SALIENCE ON ROTOR**

\[ L_n = L_k + L_v \cos 2\alpha_p \] (1st 2 Fourier Terms)

\[ L_{ip} = L_k + L_v \cos 2\alpha_p (29^\circ) \]

\[ L_{ip} = L_v \cos 2\alpha_p \]

\[ L_{ip'} = L_v \sin 2\alpha_p \] (Chart 12-12-9)

\[ L_{np} = L_v \sin 2\alpha_p \]

\[ L_{np'} = L_v \sin 2\alpha_p \]
\[
L_{q} = L_{k} + M_{k} + \frac{3}{2} L_{v} = 0.045 \ \text{h}
\]
\[
L_{d} = L_{k} + M_{k} - \frac{3}{2} L_{v} = 0.0275 \ \text{h}
\]

\[
E_g = E_{\text{no load}} = P_{\text{nm}} M_{\text{afm}} T_{r} = (377)(0.2)(200) = 15080 \ \text{V}
\]

\[
\phi = \text{terminal voltage per phase} = (L_{q} \times 5.48 ) \cdot \frac{E_g}{R_{\text{adj}} + j (R_{\text{adj}} - L_{d} L_{q})}
\]

\[
E_q = \text{quad. axis} \cdot I_q \text{ cos } \theta
\]
\[
I_d = \text{direct axis} \cdot I_m \text{ cos } \theta
\]

CONT
X₄ = DIRECT AXIS REACTANCE = \omega L_d, \ X₉ = \omega L_a

V₉ = 15.080 = (600 \angle 20°)(0,1) \cdot (600 \angle -370°) \left( 1 + j \frac{(600 \angle 37°)(0,1) = (10,4)(600 \angle -37°)}{10,500} e^{-j \cdot 280°} \right)

\text{LINE} \ V = \sqrt{3} \cdot V₉

\begin{center}
\text{EQ CIRCUIT FOR CONSTANT GAP MACHINE}
\end{center}

\[ E₉ \quad (\text{angle} = 6°) \]
\[ 10,500 \quad V₉ \]

\[ \text{EQ} \]

\[ \frac{E₉}{\omega L_d} \quad \text{MW} \quad Iₐ \quad \text{A} \]

\[ V₉ \quad \text{V} \]

\[ \frac{|E₉| - |V₉|}{V₉} \times 100\% \]

\[ \text{d) } \% \text{ REGULATION} = \frac{|E₉| - |V₉|}{V₉} \times 100\% \]

\[ 10\% \ 

\[ 10\% = \frac{45°}{10,500} \times 100\% = 43.5\% \]

\[ \begin{align*}
\text{c) } P_{\text{MECH (in)}} &= \frac{3}{2} \left( \frac{E₉ \cdot V₉ \text{ MW}}{\omega L_d} \right) + (X₄ - X₉) V₉ \text{ MW} / 2X₄ \times X₉ \\
\text{constant gap} & \quad \text{reluctance torque} \\
\text{SECOND TERM} & \quad \text{FOR CONSTANT GAP MACHINE} \\
\text{3.0 } & \quad 9.3 \text{ MW} \\
\end{align*} \]

\[ \text{f) } P_{\text{CONVERTED ELECTRIC}} = P_{\text{MECH}} = 9.3 \text{ MW} \]
6) \( P_{\text{total}}(E) = I^2 R a / 2 \)

\[
P_{\text{total}} = 3 N P(E) = \frac{3}{2}
\]

\[
P_{\text{E0}} = (9300) \text{KW} - \left(\frac{3}{2}\right)(600)^2 (1) \rho = \frac{3}{2} \frac{I_a^2}{R a} \]

\[
\eta = 0.924 C
\]

\[
V_a = \frac{3}{2} (V_a T_a) = 9450 > PF = 0.98
\]

\[
\gamma = \frac{P_{\text{MECH}}}{\omega_m} \quad \rho = 2
\]

\[
\omega_m = \frac{377}{2} \frac{2}{3} \times 10^6
\]

\[
\Rightarrow \gamma = 9.3 \times 10^6 \quad \text{Torq} = 50,000 \text{ ft-lb}
\]

\[
\text{TORS}
\]

LET \( \rho = 4 \) (POLE PAIRS) (CHAP. 5)

MON

B. MACHINES

FROM WINDING
A short motor with the following constants is connected to an A.C. voltage: 340 A in 377V. Find the average torque. Repeat, assuming D.C. excitation @ 240 V (RMS of 340 V).

\[ R_p = 50, L_{ef} = 25, K_p = 0.9, R_a = 0.1, L_{ff} = 0.01, J_m = 10 \text{ kg-m}^2 \]

\[ T = K_p i_a(t) i_f(t) \]
MAXWELL STRESS TENSOR

\[
\bar{\sigma}_m = \begin{bmatrix}
\frac{B_x^2 - B_y^2 - B_z^2}{2 \mu_0} & \frac{B_x B_y}{\mu_0} & \frac{B_x B_z}{\mu_0} \\
\frac{B_x B_y}{\mu_0} & \frac{B_y^2 - B_z^2 - B_x^2}{2 \mu_0} & \frac{B_y B_z}{\mu_0} \\
\frac{B_x B_z}{\mu_0} & \frac{B_y B_z}{\mu_0} & \frac{B_z^2 - B_x^2 - B_y^2}{2 \mu_0}
\end{bmatrix}
\]

SIGNIFICANCE: \( \bar{\sigma}_m \cdot \bar{a}_n = \bar{p} = \text{FORCE AREA ACTING ON SURFACE WHOSE NORMAL IS } \bar{a}_n \)

EXAMPLE:

\( \bar{H} = N' \bar{I} \bar{a}_z \)

\( \bar{\sigma}_{\text{inside}} = \begin{bmatrix}
0 & -\frac{B_z^2}{2 \mu_0} & 0 \\
-\frac{B_z^2}{2 \mu_0} & 0 & 0 \\
0 & 0 & \frac{B_z^2}{2 \mu_0}
\end{bmatrix} \)

\( \bar{\sigma}_{\text{outside}} = 0 \)

\( \bar{p} = (\bar{\sigma}_{\text{out}} - \bar{\sigma}_{\text{in}}) \cdot \bar{a}_n \) \( \Theta \) (cont.)
\[ \tilde{\tau} = B_{e}^{2}/2\mu_{0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \phi \\ \sin \phi \\ 0 \end{bmatrix} \]

\[ = \frac{B_{e}^{2}}{2\mu_{0}} \begin{bmatrix} \cos \phi \\ \sin \phi \\ 0 \end{bmatrix} = \frac{B_{e}^{2}}{2\mu_{0}} \tilde{a}_{r} \]

\[ = \frac{B_{e}^{2} \mu_{0}}{2} (N'I)^{2} \tilde{a}_{r} \]

**Linear Induction Machine**

**Stator**

\[ \mu = \infty \]

\[ \Delta \]

\[ a_{x} \]

\[ a_{y} \]

\[ q_{y} \]

\[ y = 0 \]

\[ y = \bar{y} \]

\[ \Delta \]

\[ F_{e} \]

\[ \mu = \infty \]

\[ \bar{\tau} = \tilde{\sigma}_{m} \cdot \bar{a}_{y} \]

\[ = \begin{bmatrix} B_{x}^{2} - B_{y}^{2}/2\mu_{0} & \frac{B_{x}B_{y}}{\mu_{0}} & 0 \\ \frac{B_{x}B_{y}}{\mu_{0}} & B_{y}^{2} - B_{x}^{2}/2\mu_{0} & -B_{x}^{2} + B_{y}^{2}/2\mu_{0} \\ 0 & -B_{x}^{2} + B_{y}^{2}/2\mu_{0} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]

\[ = \begin{bmatrix} B_{x}B_{y}/\mu_{0} \\ B_{y}^{2}/2\mu_{0} \end{bmatrix} \]

\[ \Rightarrow \tilde{\tau} = \frac{B_{x}B_{y}}{\mu_{0}} \tilde{a}_{x} + \frac{B_{y}^{2} - B_{x}^{2}}{2\mu_{0}} \tilde{a}_{y} \]

For \( y = 0 \)

For \( y = \bar{y} \), let \( y = -y \) (change sign on \( \tilde{a}_{n} \))
\[ \langle p \rangle = \alpha_x R e \left[ \frac{B_y B^*_x}{\mu_0} \right] + \alpha_y R e \left[ \frac{B_y B^*_x-B_x B^*_y}{2 \mu_0} \right] \]

(MUG AND CHUG AND VWALA!)

\[ \Rightarrow \langle p \rangle = \frac{\mu_0 (N' I_m)^2}{q} \left[ \frac{2 s R_m \alpha_x + [1-(5 R_m)^2] \alpha_y}{\sinh^2 k g + (5 R_m \cosh k g)^2} \right] \]

\[ s = \frac{V_{syn} - V_{in}}{V_{syn}} \quad \text{(SLIP)} \]

INTEGRATING OVER I FOR X, Y Z TO FIND I =

\[ \Rightarrow \text{MULTIPLY} \langle p \rangle \text{ BY} \lambda \lambda \delta \]

\[ M\text{O\textsc{N\textsc{D\textsc{A\textsc{Y}}} (15 - 2, 15 - 3) \]

\text{
\begin{tabular}{c}
\text{LAST LAB} \\
\text{DIFFERENTIAL \Rightarrow OPPOSING FLUXES,}
\end{tabular}
}

\text{TAKE HOME}

\begin{tabular}{c}
\text{\begin{tabular}{c}
\text{(PROB. I)}
\end{tabular}}
\end{tabular}

\text{SAME AS BEFORE}

\text{\begin{tabular}{c}
\text{SOLUTION} \: \hat{\alpha}(x, t), \: \omega(x, t)
\end{tabular}}

\text{USE} \: Q \: \text{TO SOLVE} \Rightarrow \hat{\alpha}(x, Q), \: \omega(x, Q)
Linear Induction

18 kHz  \Rightarrow \lambda = \frac{3 \times 10^8}{18,000} \approx 10 \text{ miles} \approx 15 \text{ km}

\text{(For science)}
\[
\left( \frac{F_x}{\lambda_0 N' V_{SYN}} \right)^2 = \frac{S R'_m}{(\coth k g)^2 + (S R'_m \sinh k g)^2}
\]

\[p_g = 1 - 226\]

**Per unit Sync Speed**

\[
S = \frac{V_{SYN} - V_M}{V_{SYN}} = 1 - \text{per. unit. syn. speed}
\]

\[
R'_m = \mu_0 \Delta V_{SYN} \left( \frac{\text{hysteresis characteristics}}{\text{normalized thickness}} \right)
\]
(\frac{F_x}{\lambda_0 \mu_0 (N'V_{\text{syn}})^2}) = \frac{S R_m'}{(\cosh k_{\theta})^2 + (S R_m' \sinh k_{\theta})^2}

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PER UNIT SYNC-SPEED = \frac{V_m}{V_{\text{sync}}}

S = \frac{V_{\text{syn}} - V_m}{V_{\text{syn}}} = 1 - \text{PER. UNIT. SYN. SPEED}

R_m' = \mu_0 \Delta V_{\text{syn}} (\text{MAGNETIC REYNOLDS \#)}

\text{(NORMALIZED THICKNESS)}

\text{ROTOR THICKNESS}
\[
W = 2.78 \times 4.45 \, \text{nw} = 8.90 \, \text{nw} \\
F_8 = W \frac{a}{b} = 8.90 \left(\frac{4}{8}\right) = 4.45 \, \text{nw} \\
F_8 = -\frac{2A^2}{N^2 \mu_0 A} \\
= -\frac{2A^2}{N^2 \mu_0 A} \\
L = \frac{N^2 \mu_0 A}{2} \\
\Rightarrow F_{\theta} = -\frac{N^2 \mu_0 A}{2} \cdot \frac{A^2}{N^2 \mu_0 A} \\
\Rightarrow L^2 = \frac{F_{\theta}}{\mu_0 A} \left(\frac{L}{N}\right)^2 \\
\Rightarrow \left|L\right| = \sqrt{\frac{F_{\theta}}{\mu_0 A}} \cdot \frac{L}{N} \\
= \sqrt{\frac{4.45 \, \text{nw} \cdot 4 \times 10^{-9} \, \text{m}^2}{100 \cdot 2 \times 10^{-4} \, \text{m}^2}} \cdot \frac{2.5 \times 10^{-2} \, \text{m}}{100} = 3.3246 \, \text{A} 
\]
4.14) \[ i = 3 \lambda^2 + \lambda (5 - x)^2 \]

a) \[ W_m = \int_0^\lambda i (\lambda, x) \, d\lambda \]
\[ = \int_0^\lambda (3 \lambda^2 + \lambda (5 - x)^2) \, d\lambda \]
\[ = \lambda^3 + \frac{\lambda^2}{2} (5 - x)^2 \quad \text{erg} \quad (CGS) \]

b) \[ W_m' = i_{\text{max}} \lambda_{\text{max}} = W_m \]
\[ = (3 \lambda^2 + \lambda (5 - x)^2) \lambda - \left( \lambda^3 + \frac{\lambda^2}{2} (5 - x)^2 \right) \]
\[ = (3 \lambda^2 - \lambda^3) + \left( \lambda^2 (5 - x)^2 - \frac{\lambda^2}{2} (5 - x)^2 \right) \]
\[ = 2 \lambda^3 + \frac{\lambda^2}{2} (5 - x)^2 \quad \text{erg} \]

c) \[ f = \frac{\delta W_m}{\delta x} \]
\[ = \frac{\delta}{\delta x} \left( \lambda^3 + \frac{\lambda^2}{2} (5 - x)^2 \right) \]
\[ = \lambda^3 + \frac{\lambda^2}{2} (5 - x) \]
\[ = \lambda^2 (5 - x) \quad \text{dyne} \]
4.17) \( L_1 = 5 + 2.5x \); \( L_2 = 1 + 2(1-x)^2 \); \( L_{12} = 3 + (1 + x) \)
\( I_1 = 15 \text{ A} \); \( I_2 = 10 \text{ A} \)

a) \( W_m = \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} \)
\[ = \frac{(5 + 2.5x)(1.5)^2}{2} + (1 + 2(1-x)^2)(10)^2 \]
\[ = 10x^2 + 2942.5x + 311.5 \]

b) \( f = \frac{\delta W_m}{\delta x} \)
\[ = 20x \]
\[ = 2942.5 \text{ nt} \]

(checked mark)

c) \( \omega = \int_{x_1}^{x_2} f(x) \, dx \)
\[ = \int_{0.05}^{0.65} (20x + 2942.5) \, dx \]
\[ = \left[ 10x^2 + 2942.5x \right]_{0.05}^{0.65} \]
\[ = 147.15 \text{ J} \]

(checked mark)
4.21) \[ g = 8 \text{ cm} \]
\[ L = 1.8 \text{ m} \]
\[ i = 100 \text{ A} \]

a) \[ W_m = \frac{L^2 i^2}{2} = \frac{1.8 \times 10^4}{2} = 9000 \text{ J} \]

b) \[ f = \frac{\lambda^2}{2N^2 \mu_0 \mu} = \frac{\lambda^2}{2X^2} \]
\[ \Rightarrow f = \frac{1}{2X^2} = \frac{1}{2N^2 \mu_0 \mu} = \frac{\mu_0 N^2 A i^2}{2X^2} \]
\[ = \frac{1.8 \times 10^4}{2X^2} = \frac{1.6 \times 10^{-2}}{2X^2} \]
\[ = -1.25 \times 10^6 \text{ Hz} \]

\[ c) L \propto \frac{1}{X} \Rightarrow LX = \text{constant} \]
\[ L_1 x_1 = L_2 x_2 \]
\[ (1,8) \times (1,8) = L_2 \times (3,6) \Rightarrow L_2 = 3,6 \text{ m} \]
\[ f_2 = \frac{L_2 i^2}{2X} = \frac{(3,6) \times 10^4}{2 \times (4) \times 10^{-2}} = 4,5 \times 10^6 \text{ Hz} \]

d) \[ W_b = \int x_2 L_2 i^2 \frac{1}{X} dX \]
\[ = \frac{1}{2X^2} \int L_2 x_2^2 i^2 \mu_0 N^2 A/ \frac{X}{2} \]
\[ = \frac{1}{2X^2} \left[ \frac{X}{2} x_i i^2 \mu_0 N^2 A \right]_X = \frac{1}{2X^2} \left[ \frac{X}{2} \times 10^{-2} \right] \]
\[ = \frac{1}{2X^2} \left[ \frac{10^4 L}{X} = A \times 10^{-2} \right] \]
\[ = 9000 \text{ J} \]
EE 353, ENERGY CONVERSION

LAB PROJECT NO. 2

1. Sketch the circuit used and describe the purpose of each instrument.

2. In the "watt-meter" method describe the procedures used.

3. Explain how the watt-meter is able to measure Fourier components. Be clear, not too verbose, and accurate. Use enough analysis to explain the theory.

4. Contrast the digital method, and its results, with the watt-meter method. Which is easier for use? Which is more accurate?

5. What are your results? Are they consistent with the analysis of Chapter 8?
WATTMETER USED TO COMPUTE FOURIER COMPONENTS BY "WATTMETER" METHOD
a) CRO USED TO DISPLAY "CURRENT" WAVEFORM FOR DIGITAL ANALYSIS
b) AMP USED TO AMPLIFY VOLTAGE AMPLITUDE FROM OSC.

c) AMMETER USED TO KEEP I CON. CONSTANT, THEREFORE KEEPING RELATIVE FREQUENCY COMPONENTS OF I CON. CONSTANT
d) OSCILLATOR USED TO VARY FREQUENCY OF VOLTAGE COMPONENT IN THE WATTMETER TO VALUES MATCHING THE HARMONICS OF THE CURRENT e) VOLT METER USED TO DETERMINE THE VALUES OF THE "CURRENT" HARMONIC COMPONENTS

(NOTE: 12.0 VOLTS USED WHEN PHOTOGRAPH OF WAVEFORM WAS MADE FOR DIGITAL ANALYSIS)

2) THE PURPOSE OF THE WATTMETER METHOD IS TO DETERMINE THE FREQUENCY COMPONENTS OF THE CURRENT WAVEFORM, WHICH WAS DISTORTED FROM A SINUSOIDAL BY THE NON-LINEAR CHARACTERISTICS OF THE TRANSFORMER, AS PICTORALLY DESCRIBED FROM A HYSTERESIS CURVE ON PAGE 8-56 OF THE TEXT. THE SINUSOIDAL VOLTAGE \( V(t) = V_0 \cos(\omega t + \phi) \), AND THE PERIODIC CURRENT \( i(t) \) EXPRESSED IN A FOURIER SERIES \( i(t) = \sum C_n \cos(\omega n t + \phi_n) \) ARE FED INTO THE VOLTAGE AND CURRENT COILS OF THE WATTMETER, THESE EXPRESSIONS ALONG WITH THE DIFFERENTIAL EQUATION FOR TORQUE OF THE WATTMETER NEEDLE: \( \gamma(t) = AV(t) \cdot i(t) = j \phi + d\phi + k_2 \phi \), YIELD FOR A SINGLE COMPONENT, DISREGARDING TRANSIENTS:

\[
\theta(t) = \frac{AV}{20} \left[ \frac{\cos(\omega_0 t + \psi_1) + \psi_2}{(\psi_0)^2 + \omega_0^2} + \frac{\cos(\omega_1 t + \psi_1) + \psi_2}{(\psi_0)^2 + \omega_1^2} \right]
\]

\( \psi_1 \) and \( \psi_2 \) are phase angles, determined by parameters \( \psi_0 \) and \( \omega_0 \).

(THE SOLUTION FROM "SKETCH OF THEORY WATTMETER" HANDOUT)

NOW \( \theta(t) \) IS OF COURSE PROPORTIONAL TO WATTAGE, ANYWAY, IT CAN BE SEEN THAT ONLY WHEN \( \omega_0 = \omega_1 = \omega_2 \) WILL THE WATTAGE HAVE AN NON-ZERO AVERAGE POWER IN TIME. THUS, BY ADJUSTING THE VOLTAGE FREQUENCY TO THE VARIOUS HARMONICS OF THE CURRENT WAVEFORM (TO WHERE THE "BEAT FREQUENCY" \( \omega_0 = 0 \)), THE CONTRIBUTION OF THAT FREQUENCY, i.e. THE FOURIER COEFFICIENT, MAY BE COMPUTED. THE WATTMETER, ACTING AS A LOW PASS FILTER, DUE TO NEEDLE INERTIA AND FRICTION, WILL OF COURSE NOT RESPOND TO HIGH FREQUENCIES,
2) Oscillator set to various harmonics of 60 Hz line voltage to setting on which a reading occurred on wattmeter. Values of frequency, current, voltage and power were then recorded from the corresponding meters depicted in figure in part (1).

DATA:

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<tr>
<th>n</th>
<th>f of V (Hz)</th>
<th>V_{RMS} (V)</th>
<th>P_{WATTS}</th>
<th>I_{RMS} (AMPS)</th>
<th>I_{n(pp)} = (P/V_{RMS})^{1/2}</th>
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\( \text{NF} = 43 \)
\( \text{NW} = 10 \)

\( \text{DC} = -0.000 \)
\( \text{F0} = 59.880 \)

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EXECUTION TIME 0283
DIGITAL ANALYSIS
4) The wattmeter method, although easier to use, does not yield as good results as the digital method. The reconstruction of the curve, as represented by the computer plot, by the Fourier components computed digitally, is nearly identical to that of the original curve, implying good representation of the digital components in describing the harmonics of the current waveform. The frequency spectrum plot of the two methods shows that the wattmeter results are not as favorable.

5) The current waveform contains only odd harmonics, which can be seen from its symmetry: \( i(t) = -i(t + \pi) \). This same waveform is shown, derived from the hysteresis curve on page 8-56 of the text.

Lab partners:
Randy Miller
Bob Shaw
TO
DR. SABBAGH
FOURIER SERIES

ORIGINAL POINTS

ID: 8538
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EXECUTION TIME 0.283

ENERGY CONVERSION
LAB PROJ. 2
FOURIER ANALYSIS

NF = 43           TDEL = 0.400 MS
NW = 10           START = 0.000 MS
DC = -0.000
FO = 59.880

PROGRAM WRITTEN
BY RANDY MILLER
\[
\begin{align*}
R &= \frac{L}{ \mu A} \\
R_A &= \frac{(15'')(4\pi \times 10^{-7})w}{(45')(2.54 \times 10^{-3} m)} = 1.175 \times 10^5 A/W \\
R_F &= \frac{(4 \times 10^2)(4\pi \times 10^{-7})w}{(45')(2.54 \times 10^{-3} m)} = 3.133 \times 10^5 A/W \\
R_{eq} &= R_A + R_F = 1.488 \times 10^6 A/W \\
\sigma_{eq} &= \rho R_{eq} = (5 \times 10^{-4})(1.488 \times 10^6) = 744.08 A-T \\
\sigma_A &= \rho R_A \\
\sigma_F &= \rho R_F \\
\sigma_{eq} &= \frac{(5 \times 10^{-4})(1.488 \times 10^6)}{\sigma_{eq}} = 587.43 A-T \\
\sigma_{eq} &= 1.7895 (\approx 78.95\%) 
\end{align*}
\]
\[ L = \frac{\mu_0 N^2 A^2}{\mu} \]

\[ A = 1.2 \times 10^{-2} \text{ m} \times \frac{1}{2} = 3 \times 10^{-3} \text{ m}^2 \]

\[ L = \frac{\mu_0 N^2 A^2}{\mu} \]

\[ L = \frac{(1)^2 (9.14 \times 10^{-2}) (3 \times 10^{-3})}{3.76 \times 9.9 \mu \text{ H/m}} = 3.74 \text{ nH/m} \]

\[ L = \mu_0 l_n \left( \frac{15}{2} \right) = 1.9 \times 10^{-7} \mu_0 \text{ H/m} \]

\[ \sqrt{2.5} = 1.9 \times 10^{-7} \mu_0 \text{ H/m} \]

\[ \times \]
(3-6) \[ E_4 = 200 \text{ A} \cdot \text{m} \]

\[ B = 1 \text{ w/ m}^2 \]

\[ A) \quad B = \frac{\mu_0 NI}{2\pi R} + \frac{\mu_0 I_A}{2\pi R} \Rightarrow I_A = \left( B - \frac{\mu_0 NI}{2\pi R} \right) \frac{2\pi R}{\mu_0} = \frac{2\pi RB}{\mu_0} = 10^5 \cdot 200 = 10^7 \text{ A} \]

\[ B) \quad \sigma = \frac{I_A}{2\pi R} = \frac{10^5}{2\pi (2.0)} = 8.05 \times 10^4 \text{ A/m} \]

\[ C) \quad \chi = \sigma NI \Rightarrow \chi = \frac{(8.05 \times 10^4) \times 2\pi (2.0)}{200} = 5.05 \times 10^2 \]

\[ D) \quad \mu = \mu_0 \left( \chi + 1 \right) = 4\pi \times 10^{-7} (5.06) = 6.36 \times 10^{-4} \frac{H}{\text{m}} \]
L₁ = 100 CMS SHEET STEEL  
L₂ = 50 CMS CAST STEEL  
L₃ = 0.2 CMS AIR GAP  
A₁ = 5 sq cm  
A₂ = A₃ = 4 sq cm.  
ρ = 60 kLINES

a) B₃ = B₂ = \( \frac{\Phi}{A} \)  
   = \( \frac{6 \times 10^{-4}}{4 \times 10^{-4}} \) = 1.5 W/m²  

H₅ = \( \frac{B}{\mu_0} \) = \( \frac{1.5}{4 \pi \times 10^{-7}} \) = 1.19 \times 10⁶ A/m  

δₗ₃ = H₅ L₃ = (2 \times 10^{-3}) \times 2.38 \times 10^3 = 5.56 \times 10^{-3} A-T  

H₃ = 4 \times 10^{-3} A/m  

δₗ₃ = H₃ L₂ = 2 \times 10^3 A-T  

δₗ₃ = 4.38 \times 10^3 AT

b) \( Φ_{SS} \) = 4.38 \times 10^3 A-T  

H₅ = 4.38 \times 10^3 A/m  

B₅ = 1.55 W/m²  

P₅ = \( (1.55) (5 \times 10^{-4}) \) = 7.75 \times 10^{-4} W  

c) P₅ = 4.125 \times 10^{-4} W

d) N \Phi = 300 \times (13.75 \times 10^{-4}) = 4.125 \times 10^{-4} W

e) L = \frac{N \Phi}{I} = \frac{4.125}{141.6} = 28.3 mH
8-4) WITHOUT TRANSFORMER.

(a) \[ P_L = \frac{V_L^2}{R_L} = \left( \frac{R_L}{R_L + R_S} E_g \right)^2 \frac{1}{R_L} \]

\[ = R_L \left( \frac{E_g}{R_L + R_S} \right)^2 \]

\[ = (5 \, \Omega) \left( \frac{100 \, \text{V}}{2.005 \, \Omega} \right)^2 \]

\[ = 2.494 \, \text{Watts} \]
WITH TRANSFORMER

\[ E_g = [R_g + R_1 + s(L_1 + aL_{12})]i_1 - sL_{12}i_2 \]
\[ aV_L = [a^2R_2 + s(L_{12} + aL_2)]i_2 - sL_{12}i_1 \]
\[ v_L = \frac{aR_2 + s(L_{12} + aL_2)}{a^2R_L}i_2 + sL_{12}i_1 = A_3i_2 + A_4i_1 \]
\[ aV_L = a^2R_L i_2 \Rightarrow V_L = aR_L i_2 = A_5i_2 \]

\[ \begin{aligned}
    i_1 &= \frac{1}{A_1} E_g = \frac{A_2}{A_1} i_2 \\
    V_L &= A_3i_2 + A_4i_1 \\
    i_2 &= \frac{1}{A_5} V_L
\end{aligned} \]

\[ \mathcal{Q} = 1 - \left[ \frac{A_4A_3}{A_1A_5} \right] = 1 - \frac{A_4A_3}{A_1A_5} \]
\[ T_1 = \frac{A_4}{A_1} \quad \Delta_1 = 1 \]
\[ V_L = \frac{A_4}{A_1} \left[ 1 + \frac{A_4A_3}{A_1A_5} \right] \]

\[ \text{Power at Load} = \left( \frac{V_L}{E_g} \right)^2 \frac{E_g^2}{R_L} \]

\[ \text{NOTE: COMPUTER SOLUTION MADE WITH SHUNT INDUCTANCE L}_{12} \]
\[ \text{(OPPOSED TO QL}_{12}) \text{ AS ILLUSTRATED IN FIGURE 8-14 OF TEXT) \]

\[ \text{NOTE: COMPUTER SOLUTION MADE WITH SHUNT INDUCTIVE REACTANCE OF SL}_{12} \]
\[ \text{(AS OPPOSED TO QSL}_{12}) \]
\[ \text{AS DEPICTED IN FIGURE 8-14 OF TEXT) \]

\[ \text{MAXIMUM PWR, DELIVERED TO TRANSFORMER \at 63.1 HZ = 1 WATT, AS OPPOSED TO PREVIOUS PWR. OF .25 WATTS, USING SHUNT REACTANCE OF SL}_{12}, \]
\[ \text{SIMILAR RESULTS OBTAINED USING QSL}_{12} \text{ AS SHUNT REACTANCE,} \]
\[ \text{BUT AT FREQ. OF 3.3 HZ) \]
PROGRAM EE
DIMENSION I

000002  COMPLEX S,

000002  WRITE(5,19

000006  READ(2,20)

000014  READ(2,20)

000022  READ(2,20)

000030  READ(2,20)

000036  READ(2,20)

000044  READ(2,20)

000052  READ(2,20)

000060  READ(2,20)

000066  FM=0.0

000067  PMAX=0.0

000067  R=10.**(1.0+

000075  XL1=XL1SC:/

000077  XL2=XL1

000077  XL12=XL1DC.

000102  TPI=8.*ATA;

000104  ND=(25.*D)·

000110  DO 90 J=1,1

000114  S=CMPLX(0.*

000117  A1=R$+R1+S:

000130  A2=-S*A*X:

000134  A3=-(*R2):

000150  A4=S*XL12

000154  A5=CMPLX(A:

000157  AV=A4/(A1*·

000177  G1=CMPLX(J

000222  PWR(J)=EG*

000226  IF(PWR(J)-1

000231  PMAX=PWR(J)

000233  FM=F

000235  FR(J)=ALOG:

000241  WRITE(5,21)

000253  F=F*R

000255  90 CONTINUE

000257  WRITE(5,22)

000267  WRITE(5,23)

000273  CALL SETPL1

000302  WRITE(5,23)

000306  CALL SETPL1

000315  19 FORMAT('7R

000315  *7X,'FREQUENCY

000315  ')

000315  20 FORMAT(F15.

000315  21 FORMAT(5X,1

000315  22 FORMAT(7X

000315  23 FORMAT(7X

000315  STOP

000317  END

000317  END
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MAXIMUM POWER= 9.99936145E+12
c) \( R_p = \frac{(R_5 + R_1) a^2 (R_3 + R_4)}{R_5 + R_1 + a^2 (R_2 + R_4)} \)

\[ = \frac{(2.100)(20)^2(5.25)}{(2.100) + (20)^2(5.25)} \]

\[ = 1050 \Omega \]

\( L_E = L_1 + a^2 L_2 \)

\[ = (0.025)(1 + (20)^2) \]

\[ = 10.025 \text{ H} \]

\( R_E = \frac{R_5 + R_1 + a^2 (R_2 + R_4)}{2} \)

\[ = (2.100) + (20)^2(5.25) \]

\[ = 4200 \Omega \]

\( \omega_L = \frac{R_p}{a L_1} \)

\[ = \frac{(1050)}{(20)(5)} \]

\[ = 10.5 \text{ rad/sec} \]

\( \omega_H = \frac{R_S}{L_{eq}} \)

\[ = \frac{(4200)}{(10.025)} \]

\[ = 418.953 \text{ rad/sec} \]

**Band Pass Frequencies**

\( \omega_L < \omega < \omega_H \)

\[ 10.5 \text{ rad/sec} < \omega < 418.953 \text{ rad/sec} \]

\[ \Rightarrow 1.67113 \text{ kHz} < f < 66.6784 \text{ Hz} \]
1. The current sheet of Fig 1-5 (any notes, p.18) is replaced by an axially symmetric distributed current sheet \( J_s(r) = -J_{s0} \frac{r}{R_s} \).

Show that the azimuthal current density is

\[
\bar{J}_\phi(r, \theta) = \bar{J}_{s0} \frac{R_s}{r} \left[ \left( \frac{k_s^2}{r^2} + \frac{1}{r^4} \right) \cos \theta \right. \\
+ \left. \frac{k_s}{r^4} \frac{R_s^2}{k_s^2} \right] \left( \frac{k_s^2}{r^2} - \frac{k_s^2}{R_s^2} \right) \sin \theta.
\]

2. Derive an expression for \( L_{s0} \), using the results of 1), and sketch.

3. Derive an expression for torque and pitch.

\[\text{[Equations and derivations]}\]
1. Since we are concerned only with the negative axis, let $y_0 = 0$.

Then, $(1)$ becomes

$$\frac{y_n}{y_0} (m + n - 1) = 0 \quad \text{for all} \ n.$$  

and $(2)$ becomes

$$\frac{y_{n+1}}{y_n} (m + n) = 0 \quad \text{for all} \ n.$$  

Equation $(1)$ is:

$$\frac{y_n}{y_0} (m + n - 1) = 0.$$  

Multiplying the equation by $e^o$, and integrating with respect to $y$, from $-\pi$ to $\pi$, we get (remember your integrals of sine and cosine products)

$$\Pi_n \left( \int_{-\pi}^{\pi} (e^o \sin^2 \theta) \cos \theta \, d\theta \right) = 0 \quad \text{for all} \ n.$$  

Equations $(1)$ and $(3)$ imply that $\frac{y_{n-1}}{y_n} = 0$ for all $n$.

Next, multiplying $(1)$ by $\sin \theta$ and integrating with respect to $\theta$, from $-\pi$ to $\pi$:

$$\Pi_n \left( \int_{-\pi}^{\pi} (e^o \sin \theta \cos^2 \theta - \sin \theta \cos \theta \cos^2 \theta) \, d\theta \right) = 0 \quad \text{for all} \ n.$$  

Thus:

$$\Pi_n (e^o \sin \theta - \sin \theta \cos^2 \theta) = 0 \quad \text{for all} \ n.$$  

Equations $(1)$ and $(2)$ imply that $b_n = d_n = 0$, all $n \neq 0$.  

$(1)$ $(3)$ and $(2)$ imply that $b_n = d_n = 0$, all $n \neq 0$.  

Additional equations:

$$b_n = 0 \quad \text{all} \ n.$$  

$$d_n = 0 \quad \text{all} \ n.$$
\[ a_1 = \frac{\frac{1}{\left(\frac{1}{R_c^2} - \frac{1}{R_o^2}\right)} - \frac{1}{R_c^2}}{\left(\frac{1}{R_c^2} - \frac{1}{R_o^2}\right)} = \frac{\frac{1}{R_c^2}}{\left(\frac{1}{R_c^2} - \frac{1}{R_o^2}\right)} \]

\[ a_2 \left(R, \theta\right) = \frac{M_0 J_{5a}}{\left(\frac{1}{R_c^2} - \frac{1}{R_o^2}\right)} \left[ \frac{k}{R_c^2} + \frac{1}{R_c^2} \right] \sin \theta \]

\[ \bar{B} = \frac{1}{R_c^2} \frac{\partial A_2}{\partial R} - a_2 \frac{\partial A_2}{\partial \theta} = \frac{\partial}{\partial R} \left( \frac{M_0 J_{5a}}{\left(\frac{1}{R_c^2} - \frac{1}{R_o^2}\right)} \left[ \frac{1}{R_c^2} + \frac{1}{R_c^2} \right] \cos \theta \right) + a_2 \frac{M_0 J_{5a}}{\left(\frac{1}{R_c^2} - \frac{1}{R_o^2}\right)} \left[ \frac{1}{R_c^2} - \frac{1}{R_o^2} \right] \sin \theta \]

\[ \bar{B} = \frac{\frac{M_0 R_a J_{5a}}{R_c^2}}{\frac{1}{R_c^2} + \frac{1}{R_c^2}} \left[ \frac{\frac{1}{R_c^2} + \frac{1}{R_c^2}}{\frac{R_a}{R_c^2} - \frac{1}{R_a}} \right] \cos \theta \]

\[ + \frac{\frac{M_0 R_a J_{5a}}{R_c^2}}{\frac{1}{R_c^2} + \frac{1}{R_c^2}} \left[ \frac{\frac{1}{R_a} - \frac{1}{R_c^2}}{\frac{R_a}{R_c^2} - \frac{1}{R_a}} \right] \sin \theta \]

\[ \Phi_{fa} = N_f \int_{\theta - \kappa}^{\theta + \kappa} B_r \left(R, \theta\right) R_d d\theta = 2 N_f \frac{M_0 R_a J_{5a}}{R_c^2} \left[ \frac{1}{\frac{R_a}{R_c^2} - \frac{1}{R_a}} \right] \cos \theta \]

\[ = \frac{4 N_f M_0 R_a J_{5a}}{\left(\frac{R_a}{R_c^2} - \frac{1}{R_a}\right) \sin \kappa} \]

\[ = \frac{4 N_f M_0 R_a J_{5a}}{\left(\frac{R_a}{R_c^2} - \frac{1}{R_a}\right) \sin \kappa} \]
\[ T = \lambda \rho \kappa \frac{d N_{\kappa}}{d \kappa} \]
18. Calculate the force in tons that is necessary to hold two charges of 1 coulomb at a separation of 1 meter. \( \text{in} \times 10^{-4} \) lbs.

19. Calculate the distance of separation of two electrons in a vacuum for which the force between them is equal to the gravitational force on one of them at the earth's surface.

20. If \( \phi = \sqrt{x^2 + y^2} \) volts in the xy plane, determine the magnitude and direction of the electric field at the point (1,1).

21. A p-n semiconductor junction is shown. Because of the diffusion of charge carriers, a layer of positive charge accumulates on one side of the junction, and an equal layer of negative charge accumulates on the other. If an electron moves into the region between the two charge layers, it is urged to the right by the electric field. For an electron on the right to cross the junction, it must have a large enough velocity to penetrate the retarding effect of the field (called a "potential barrier"). Calculate the minimum velocity for this to occur if the density of each charge layer is \( 4.25 \times 10^{-6} \) coulombs/m², the layers are 10⁻³ cm apart, and the dielectric constant of the material \( K=2 \).

22. If corona discharge occurs at 30 kV/cm in air, what is the smallest radius of a sphere of a van de Graaf generator if it is to be charged to 5 million volts?

23. An electron beam in a cathode ray tube is circular in cross section with a radius of 1 mm. Calculate the force that the charge in the beam exerts on an electron that is located at the surface of the beam. Calculate the distance that such an electron will move outward (defocusing the beam) in moving 0.25 m longitudinally if it has a longitudinal velocity of \( 22.9 \times 10^6 \) m/sec and a beam current of 40 microamperes.

24. 3.1

25. 3.3

26. 3.4

27. 3.5

28. 3.6

29. 3.7

30. 3.9

31. 3.16

32. 3.17
Solution of Some Problems

1. Since we are concerned with only the positive $k_F$, let $k_F > 0$.

Now, \( (1 - 98) \) (my notes, p. 17) becomes

\[ a_m R_F^{(m)} - \epsilon_m k_F^{-(m+1)} = 0 \quad \text{(all } m) \]

and \( (1 - 99) \) becomes

\[ b_m k_F^{-(m+1)} - d_m k_F^{-(m+1)} = 0 \quad \text{(all } m) \]

Equation \( (1 - 96) \) is:

\[ \sum_{n=1}^{\infty} \left( a_n \kappa^{(n-1)} - c_n \kappa^{(n+1)} \right) \cos n \theta \rightarrow \sum_{n=1}^{\infty} \left( b_n \kappa^{(n-1)} - d_n \kappa^{(n+1)} \right) \sin n \theta \]

\[ \Rightarrow -2a_0 T_{Sa}(0) = 4 \mu_0 T_{Sa} \sin \theta \]

Multiplying this equation by \( \cos \theta \) and integrating with respect to \( \theta \) from \(-\pi\) to \(+\pi\), we get (remember your integrals of cosine and sine products):

\[ \int_{-\pi}^{\pi} \left( a_n \cos (n-1) - c_n \cos (n+1) \right) \cos n \theta \, d\theta = 0 \quad \text{for all } n \]

Equations (1) and (3) imply that \( a_m = c_m = 0 \) for all \( m \).

Next, multiply \((1)\) by \( \sin \theta \) and integrate with respect to \( \theta \) from \(-\pi\) to \( +\pi\):

\[ \int_{-\pi}^{\pi} \left( b_n \sin (n-1) - d_n \sin (n+1) \right) \cos n \theta \, d\theta = 0 \quad \text{all } n \neq 1 \]

\[ \int_{-\pi}^{\pi} \left( b_1 - d_1 \kappa^2 \right) \cos n \theta \, d\theta = \frac{4 \mu_0 T_{Sa}}{n} \]

(9a) and (2) imply that \( b_n = d_n = 0 \) for all \( n \neq 1 \), (4) (6) and (12)

\[ b_1 - d_1 \kappa^2 = 0 \quad \Rightarrow \quad b_1 = \frac{\mu_0 T_{Sa}}{\frac{1}{k_F^2} - \frac{1}{\kappa^2}} \]

\[ b_1 - d_1 \kappa^2 = \mu_0 T_{Sa} \]
\[
\psi_t = \frac{1}{1 - \frac{1}{R_{\psi}^2} - \frac{1}{R_{\alpha}^2}} = \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{\left( \frac{1}{R_{\psi}^2} - \frac{1}{R_{\alpha}^2} \right)}.
\]

\[
\phi_2 (r, \theta) = \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{\left( \frac{1}{R_{\psi}^2} - \frac{1}{R_{\alpha}^2} \right)} \left[ \frac{1}{R_{\psi}^2} + \frac{1}{r^2} \right] \sin \theta.
\]

\[
\bar{B} = \bar{a}_r \frac{1}{r} \frac{\partial a_r}{\partial \theta} - \bar{a}_\theta \frac{\partial a_r}{\partial r} = \bar{a}_r \left( \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{\left( \frac{1}{R_{\psi}^2} - \frac{1}{R_{\alpha}^2} \right)} \right) \left[ \frac{1}{R_{\psi}^2} + \frac{1}{r^2} \right] \cos \theta
\]

\[
+ \bar{a}_\theta \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{\left( \frac{1}{R_{\psi}^2} - \frac{1}{R_{\alpha}^2} \right)} \left[ \frac{1}{R_{\psi}^2} - \frac{1}{r^2} \right] \sin \theta.
\]

\[
\psi_a = \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{r} \left[ \frac{(\lambda_t)^2 + (\lambda_e)^2}{(\lambda_t)^2 - (\lambda_e)^2} \right] \cos \theta.
\]

\[
+ \bar{a}_\theta \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{r} \left[ \frac{(\lambda_t)^2 - (\lambda_e)^2}{(\lambda_t)^2 - (\lambda_e)^2} \right] \sin \theta.
\]

\[
\int_{-\pi}^{\pi} \psi_a d\theta = \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{R_{\psi} \left[ (\lambda_t)^2 - (\lambda_e)^2 \right]} \left[ \frac{1}{r} \cos \theta \right]_{-\pi}^{\pi}
\]

\[
= \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{R_{\psi} \left[ (\lambda_t)^2 - (\lambda_e)^2 \right]} \left[ \frac{1}{r} \cos \theta \right]_{-\pi}^{\pi}
\]

\[
= \frac{4 \lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{R_{\psi} \left[ (\lambda_t)^2 - (\lambda_e)^2 \right]} \sin \theta.
\]

\[
L_{fa} = \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{R_{\psi} \left[ (\lambda_t)^2 - (\lambda_e)^2 \right]} \sin \theta.
\]

\[
= \frac{\lambda_0 \lambda_{1/2} \tilde{\mathcal{N}}_{\sigma}}{R_{\psi} \left[ (\lambda_t)^2 - (\lambda_e)^2 \right]} \sin \theta.
\]
EE-353 ENERGY CONVERSION

MIDTERM EXAM (Winter 1971-72)

Be it jewel or toy,
Not the prize gives the joy,
But the striving to win the prize.

Pisistratus Caxton—
The Boatman

Somebody said that it couldn't be done,
But he with a chuckle replied
That "maybe it couldn't," but he
would be one
Who wouldn't say so till he'd tried.

So he buckled right in with the trace
of a grin
On his face. If he worried he hid it.
He started to sing as he tackled the
thing
That couldn't be done, and he did it.

Edgar A. Guest—It Couldn't
Be Done
2. 11-13. The origin for \( \pi \) is as depicted in Figure 4-13. The spring is completely relaxed at \( \pi = 0 \). There is no external force. Determine the d.c. and a.c. (linearized) equations

\[
\dot{\theta}(t) = V_0 + V_{es}(t), \quad \dot{\theta}(t) = V_0 + \frac{dV_{es}(t)}{dt},
\]

where \( V_0 \) is the d.c. component and \( V_{es}(t) \) are the a.c. signals. Be sure that you correlate the sign of the inductance with the assumed positive directions of \( I_0 \) and \( I_2 \).

3. 12-7.

4. A two-port network may be represented using the hybrid per

\[
Z_{11}, \quad G_{21}, \quad Z_{22}, \quad Y_{22}
\]

in the following way:

\[
V_1 = \frac{1}{Z_{11}} I_1 \quad \text{and} \quad V_2 = \frac{1}{G_{21}} I_2
\]

Explain the meaning of these parameters and the physical meaning of measuring them. Let \( V_2 = \alpha^2 I_2 \), \( V_2 = \frac{I_2}{\alpha} \), as in the RL current analogy.

5. Determine the hybrid parameters of Prob. 4 for the equivalent circuit of the hand-out of Chap. 11. Express your answer in terms of \( R, \quad L, \quad C, \quad \phi \), \( L \).

6. A conservative transducer is one for which the average a.c. input into both ports is zero (i.e., there is no dissipation within the transducer). Thus, letting \( V_1, \quad I_1, \quad V_2, \quad I_2 \) be phasor representations of sinusoids, we have

\[
\overline{P_{\text{avg}}} = \frac{1}{2} \text{Re} \left[ V_1 I_1^\ast + V_2 I_2^\ast \right] = 0 \quad (* = \text{conjugate})
\]

(a) Show that for such a transducer the hybrid parameters of Problem 4 must satisfy

\[
Z_{11} = jX_1, \quad Y_{22} = jX_2, \quad H_{21} = -jY_{12}^\ast
\]

where \( X_1 \) and \( X_2 \) are real. Explain what these results mean in terms of the parameters making up the transducer.
7. A force pulse $F_2(t)$ is applied to the transducer of problem (11-1). If the electrical output (port 1) is open circuited, determine the output voltage, $v_3(t)$. Assume that the system starts from rest at the quiescent point.

$$\theta_0 = 100 \cdot \alpha, \quad \lambda_0 = \lambda \cdot \alpha$$

$$L_0 = \frac{h}{k}, \quad b = 0.0 \frac{\text{m} \cdot \text{s}}{\text{m}}, \quad \frac{f_2(t)}{F_2}$$

$$M = 0.1 \text{kg}, \quad a = -0.1 \text{m/s}$$

$$X_0 = 0.1 \text{m}, \quad K_S = \frac{5 \text{N}}{\text{m}}$$

8. Using the same data as in problem 7, this time excite the transducer from the electrical side and calculate the velocity output $x_2$. There is no mechanical load applied to part 2. The electrical input, $v_1(t)$, is of the same form as $F(t)$, above, with the constant value being $E_0$ volts.
"... THE WISDOM OF THIS WORLD IS FOLLY WITH GOD..."

CORINTHIANS 1, 3.19

"Let us pray"
1) To find the energy and the co-energy, a relationship must be found between $\lambda$ and $\lambda$. Then
\[
\text{co-energy} = \int_0^L \lambda \, dl
\]
and
\[
\text{energy} = \int_0^L \frac{d\lambda}{\mu}
\]
Current may be found by dividing the sum of the mmf's of the gap and the steel by the turns ratio. The mmf of the steel in turn, may be found by:
\[
\frac{A}{\lambda} = H \cdot \ell
\]
where $\ell$ is the length of the sheet steel, the mmf of the gap may be found from:
\[
\frac{B}{\lambda} = B x / \mu
\]
where $x$ is the gap length, corresponding values of $B$ and $H$ may be read from the $B$-H curve for sheet steel on page 3-22 of text.
$\lambda$ may be solved for from:
\[
\lambda = \frac{N \cdot A}{\mu}
\]
where $N$ is the turns ratio, and $A$ is the cross sectional area.

Thus, for each co-ordinate of the B-H curve, with a given gap length, a corresponding $\lambda$, $\frac{B}{\lambda}$ and $\frac{A}{\lambda}$ may be computed. $\frac{B}{\lambda}$ and $\frac{A}{\lambda}$ then yield a corresponding value of $\lambda$. The energy and co-energy is then found by numerical integration on the resultant $\lambda$-$\lambda$ curve.
PROGRAM EE353A (INPUT, OUTPUT, TAPE2=INPUT, TAPE5=OUTPUT)

000002 DIMENSION G(1001), F(1001), PWR(1001)

000002 COMPLEX $, A1, A2, A3, A4, A5, AV

000002 WRITE(5, 19)

000006 READ(2, 20) KG

000014 READ(2, 20) KG

000022 READ(2, 20) XLISC

000030 READ(2, 20) XL1UC

000036 READ(2, 20) IA

000044 READ(2, 20) IG

000052 READ(2, 20) F

000060 READ(2, 20) ID

000066 FM = 0.0

000067 PMAX = 0.0

000067 R = 10.*** (1. / 25.1)

000075 XL = XLISC / 2.

000077 XL2 = XL1

000077 XL12 = XL1UC / A

000102 TPI = B * ATAN(1.1)

000104 ND = (25.0 * D) + 1.

000110 DO 90 J = 1, ND

000114 S = CMPLX(10.0, TPI * F)

000117 A1 = KG * R * S + (XL1 * A * XL2) / A

000130 A2 = S * XL12

000145 A3 = (A * R + S * (XL12 + A * XL2)) / A

000150 A4 = S * XL12

000154 A5 = CMPLX(A * RL, 0.0)


000217 G(J) = CABSTAV

000222 PWR(J) = EG * EG * (G(J) * G(J) / RL)

000226 TPI(PWR(J) - PMAX) * 15.15, 10

000231 10 PMAX = PWR(J)

000233 FM = F

000235 FR(J) = ALDOG(I(F)

000241 WRITE(5, 21) F, G(J), PWR(J)

000253 F = F + R

000253 90 CONTINUE

000257 WRITE(6, 22) PMAX, FM

000267 WRITE(5, 23)

000273 CALL SETPLT(1, FR, G, ND, 10, 11, 12, 13, PWR) VS AV

000302 WRITE(5, 23)

000306 CALL SETPLT(1, FR, PWR, ND, 10, 11, 12, 13, PWR) VS PWR

000315 19 FORMAT('FROBERT J. MARKS 11" ENERGY CONVERSION/" EE353/\",\n
*7X, 'FREQUENCY (HZ)\", 8X, 'VOLTAGE GAIN\", 8X, 'POWER\")

000315 20 FORMAT(F15.8)

000315 21 FORMAT(5X, E15.8, 5X, E15.8, 5X, E15.8)

000315 22 FORMAT('MAXIMUM POWER='\", E15.8\", \" WATTS AT \", E15.8\", HZ\")

000315 23 FORMAT('T")

000315 STOP

000317 END

PROGRAM LENGTH INCLUDING I/O BUFFERS
10455

UNUSED COMPILER SPACE
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**Maximum Power**: 9.99936145e-01 Watts at 6.30957344e+01 Hz
PROGRAM ENCON (INPUT,OUTPUT,TAPE2=INPUT,TAPE5=OUTPUT)

DIMENSION B(15),H(15),XL(15),CUR(15),G(3),E(3),CE(3),XCUR(15),
IXXL(15)

C SLL=LENGTH OF SHEET STEEL
C CSA=CROSS SECTIONAL AREA

READ(2,10)SLL,CSA,TURNS

BINC=0.1

UO=16.*ATAN(1.)*(10.***(-7.))

C READING IN VALUES OF H, AND CORRESPONDING B READ FROM B-H CURVE

DO 90 N=1,15

READ(2,11)H(N)

B(N)=X(N-1)

C COMPUTING XL-I RELATION FOR DIFFERENT GAPS

DO 93 J=1,3

READ(2,11)G(J)

INDIVIDUAL RELATIONS OF XL-I

DO 92 K=1,15

XL(J,K)=TURNS*CSA*B(K)

FS=H(K)*SLL

FG=I::l(K)*G(J)/UO

CUR= CURRENT

CUR(J,K)=(FS+FG)/TURNS

CONTINUE

C FINDING CO-ENERGY (CE) AND ENERGY (E) BY TRAPEZOIDAL INTEGRATION

CF(J)=0.0

E(J)=0.0

DO 93 L=1,14

ACE=(CUR(J,L+1)-CUR(J,L))*(XL(J,L+1)-XL(J,L))/2.

CE(J)=CE(J)+ACE

AE=(XL(J,L+1)-XL(J,L))*(CUR(J,L+1)+CUR(J,L))/2.

E(J)=E(J)+AE

CONTINUE

C ECHOING DATA

WRITE(5,12)SSL,CSA,TURNS

WRITE(5,13)B(M),H(M)

CONTINUE

C PRINTING RESULTS

WRITE(5,14)NN,G(NN)

WRITE(5,15)CUR(NN,MM),XL(NN,MM)

XCUR(MM)=CUR(NN,MM)

XXL(MM)=XL(NN,MM)

CONTINUE

WRITE(5,17)E(NN),CE(NN)

CONTINUE

CALL SETPLT(1,XCUR,XXL,15,1H*,8,8HLAM VS 1)

CONTINUE

CALL EX

FORMAT(3F10.4)

FORMAT(2X,E15.8,5X,E15.9)

FORMAT(2X,SHEET STEEL LENGTH,E15.9)

FORMAT(2X,CROSS SECTIONAL AREA,E15.9)

FORMAT(2X,TURNS,E15.9)

FORMAT(3X,T19X,H*,/)

FORMAT(2X,E15.8,5X,E15.9)

FORMAT(2X,SHEET STEEL LENGTH,E15.9)

FORMAT(2X,CROSS SECTIONAL AREA,E15.9)

FORMAT(2X,TURNS,E15.9)

FORMAT(3X,T19X,H*)
PROGRAM LENGTH INCLUDING I/O BUFFERS
002717

UNUSED COMPILER SPACE
006000

LOAD MAP.  01/17/72. 18.10.13.  PAGE 1

FL REQUIRED TO LOAD  27100
FL REQUIRED TO RUN  20600
INITIAL TRANSFER TO ENCON  103

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ROBERT J. MARKS II
ENERGY CONVERSION

ALL VALUES MKS UNITS

SHEET STEEL LENGTH = 5.00000000E-01
CROSS SECTIONAL AREA = 1.00000000E-03
TURNS = 1.00000000E+03

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B-H CO-ORDINATES READ FROM
B-H CURVE FOR SHEET STEEL
ON PAGE 3-22 OF TEXT.
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**ENERGY** = 5.57679611E-01  
**CO-ENERGY** = 1.44717961E+00

\[
\text{ENERGY} \times \text{CO-ENERGY} = \lambda_m \cdot \lambda_m
\]

\[
(0.53) \times (1.45) \leq (1.40)(1.43)
\]

\[
2.03 \leq 2.00
\]
\begin{align*}
\text{ENERGY} &= 9.47609221E-01 \\
\text{CO-ENERGY} &= 1.83710922E+00
\end{align*}

\text{GAP(2)} = 1.00000000E-03

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\text{ENERGY} = 9.47609221E-01 \\
\text{CO-ENERGY} = 1.83710922E+00
\[ \text{GAP}(3) = 0. \]

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\[ \text{ENERGY} = 1.67750000E-01 \quad \text{CD-ENERGY} = 1.05725000E+00 \]

\[ \text{ENERGY} + \text{CD-ENERGY} = \frac{m}{L} \sum \frac{m}{L} \frac{1}{L} (1.06) \]

\[ \frac{1}{1.23} = 1.225 \]
\[ F = \frac{6W}{6x} = \frac{(1.837 - 1.057)}{(1.0 - 0.0)} \times 10^3 = 780 \text{ nt} \]

Can't use this formula because limited and I feel are not finished.

must use \[ F = -\frac{\partial W}{\partial x} \].
\[
\begin{align*}
\lambda_1 &= \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, (\text{Eq. 4-91, 4-92}) \\
\lambda_2 &= \begin{bmatrix} L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}
\end{align*}
\]

\[L_{11} = a (l^2 - x^2) \quad a = \frac{N_1^2 \mu_0}{2B^2}, (\text{Eq. 4-100, 4-101}) \Rightarrow \frac{dL_{11}}{dx} = -2ax\]

\[L_{12} = L_{21} = bx (l-x) \quad b = \frac{N_1 N_2 \mu_0}{2B^2}, (\text{Eq. 4-101, 4-105}) \Rightarrow \frac{dL_{12}}{dx} = b(l-2x)\]

\[L_{22} = cx (2l-x) \quad c = \frac{N_2^2 \mu_0}{2B^2}, (\text{Eq. 4-110, 4-111}) \Rightarrow \frac{dL_{22}}{dx} = 2c(l-x)\]

\[V_i = R_i i_1 + \frac{d\lambda_1}{dt}\]

\[= R_i i_1 + \frac{d}{dt} \left[ L_{11} i_1 - L_{12} i_2 \right]\]

\[= R_i i_1 + \frac{d}{dt} \left( L_{11} i_1 - i_1 \frac{dL_{11}}{dt} - i_2 \frac{dL_{12}}{dt} \right)\]

\[= R_i i_1 + L_{11} \frac{di_1}{dt} + i_1 \frac{dL_{11}}{dx} \frac{dx}{dt} + i_2 \frac{dL_{12}}{dx} \frac{dx}{dt}\]

\[= R_i i_1 + L_{11} \frac{di_1}{dt} + i_1 \frac{dL_{11}}{dx} \frac{dx}{dt} + L_{12} \frac{di_2}{dt} + i_2 \frac{dL_{12}}{dx} \frac{dx}{dt}\]

\[= R_i i_1 + a (l^2 - x^2) \frac{di_1}{dx} + bx (l-x) \frac{di_2}{dx} \frac{dx}{dt} \left[ i_2 \frac{dL_{12}}{dx} \frac{dx}{dt} + i_2 2ax \right]\]

\[\text{LET} \quad V_i = V_{01} + V_{11}\]

\[i_1 = i_{01} + i_{11}\]

\[x = x_0 + x_1\]

\[i_2 = i_{02} + i_{12}\]

\[\text{WHERE THE D IN THE FIRST DIGIT DESIGNATES A D.C. VALUE, AND 1 AN A.C. VALUE}\]
From fifth step in determining \( V_x, V_y \) may be derived. By replacing, all \( I \) subscripts with 25, and

\[
V_x = R_1 I + I_2 b x_0 \frac{dI}{dx} + b x_0 x_0 \frac{d^2 I}{dx^2} + \frac{b}{2} (x_0 - b) x_0 \frac{d^2 I}{dx^2}.
\]

\[
V_y = R_2 I + I_2 b x_0 \frac{dI}{dx} + b x_0 x_0 \frac{d^2 I}{dx^2} + \frac{b}{2} (x_0 - b) x_0 \frac{d^2 I}{dx^2}.
\]

\[
I = \frac{1}{b \cdot 2 x_0} \int \left[ \frac{b}{2} (x_0 - b) \cdot x_0 \cdot \frac{d^2 I}{dx^2} + b \cdot x_0 \cdot \frac{d^2 I}{dx^2} \right] dx,
\]

\[
I = \frac{1}{b \cdot 2 x_0} \int \left[ \frac{b}{2} (x_0 - b) \cdot x_0 \cdot \frac{d^2 I}{dx^2} + b \cdot x_0 \cdot \frac{d^2 I}{dx^2} \right] dx.
\]

Equations A-E and D-C components.
FREE BODY OF MASS, (HORIZONTAL COMPONENTS)

\[ F_{SP} = \text{FORCE FROM SPRING} = kx \]
\[ F_{DP} = \text{FORCE FROM DASH POT} = -b \frac{dx}{dt} \]
\[ F_{P1} = \text{FORCE FROM POLE 1, ATTEMPTING TO ALIGN MASS VERTICALLY} = \frac{i_1^2}{2} \frac{dL_1}{dx} \]
\[ F_2 = \text{INERTIAL FORCE} = M \frac{d^2x}{dt^2} \]
\[ F_{P2} = \text{FORCE FROM POLE 2} = \frac{i_2^2}{2} \frac{dL_2}{dx} \]
\[ F_{P12} = \text{FORCE FROM CENTRAL POLE} = i_1, i_2 \frac{dL_{12}}{dx} \]

\[ \Sigma F = 0 = F_2 + F_{P2} + F_{P12} - F_{SP} - F_{DP} - F_{P1} \]
\[ = M \frac{d^2x}{dt^2} + \frac{i_2^2}{2} \frac{dL_2}{dx} + i_1, i_2 \frac{dL_{12}}{dx} - kx - b \frac{dx}{dt} \frac{i_1^2}{2} \frac{dL_1}{dx} \]
\[ = M \frac{d^2x}{dt^2} + \frac{i_2^2}{2} c(x - l) + i_1, i_2 b(l - 2x) - kx - b \frac{dx}{dt} + i_1^2 \frac{dL_1}{dx} \]

DIVIDING INTO COMPONENTS:
\[ = M \frac{d^2x}{dt^2} + (i_0 + i_1) \frac{dL}{dt} - (I_{02} + I_{12}) c(l - x) + (I_{01} + I_{11}) (I_{02} + I_{12}) b \frac{dx}{dt} \]
\[ - (I_{01} + I_{11}) 2(c(l - x)) - k(x + x_0) - b \frac{dx}{dt} + (I_{01} + I_{11}) 2a(x + x_0) \]

STRIKING OUT TERMS CONTAINING PRODUCTS OF A-C COMPONENTS
\[ O = M \frac{d^2x}{dt^2} + c(l I_{02} + 2l I_{12} - c I_{02} x_0 - 2c l I_{12} x_0 - c I_{02} x_1) \]
\[ + b(l I_{01} + b l I_{02} + b l I_{12}) + I_{11} x_0 - 2I_{01} x_0 - 2I_{11} x_1 + 2I_{11} x_0 \]
\[ - k(x_0 - k x_1) - b \frac{dx}{dt} + a I_{02} x_0 + a I_{12} x_1 + a x_0 + 2a I_{01} x_0 \]
\[ = M \frac{d^2x}{dt^2} + c \left[ l I_{02} - 2 l I_{02} x_0 - 2 I_{01} x_0 - 2 I_{12} x_0 - I_{02} x_1 \right] \]
\[ + b \left[ l I_{01} + l I_{02} + l I_{12} + l I_{11} x_0 - 2 l I_{11} x_0 - 2 I_{11} x_1 + 2 I_{11} x_0 \right] \]
\[ - k(x_0 - k x_1) - b \frac{dx}{dt} \]

D.C. VALUES:
\[ O = c l I_{02} - c I_{02} x_0 + b l I_{01} x_0 - 2 b l I_{02} x_0 \]

A.C. VALUES:
\[ O = M \frac{d^2x}{dt^2} + c \left[ 2 l I_{12} - 2 l I_{12} x_0 - I_{12} x_1 \right] \]
\[ + b \left[ I_{01} x_0 + l I_{12} + l I_{11} x_0 \right] + a \left[ I_{02} x_1 + 2 I_{01} I_{11} x_0 \right] \]
\[ - 2I_{01} x_1 + 2I_{11} x_0 - b \frac{dx}{dt} - k x_1 \]
2) \( L = L_0 + L_2 \cos 4\alpha + L_6 \cos 12\alpha \)
\( \alpha = \omega_{mt} t \)

\[ \begin{align*}
\frac{dL}{dt} & = 4\omega_m \left[ L_2 \sin 4(\omega_{mt} t + \delta) + 3L_6 \sin 12(\omega_{mt} t + \delta) \right] \\
i(t) & = I_m \sin \omega_{mt} t \Rightarrow I^2(t) = \frac{I_m^2 \sin^2 \omega_{mt} t}{2}
\end{align*} \]

\( \tau(t) = -\frac{d^2(t)}{dt^2} \cdot \frac{dL(t)}{dt} \)

\[ \begin{align*}
\tau(t) & = -\frac{I_m^2}{2} \left[ 1 - \cos 2\omega_{mt} t \right] - 4\omega_m \left[ L_2 \sin 4(\omega_{mt} t + \delta) + 3L_6 \sin 12(\omega_{mt} t + \delta) \right]
\end{align*} \]

The frequencies present in the above expression for torque are \( 4\omega_m, 12\omega_m, 14\omega_m - 2\omega, 14\omega_m + 2\omega, 12\omega_m - 2\omega, \) and \( 12\omega_m + 2\omega. \) Any function which can be expressed as a series of impulses in the frequency domain (as \( \tau(t) \) does), must be periodic. Let \( T_{\tau} \) be the period of the torque wave. Thus, the corresponding period of each frequency component \( (T_{\tau}/T_c = 2\pi/\omega_c) \) is evenly divisible into \( T_{\tau} \) (i.e. \( T_{\tau}/T_c = I_c \exists I_c \in \text{non-negative integer} \)).

For \( \tau(t) \) to have a non-zero value, its integral over \( T_{\tau} \) must have a non-zero value. All pure sinusoid terms in \( \tau(t) \) will yield a DC torque over \( T_{\tau} \), in that any sinusoid’s DC value over a non-negative integer multiple of its period is zero. The terms expressed as the product of two sinusoids will yield a DC value iff they are orthogonal.

Thus:

\[
2\omega = 4\omega_m \Rightarrow \omega_m = \frac{1}{2} \omega
\]
or
\[
2\omega = 12\omega_m \Rightarrow \omega_m = \frac{1}{6} \omega
\]

For the machine to supply an average torque.
The relationship between electric and mechanical angles is

$$\psi_e = P\phi$$

where \( P \) is the number of pole pairs. Differentiating both sides:

$$\omega = \frac{d\psi_e}{dt} = P\omega_m$$

It has been established that \( \omega_m = \frac{1}{\alpha} \omega \), where \( \alpha = 2 \) or \( 6 \), if the system is expected to yield any average torque.

Thus

$$\omega = P\omega_m = P \left( \frac{1}{\alpha} \omega \right)$$

\( \Rightarrow P = \alpha \)

Ergo, the system must have either 2 or 6 pole pairs in order for any average torque to be delivered. As it stands, the system has 2 pole pairs, average torque being supplied by the \( \omega_m = 2\omega \) term, and it's third harmonic \( (6\omega) \) roughly.

[Diagram of voltages and torques]

Q.C. Torque

$$\sin \omega_m t \text{ (volts)}$$

$$\sin 2\omega_m t$$
4) \( V_1 = I_1 Z_{11} + V_2 G_{12} \)
\( I_2 = I_1 H_{21} + V_2 Y_{22} \)
\( V_2 = \alpha x_0 \)
\( I_2 = \gamma_2 / \alpha \)

The two equations represent the two-port network illustrated. The remaining two equations for \( V_2 \) and \( I_2 \) suggest an equivalent electro-mechanical circuit, in electrical units (\( V_2 \) has units of volts, \( I_2 \) units of amps, etc.).

\[ I_2 = \frac{F_2}{\alpha} \Rightarrow \alpha \text{ has units of force (force/time)} \]
\[ V_2 = \alpha \dot{x} \Rightarrow \text{voltage} = \left( \frac{\text{force-time}}{\text{charge}} \right) \left( \frac{\text{distance}}{\text{time}} \right) = \left( \frac{\text{energy}}{\text{charge}} \right) \frac{\text{distance}}{\text{time}} = \text{voltage} \]

By shorting out \( V_2 \) terminal (thus holding \( \dot{x} \), the velocity of the mechanical side, zero), \( Z_{11} \) and \( H_{21} \) may be measured as follows:

\[ V_1 = Z_{11} I_1 \bigg|_{V_2=0} \Rightarrow Z_{11} = \frac{V_1}{I_1} \bigg|_{V_2=0} = \frac{V}{I} \]

\[ I_2 = I_1 H_{21} \bigg|_{V_2=0} \Rightarrow H_{21} = \frac{I_2}{I_1} \bigg|_{V_2=0} = \frac{1}{\alpha} \left( \frac{F_2}{I_1} \right) \bigg|_{V_2=0} = \frac{1}{\alpha} \text{H}_m \]

By opening the electrical terminals, forcing \( I_1 = 0 \), \( G_{12} \) and \( Y_{22} \) may be measured as follows:

\[ V_1 = V_2 G_{12} \bigg|_{I_1=0} \Rightarrow G_{12} = \frac{V_1}{V_2} \bigg|_{I_1=0} = \frac{1}{\alpha} \left( \frac{V}{x_2} \right) \bigg|_{I_1=0} = \frac{1}{\alpha} \text{G}_e \]

\[ I_2 = Y_{22} V_2 \bigg|_{I_1=0} \Rightarrow Y_{22} = \frac{I_2}{V_2} \bigg|_{I_1=0} = \frac{1}{\alpha^2} \left( \frac{F_2}{x_2} \right) \bigg|_{I_1=0} = \frac{1}{\alpha^2} \text{Y}_m \]

Thus

\[ Z_{11} = Z_C \]
\[ \Rightarrow Z_{11} = \left( \frac{V}{I} \right) \bigg|_{V_2=0} = \frac{V}{I} \]

\[ H_{21} = \frac{1}{\alpha} \text{H}_m \]
\[ \Rightarrow H_{21} = \left( \frac{F_2}{I_1} \right) \bigg|_{V_2=0} = \frac{F_2}{I_1} \]

\[ G_{12} = \frac{1}{\alpha} \text{G}_e \]
\[ \Rightarrow G_{12} = \left( \frac{V}{x_2} \right) \bigg|_{I_1=0} = \frac{V}{x_2} \]

\[ Y_{22} = \frac{1}{\alpha^2} \text{Y}_m \]
\[ \Rightarrow Y_{22} = \left( \frac{F_2}{x_2} \right) \bigg|_{I_1=0} = \frac{F_2}{x_2} \]

\[ Z_{11} = Z_C \]
\[ H_{21} = \frac{1}{\alpha} \text{H}_m \]
\[ G_{12} = \frac{1}{\alpha} \text{G}_e \]
\[ Y_{22} = \frac{1}{\alpha^2} \text{Y}_m \]

\[ \Theta \quad \text{Z}_C = \left( \frac{V}{I} \right) \bigg|_{V_2=0} \]
\[ \Theta \quad \text{H}_m = \left( \frac{F_2}{I_1} \right) \bigg|_{V_2=0} \]
\[ \Theta \quad \text{G}_e = \left( \frac{V}{x_2} \right) \bigg|_{I_1=0} \]
\[ \Theta \quad \text{Y}_m = \left( \frac{F_2}{x_2} \right) \bigg|_{I_1=0} \]

\[ \text{Z}_C = \left( \frac{V}{I} \right) \bigg|_{V_2=0} \]
\[ \text{H}_m = \left( \frac{F_2}{I_1} \right) \bigg|_{V_2=0} \]
\[ \text{G}_e = \left( \frac{V}{x_2} \right) \bigg|_{I_1=0} \]
\[ \text{Y}_m = \left( \frac{F_2}{x_2} \right) \bigg|_{I_1=0} \]

\[ \Theta \quad \text{Z}_C = \left( \frac{V}{I} \right) \bigg|_{V_2=0} \]
\[ \Theta \quad \text{H}_m = \left( \frac{F_2}{I_1} \right) \bigg|_{V_2=0} \]
\[ \Theta \quad \text{G}_e = \left( \frac{V}{x_2} \right) \bigg|_{I_1=0} \]
\[ \Theta \quad \text{Y}_m = \left( \frac{F_2}{x_2} \right) \bigg|_{I_1=0} \]

\[ \text{Z}_C = \left( \frac{V}{I} \right) \bigg|_{V_2=0} \]
\[ \text{H}_m = \left( \frac{F_2}{I_1} \right) \bigg|_{V_2=0} \]
\[ \text{G}_e = \left( \frac{V}{x_2} \right) \bigg|_{I_1=0} \]
\[ \text{Y}_m = \left( \frac{F_2}{x_2} \right) \bigg|_{I_1=0} \]
Clearly, $Z_e, H_m, G_e, \text{and } Y_m$ may be computed experimentally in the system described. Generally, each will contain resistive, as well as reactive components. Thus, measurements of amplitude and phase relationship should be made in a frequency response analysis over the spectrum of interest, expressing original two-port equations again:

1. \[ V_1 = I_1 Z_e + V_2 G_{e21} \]
   \[ = I_1 Z_e + (\alpha \dot{x}_2) \left( \frac{1}{\alpha} G_e \right) \]
   \[ = I_1 \dot{x}_2 G_e \]

2. \[ I_2 = I_1 H_{e21} + V_2 Y_{e22} \]
   \[ (F_2) = I_1 \left( H_{m2} \right) + (\alpha \dot{x}) \left( \frac{1}{\alpha} Y_m \right) \]
   \[ F_2 = I_1 H_m + \dot{x} Y_m \]

Again:

\[
\begin{align*}
\{ V_1 &= I_1 Z_e + \dot{x}_2 G_e \\
F_2 &= I_1 H_m + \dot{x} Y_m \\
\end{align*}
\]

Formal diagram, using experimental results:

Now the left port has electrical dimensions, and the right port mechanical dimensions, as opposed to the initial case where both ports boasted of electrical dimensions. $G_e$, experimentally measured, is a conversion factor of sorts. It converts the mechanical unit of velocity into the electrical unit of volts, in that $G_{12} = \frac{1}{\alpha} G_e$, and $G_{12}$ is unitless. $G_e$ has units of $\alpha$ (= volt/velocity). Similarly, the other "conversion factor", $H_m$, electrical to mechanical (current to velocity) has dimensions of $\alpha$ (= velocity/amp = volt/velocity).

$Z_e$ is the electrical impedance, $Y_m$ is the mechanical admittance, in the electromechanical force-current analogy, $R$ might be analogous to the (mechanical damping factor), $C$ to mass, 1/L to spring constant, etc.
5) CIRCUIT:

\[ I_1 = \frac{R}{L_0} I_1 + 4 \cdot I_2 \]

\[ V_1 = C \cdot \frac{d}{dt} E + L \cdot \frac{d}{dt} I_2 \]

**LET**

\[ Z_S = R + S L_0 \]

**AND**

\[ Y_P = SC + S C + L \]

\[ \begin{bmatrix} Z_S \end{bmatrix} = \begin{bmatrix} 4 \cdot I_2 \\ Z_S \end{bmatrix} \]

\[ Y_P \]

**THUS**

\[ V_1 = Z_S I_1 + V_2 \]

\[ I_2 = I_1 + Y_P V_2 \]

**HYBRID EQUATIONS:**

\[ V_1 = Z_{11} I_1 + G_{12} V_2 \]

\[ I_2 = H_{21} I_1 + Y_{22} V_2 \]

\[ Z_{11}(s) = Z_S(s) = R + S L_0 \]

\[ G_{12}(s) = 1 = \frac{V_1}{I_1} \big|_{V_2=0} \]

\[ H_{21}(s) = -1 = \frac{I_2}{V_2} \big|_{I_1=0} \]

\[ Y_{22}(s) = SC + S L = \frac{V_2}{I_1} \big|_{I_1=0} \]

\[ \frac{20}{20} \]
6) \( V_1 = I_1 Z_{ii} + V_2 G_{12} \)
\( I_2 = I_1 H_{21} + V_2 Y_{22} \)

FOR A CONSERVATIVE TRANSDUCER:

\[
\begin{align*}
P_{\text{ave}} &= \frac{1}{2} R_e \left[ V_1 I_1^* + V_2 I_2^* \right] = 0 \\
&= \frac{1}{2} R_e \left[ V_1 I_1^* \right] + \frac{1}{2} R_e \left[ V_2 I_2^* \right] \\
&= \frac{1}{2} R_e \left[ V_1 I_1^* \right] + \frac{1}{2} R_e \left[ V_2 I_2^* \right] \\
&= \frac{1}{2} R_e \left[ (I_1 Z_{ii}) I_1^* + (V_2 G_{12}) I_1^* + (I_1 H_{21}) V_2^* + (V_2 Y_{22}) V_2^* \right] = 0
\end{align*}
\]

SHORTING SECOND PORT OF CONSERVATIVE TRANSDUCER \( \Rightarrow V_2 = V_2^* = 0 \)

\[
\Rightarrow P_{\text{ave}} \bigg|_{V_2=0} = \frac{1}{2} R_e \left[ Z_{ii} I_1^2 \right]
\]

THE QUANTITY \( I_1^2 \) IS PURE REAL, THUS FOR \( P_{\text{ave}} \bigg|_{V_2=0} = 0 \), \( Z_{ii} \) MUST BE PURE IMAGINARY, YIELDING A PURE IMAGINARY \( Z_{ii} I_1^2 \) TERM, THE REAL COMPONENT OF WHICH IS OF COURSE ZERO.

IN THAT \( Z_{ii} \) HAS UNITS OF IMPEDANCE, ITS IMAGINARY COMPONENT WILL BE A REACTANCE \( jX_1 \).

\[
\Rightarrow Z_{ii} = jX_1
\]

OPEN CIRCUITING FIRST PORT OF THE CONSERVATIVE TRANSDUCER \( \Rightarrow I_1 = I_1^* = 0 \)

\[
\Rightarrow P_{\text{ave}} \bigg|_{I_1=0} = \frac{1}{2} R_e \left[ Y_{22} V_2 V_2^* \right]
\]

BY THE SAME ARGUMENT, ADMITTANCE \( Y_{22} \) MUST CONSIST OF A PURELY SUSCEPTIVE COMPONENT:

\[
\Rightarrow Y_{22} = jB_2
\]

THE POWER EQUATION THEREFORE BECOMES:

\[
P_{\text{ave}} = \frac{1}{2} R_e \left[ G_{12} (V_2 I_1^*) + H_{21} (V_2^* I_1) \right] = 0
\]

\[
\Rightarrow H_{21} (V_2^* I_1) = -\frac{R_e}{G_{12}} (V_2 I_1^*)
\]

LET \( H_{21} = a + j b \)
\[ G_{12} = c + j d \]
\( V_2^* I_1 = e + j f \)

\[
(a + j b) (e + j f) = (c + j d) (e - j f)
\]

\[
(a e - b f) + j (b e + a f) = -(c e + d f) + j (d e - c f)
\]

EQUATING REAL AND IMAGINARY:

\[
\begin{align*}
&ae - bf = -ce - df \\
&be + af = de + cf
\end{align*}
\]

\[
\Rightarrow \begin{cases}
ae - bf = -ce - df \\
be + af = de + cf
\end{cases} \Rightarrow \begin{cases}
a = -c \\
b = d
\end{cases}
\]

\[
\Rightarrow H_{21} = -G_{12}^* 
\]

What does this mean in terms of the elements making up the system?
7) THE A-C ELECTRO-MECHANICAL EQUATIONS DESCRIBING SYSTEM OF PROBLEM (II-1) WERE FOUND TO BE

\[ C_1(t) = R_1 I(t) + L_0 \left(1 + \alpha x_0^2\right) \frac{dI(t)}{dt} + 2 \alpha L_0 I_0 x_0 \frac{dx_1(t)}{dt} \]

\[ f_1(t) = M \frac{d^2x(t)}{dt^2} + b \frac{dx(t)}{dt} + \left(k_5 - \alpha L_0 I_0^2\right) x(t) - 2 \alpha L_0 I_0 x_0 \dot{x}(t) \]

LETTING \[ \alpha = 2 \alpha L_0 I_0 x_0 \]
\[ L = L_0 \left(1 + \alpha x_0^2\right) \]
\[ C_n = (k_5 - \alpha L_0 I_0^2)^{-1} \]

THE EQUIVALENT ELECTRO-MECHANICAL CIRCUIT WOULD BE

\[
\begin{array}{c}
\dot{I}_1(t) = \frac{E_1(t)}{R_1 + \alpha L_0 x_0} \\
E_1(t) = M \frac{d^2x(t)}{dt^2} + b \frac{dx(t)}{dt} + \left(k_5 - \alpha L_0 I_0^2\right) x(t) - 2 \alpha L_0 I_0 x_0 \dot{x}(t)
\end{array}
\]

ALSO LET:
\[ Z_e = R + S L \]
\[ Z_m = S M + b + C_n S \]

AND
\[ E_2 = \alpha \dot{x} \]
\[ f_1 = -\alpha I_1 \]

THE CIRCUIT THUS BECOMES (IN LAPLACE DOMAIN)

\[
\begin{array}{c}
\frac{I_1}{E_1} = \frac{Z_e}{Z_m} \\
E_2 = \alpha \dot{x} \\
-\alpha I_1 = \frac{F_1}{E_2} = \frac{-\alpha^2}{\alpha^2} \frac{F_1}{\dot{x}} = \frac{-\alpha}{\alpha^2} \left(\frac{F_3 - Z_m x}{x}\right) = \frac{Z_m}{\alpha^2} = \frac{F_3}{\alpha}
\end{array}
\]

THE CIRCUIT THEN BECOMES:

\[
\begin{array}{c}
Y_m = \frac{Z_m}{\alpha^2}
\end{array}
\]

WHERE \[ Y_m = \frac{Z_m}{\alpha^2} \]
OPENING TERMINALS ON ELECTRICAL PORT $\Rightarrow I_1 = 0$.

Thus, the circuit becomes:

$$F_2 \cdot \frac{1}{Z_m} + \alpha X = \frac{F_2}{\alpha Y_m} \Rightarrow E_1 = \frac{F_2}{\alpha Y_m} \frac{\alpha X}{Z_m} = \frac{F_2 \alpha X}{Z_m}$$

Given $f_2(t)$,

$$\Rightarrow g_2(s) = F\{f_2(t)\} = \frac{A}{s} \left(1 - e^{-sT}\right)$$

NOW

$$E_1(s) = \frac{\alpha F_2}{Z_m(s)} = \frac{\alpha F_2}{s} \left[\frac{1 - e^{-sT}}{sM + b + \frac{1}{C_m}s}\right]$$

$$= \alpha F_2 \left[\frac{1}{s^2M + bS + \frac{1}{C_m}} \right]$$

$$= \alpha F_2 \left[\frac{e^{-sT}}{s^2M + bS + \frac{1}{C_m}} \right]$$

$e^{-sT}$ corresponds to a time shift of $T$ in the time domain, thus finding the roots of the quadratic $s^2M + bS + \frac{1}{C_m}$ are of present concern.
REAL M, KS, LO, IO
DIMENSION C(3), COF(3), RR(2), RI(2)
READ(2, 50) R, LO, M, X0, IO, B, A, KS
ALPHA = 2 * A * LO * IO * X0
WRITE(5, 48)
WRITE(5, 49) ALPHA
CM = 1. / (KS - A * LO * IO * IO)
C(1) = 1. / CM
C(2) = B
C(3) = M
CALL POLRT(C, COF, 2, RR, RI, IER)
DO 20 J = 1, 3
WRITE(5, 51) J, C(J)
20 CONTINUE
DO 21 K = 1, 2
WRITE(5, 52) K, RR(K), RI(K)
21 CONTINUE
STOP
48 FORMAT('1')
49 FORMAT(' ALPHA=', E15.8)
50 FORMAT('5*F5.3')
51 FORMAT(' C(', I2, ')=', E15.8)
52 FORMAT(' RT(', I2, ')=', E15.8, ' REAL, ', E15.8, ' IMAG')
END

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END OF COMPILATION
// LOAD
\[
\begin{align*}
\text{ALPHA} &= -0.1999998E-02 \\
C(1) &= 0.5000109E-03 \\
C(2) &= 0.1000001E-01 \\
C(3) &= 0.1000000E-00 \\
\text{RT}(1) &= -0.50000004E-01 \text{ REAL, } -0.70710739E+02 \text{ IMAG} \\
\text{RT}(2) &= -0.50000004E-01 \text{ REAL, } 0.70710739E+02 \text{ IMAG} \\
S &\ 32 \text{ STOP 0000}
\end{align*}
\]
\[ S^M + b_s + \frac{1}{C_m} = (S-R)(S-R^*) = S^2 + R^2 \]

Where \( R = 0.05 - j0.7 \) \( \Rightarrow c, c^* \text{ Math Tables, 16th Edition, p. 442, 3/12} \)

\[ L^{-1}\left\{ \frac{1}{S^M + b_s + \frac{1}{C_m}} \right\} = L^{-1}\left\{ \frac{1}{(S-R)(S-R^*)} \right\} \]

\[ = \frac{1}{R-R^*} \left( e^{Rt} - e^{R^*t} \right) \Rightarrow c, c^* \text{ Math Tables, 16th Edition, p. 442, 3/12} \]

\[ = \frac{1}{j2b_1} \left[ e^{at} (e^{ibt} - e^{-ibt}) \right] \]

\[ = \frac{1}{j2b_1} \left[ e^{at} (j2\sin bt) \right] \]

\[ \Rightarrow e(t) = \frac{\alpha F_2}{b_1} \left[ e^{at} \sin bt \mu(t) - e^{a(t-T)} \sin b(t-T) \mu(t-T) \right] \]

\[ \Rightarrow e(t) = (2.82 \times 10^6) F_2 \left[ e^{-0.05t} \sin \left( +0.7 \right) t \mu(t) - e^{-0.05(t-T)} \sin \left( +0.7 \right) (t-T) \mu(t-T) \right] \]

The Equivalent Circuit Again:

\[ F \rightarrow \]

\[ + \quad + \]

\[ C = M \quad L = C_m \]

\[ G = b \]

\[ \mu(t) \]

\[ E(t) \]

The resonant frequency of the L-C tank

\[ \omega_r = \frac{1}{\sqrt{LC}} = \sqrt{\frac{1}{LM}} = \left( \frac{M}{L} \right)^{1/2} = \left( \frac{5000}{0.5} \right)^{1/2} = 70.7 \]

Which is the angular frequency of the sinusoids in the resultant expression for \( e(t) \).
8) **HAVING NO MECHANICAL LOAD WILL REDUCE THE CIRCUIT DERIVED IN PROBLEM 7 TO THE FOLLOWING:**

\[
\begin{align*}
E_I &= \left( Z_e + \frac{1}{Y_m} \right) I_I = \left( Z_e \frac{E_I}{Z_m} \right) I_I = E_I (Z_e + \frac{E_I^2}{Z_m})^{-1} \\
\end{align*}
\]

**Also:**

\[
E_I = \frac{Z_m}{Z_e + \frac{E_I^2}{Z_m}} \\
\]

**Thus:**

\[
I_I = \frac{\alpha E_I}{Z_m} = \frac{\alpha E_I}{Z_m(Z_e + \alpha^2 / Z_m)} = \frac{\alpha E_I}{Z_m Z_e + \alpha^2} \\
\]

**Again:**

\[
E_I(s) = \frac{E_I}{s} \left( 1 - e^{-sT} \right) \\
\]

\[
\Rightarrow I_I(s) = \frac{\alpha E_I}{s} \left[ \frac{1 - e^{-sT}}{(s^2 + c_2)(s + c_2) + c^2} \right] \\
= \frac{\alpha E_I}{s} \left[ \frac{1 - e^{-sT}}{R^2 s^2 + R s c_m + c_m^2 + c^2} \right] \\
= \frac{\alpha E_I}{s} \left[ \frac{1 - e^{-sT}}{A s^2 + B s^2 + C s + D} \right] \\
\]

\[
A = (L^2 + M L_O) \left( 1 + \alpha X_0^2 \right) \\
B = (L_b + L_m) \left( 1 + \alpha X_0^2 \right) + R M \\quad \text{RM} \\
C = R_b \left( 1 + \alpha X_0^2 \right) + L_b \left( 1 + \alpha X_0^2 \right) \left( k = a L_o \right) \left( 1 + \alpha X_0^2 \right) + \left( 2 a L_o X_0 \right)^2 \\
D = R_b \left( k = a L_o \right) \left( 1 + \alpha X_0^2 \right) \\
\]

**SOLVING FOR ROOTS OF THE CUBIC**
DIMENSION C(4), COF(4), RR(3), RI(3)
WRITE (5,12)
READ(2,14)R,XLO,XM,XO,XIO,B,A,XKS
XL=XLO*(1.+A*XO*XO)
C(4)=XM*XL
C(3)=B*XL+R*XM
C(2)=R*B+XL*(XKS-A*XLO*XIO*XIO)+(2.*A*XLO*XIO*XIO)**2.)
C(1)=R*(XKS-A*XLO*XIO*XIO)
CALL POLRT(C,COF,3,RR,RI,IER)
DO 21 M=1,4
WRITE(5,13)M,C(M)
21 CONTINUE
DO 20 J=1,3
WRITE(5,15)J,RR(J),RI(J)
20 CONTINUE
12 FORMAT('1')
13 FORMAT(' C(1,12,1)',E15.8)
14 FORMAT(8F5.3)
15 FORMAT(' ROOT ',I2,'=',E15.8,' REAL ',E15.8,' IMAG')
STOP
END

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END OF Compilation

// LOAD
C( 1)= 0.50000101E 05
C( 2)= 0.50050397E 03
C( 3)= 0.10309990E 02
C( 4)= 0.99899929E-01

ROOT 1=-0.50002090E-01 REAL, 0.70710739E 02 IMAG
ROOT 2=-0.50002090E-01 REAL, -0.70710739E 02 IMAG
ROOT 3=-0.10010011E 03 REAL, 0.00000000E 00 IMAG
S 32 STOP 0000

EXECUTION TIME 0111
Thus, \( A S^3 + BS^2 + CS + D = (S - R_f)(S - R_c)(S - R_c^*) \)

\[
\hat{k}^{-1}\left\{ \frac{1}{(S - R_f)(S - R_c)(S - R_c^*)} \right\} = \frac{(R_c - R_c^*) e^{R_c t} + (R_c^* - R_f) e^{R_c^* t} + (R_f - R_c) e^{R_f t}}{(R_c - R_f)(R_c - R_c^*)(R_c^* - R_f)}
\]

Let \( R_c = a + jb \) and \( R_f = c \)

\[
\hat{k}^{-1}\left\{ \frac{1}{(S - R_f)(S - R_c)(S - R_c^*)} \right\} = \frac{j2b e^{ct} + [(a-c) - jb] e^{(a+jb)t} - [(a-c) + jb] e^{(a-jb)t}}{j2b [(a-c) + jb][(a-c) - jb]}
\]

\[
= \frac{j2b e^{ct} + [(a-c) e^{(a+jb)t} - jbe^{(a+jb)t} - (a-c) e^{(a-jb)t} - jbe^{(a-jb)t}] + 2jbe^{(a-jb)t}}{j2b [(a-c)^2 + b^2]}
\]

\[
= \frac{j2be^{ct} + e^{at}[(a-c)(e^{ibt} - e^{-ibt})] - jb(e^{ibt} + e^{-ibt})}{j2b [(a-c)^2 + b^2]}
\]

\[
= \frac{e^{ct} + e^{at}[(a-c) \sin bt - \cos bt]}{[(a-c)^2 + b^2]}
\]

Thust

\[
\hat{k}(t) = \hat{k}^{-1}\{\alpha E_1 \left[ \frac{1 - e^{-st}}{(S - R_f)(S - R_c)(S - R_c^*)} \right] \} = \frac{\alpha E_1}{[(a-c)^2 + b^2]} \left\{ \{ e^{ct} + e^{at} \left( \frac{a-c}{b} \sin bt - \cos bt \right) \} \mu(t) - \{ e^{ct} + e^{at} \left( \frac{a-c}{b} \sin b(t-T) - \cos b(t-T) \right) \} \mu(t-T) \right\}
\]

\[
\hat{k}(t) = (-1.33 \times 10^{-8}) E_1 \left\{ \{ e^{-100t} + e^{-0.05t} \left( 1.41 \sin (70.7) t - \cos (70.7) t \right) \} \mu(t) - \{ e^{-100(t-T)} + e^{-0.05(t-T)} \left( 1.41 \sin (70.7)(t-T) - \cos (70.7)(t-T) \right) \mu(t-(t-T)) \right\}
\]

\[
\hat{k}(t) = \{ e^{-100(t-T)} + e^{-0.05(t-T)} \left( 1.41 \sin (70.7)(t-T) - \cos (70.7)(t-T) \right) \mu(t-(t-T)) \}
\]
\[ \begin{aligned}
\text{THUS} \quad A_5^3 + B_5^3 + C_5 + D &= (S - R_c)(S - R_c)(S - R_c^*) \\
\quad &= \left( \frac{1}{(S - R_c)(S - R_c)(S - R_c^*)} \right) = \frac{(R_c - R_c^*)e^{R_c^*t} + (R_c^* - R_c)e^{R_c t} + (R_c - R_c^*)e^{R_c^*t} - (R_c^* - R_c)e^{R_c t}}{(R_c - R_c)(R_c - R_c^*)(R_c^* - R_c)} \\
\text{LET} \quad R_c = a + jb \quad \text{AND} \quad R_c^* = a - jb
\end{aligned} \]

\[ \begin{aligned}
\Rightarrow \quad L^{-1}\left\{ \frac{1}{(S - R_c)(S - R_c)(S - R_c^*)} \right\} &= \frac{j2be^{ct} + [(a-c) - jb]e^{(a+jb)t} - [(a-c) + jb]e^{(a-jb)t}}{j2b[(a-c)^2 + b^2]} \\
&= \frac{j2be^{ct} + (a-c)e^{(a+jb)t} - jbe^{(a+jb)t} - (a-c)e^{(a-jb)t} - jbe^{(a-jb)t}}{j2b[(a-c)^2 + b^2]} \\
&= \frac{j2be^{ct} + e^{at}[(a-c)(e^{jbt} - e^{-jbt})] - jbe^{jbt + e^{-jbt}}}{j2b[(a-c)^2 + b^2]} \\
&= \frac{e^{ct} + e^{at}[(a-c)\sin(bt) - \cos(bt)]}{[(a-c)^2 + b^2]}
\end{aligned} \]

\[ \begin{aligned}
\text{THUS} \quad \dot{x}(t) &= L^{-1}\left\{ \alpha E_i \left[ 1 - e^{-st} \right] \right\} \\
&= \frac{\alpha E_i}{(a-c)^2 + b^2} \left\{ e^{ct} + e^{at} \left( \frac{a-c}{b} \sin(bt) - \cos(bt) \right) \right\} \mu(t) \\
&\quad - \left\{ e^{c(t-T)} + e^{a(t-T)} \left( \frac{a-c}{b} \sin(b(t-T)) - \cos(b(t-T)) \right) \right\} \mu(t-T)
\end{aligned} \]

\[ \begin{aligned}
\dot{x}(t) &= (-1.33 \times 10^{-5}) E_i \left\{ e^{-0.05t} + e^{0.05t} \left( +1.41 \sin(70.7)t \right. \right. \\
&\quad \left. - \cos(70.7)t \right) \right\} \mu(t) - \left\{ e^{-0.05(t-T)} + e^{0.05(t-T)} \left( +1.41 \sin((70.7)(t-T)) \right. \right. \\
&\quad \left. - \cos((70.7)(t-T)) \right) \right\} \mu(t-T)
\end{aligned} \]
1. Energy:

\[
\text{Force} = -\frac{\Delta \text{Work}}{\Delta \text{X}} = -\frac{0.61 - 0.189}{0.05 - 0.01} = -842 \text{ Newtons}
\]

2. According to Fig. 9-15 and the corresponding discussion:

\[
\begin{align*}
\alpha &= a(l^2-x^2), \quad l \geq x \geq 0, \quad a = \frac{N^2 \cdot \alpha_0}{2g} \quad (a = b = c), \\
b &= b(l-x), \quad l \geq x \geq 0, \quad b = \frac{N^2 \cdot \alpha_0}{2g} \quad (\text{the height function is a line}) \\
c &= c(l-x), \quad l \geq x \geq 0, \quad c = \frac{N^2 \cdot \alpha_0}{2g}.
\end{align*}
\]

Note that the negative sign on \(\alpha\) follows because the assumed position derivative \(\alpha_a(t), \alpha_b(t)\) in Fig. P 11-13, P 11-70 produces kinetic friction.

According to (4-147):

\[
f = \frac{\partial w_m}{\partial x} = \frac{\alpha^2}{2} \frac{d \alpha_a}{dx} + \alpha \frac{d \alpha_b}{dx} + \frac{k}{2} \frac{d \alpha_b}{dx} + \frac{k}{2} \frac{d \alpha_b}{dx} + \frac{f_i}{2} \times (2 \alpha x - \alpha x^2)
\]

Equation of dynamics: \(f = -m^2 \alpha x + a(2x-2) \alpha_a(x) + a \alpha_b(x)(l-x),\)

\[
= \begin{align*}
\alpha \frac{d^2 x}{dt^2} + D \frac{dx}{dt} + K x \\
U_a &= \frac{\alpha_a^2}{2} \\
U_b &= \frac{\alpha_b^2}{2}
\end{align*}
\]
\[ V_0 = I_0 R, \quad V_0 = I_{0b} R \Rightarrow I_{0b} = I_0 - \frac{V_0}{R}. \]

For electromagnetic force balance under d.c. conditions, there becomes

\[ F = -I_0^2 a x_0 + a (2x_0 - 1) I_0^2 + a I_0^2 (l - x_0) \geq 0, \]

\[ 0 = k x_0 = \left[ \begin{array}{c} x_0 = 0 \end{array} \right]\]

This result follows from Fig. 14 during the moment, with hinging phases, the currents will not produce net force per current by moving from the position \( x = 0 \). If, however, the contact was used as in Fig. 15, with a positive mutual inductance (\( M = 0 \) for phases connected) the \( I_{ab} = \frac{b}{a} (l - x) \), and the middle term above in the force expression would have the opposite sign.

\[ f = -I_0^2 a x_0 - a (2x_0 - 1) I_0^2 + a I_0^2 (l - x_0) = -l a I_0^2 (l - x_0) \]

which means that the d.c. force balance becomes

\[ -l a I_0^2 (l - x_0) + 2a l I_0^2 = k x_0 \Rightarrow x_0 = \frac{2a l I_0^2}{k + 2a l I_0^2}. \]

Now, if \( k = 0 \) (no spring), one that the quiescent position \( x_0 = \frac{l}{2} \)

we would have anticipated on the basis of the symmetrical system.

We continue with the negative mutual inductance, after some algebra, utilizing the facts that \( I_{0b} = I_{0a} \), \( I_a = I_{0a} - I_0 \), \( x_0 = 0 \), we find for the a.c. equations

\[ I_0 (t) = I_a x_0 + al a \frac{dx_0}{dt} \]

\[ V_0 (t) = I_0 x_0 + al a \frac{dx_0}{dt} \]

\[ 0 = I_0 (x_0(t) - x_0(t)) + a x_0 \frac{dx_0}{dt} + \frac{dx_0}{dt} + k x_0 \]

Note the differential equation above we finally note that \( I_{0b} = a x_0 \) for \( x = 0 \), above the point of the peak b.c. \( x_{0b} = 0 \) for \( -b < x < 0 \), \( I_{0b} = 0 \) for \( x < 0 \). These results follow from the identical magnetic circuit model used in the d.c. case.
\[ E = E_{\text{in}}(t), \quad E^2 = I^2 \sin^2 \omega t = \frac{E_{\text{in}}^2}{2} \left( 1 - \cos 2\omega t \right) \]

\[ L = L_{\text{in}} \left[ L_2 \sin(\omega t + \phi) + 3L_b \sin(\omega t + 4\phi) \right]. \]

\[ = -L_{\text{in}} \left[ L_2 \sin(\omega t + \phi) + 3L_b \sin(\omega t + 4\phi) \right] \]

\[ = -L_{\text{in}} \left[ \frac{L_2}{2} \phi \sin(\pi + 2\omega t + 4\phi) + \phi \sin(\pi - 2\omega t + 4\phi) \right] \]

\[ = -3L_b \phi \sin(2\pi - 2\omega t + 4\phi) + \phi \sin(2\pi - 2\omega t + 4\phi) \]

\[ \omega_{m} = \frac{\omega}{2}, \frac{\omega}{6} \]

In the first case the machine under a 4-pole (2 pair of poles) machine, while in the second it behaves as a 12-pole (6 pair of poles) machine.

\[ Z_{12} = \text{the driving point impedance of pole-1, measured with pole-2 short-circuited (} V_i = 0 \text{) or open-circuited (as conventional)} \]

\[ Z_{12} = \text{the voltage transfer ratio between pole-1 and pole-2, measured with } I_{12} = 0 \text{ (open-circuit at pole-1).} \]

\[ Y_{12} = \text{the current transfer ratio between pole-1 and pole-2, measured with } V_{12} = 0 \text{ (short-circuit at pole-1).} \]

\[ Z_{12} = \text{the driving point admittance of pole-2 and 1, measured with} \]

\[ \psi \text{ (open-circuit at pole-1).} \]
\( I_2'' \) and clearly, 
\[ \begin{align*} \phi_{12} &= B_1 \cos \theta \end{align*} \]

\( P_{12} \), open port 2. \( P_{12} \), clearly, 
\[ \begin{align*} H_{21} &= 0 \end{align*} \]

\( Y_{22} \), open port 1. \( Y_{22} \), clearly, 
\[ \begin{align*} Y_{22} &= \frac{s \epsilon + 2 + \frac{1}{s \epsilon}}{s \epsilon} \end{align*} \]

\( V_1 = I_1 R_{11} + V_2 G_{12} \Rightarrow V_1 I_1^* = I_1 I_1^* + V_2 G_{12} I_1^* \)

\( T_2 = I_1 H_{21} + V_2 Y_{21} \Rightarrow V_2 I_2^* = V_2 I_1^* H_{21} + V_2 I_2^* Y_{21} \)

\[ \frac{1}{2} \Re \left( V_2 I_2^* + V_2 I_1^* \right) = \frac{1}{2} \Re \left[ I_1 I_1^* + V_2 I_2^* + V_2 I_2^* G_{12} I_1^* \right] \]

This must hold for all \( V_2 \) and \( I_1 \). Hence, assume \( V_2 = 0 \), and 
that port 2 is not excited, \( \text{max.} \). 
\[ \frac{1}{2} \Re [I_1 I_1^* + V_2 I_2^* + V_2 I_2^* G_{12} I_1^*] = 0 \] 

Since, \( \left[ \begin{array}{c} \phi_{21} \\ \phi_{12} \end{array} \right] = j X_1 \) is purely reactive \( (X_1) \) is real, either positive or 

Next, assume that \( I_1 = 0 \), \( \text{max.} \), that port 1 is not excited, \( \text{max.} \). 
\[ \frac{1}{2} \Re [V_2 I_2^* + V_2 I_2^*] = \frac{1}{2} \Re \left\{ \frac{V_2 I_2^* - V_2 I_2^*}{2} \right\} = 0 \] 

Finally, assume that 
\[ V_2 \to 1 \text{ volt} \text{ (phase shift 0)} \text{ and } I_1 = 1 \text{ amp} \text{ (phase angle 0)} \text{.} \] 

\[ \begin{align*} &\text{Finally, assume that} \quad V_2 \to 1 \text{ volt (phase shift 0)} \quad \text{and} \quad I_1 = 1 \text{ amp (phase angle 0).} \\ &\text{Finally, assume that} \quad V_2 \to 1 \text{ volt (phase shift 0)} \quad \text{and} \quad I_1 = 1 \text{ amp (phase angle 0).} \end{align*} \]
\[ P = \frac{1}{2} \text{Re} \left[ \frac{V_1 X_1}{V_1 X_1 + V_2 Y_2 (G_{12} + H_{12}^*)} \right]. \]

\[ = \frac{1}{2} \text{Re} \left[ \frac{1}{1 + (G_{12})^*} \right]. \]

\[ = \frac{1}{2} \text{Re} \left[ \frac{1}{G_{12}^* + 1} \right]. \]

\[ = \frac{1}{2} \text{Re} \left[ \frac{1}{G_{12} + H_{12}^*} \right]. \]

\[ = \frac{1}{2} \text{Re} \left[ \frac{1}{G_{12} + H_{12}^*} \right] = 0. \]

Hence, we have \( \text{Re} \left[ G_{12} + H_{12}^* \right] = \text{Re} \left[ G_{12} + H_{12}^* \right] = 0. \) Now,

\[ G_{12} + H_{12}^* = 0 \] (because both real and imaginary parts of \( G_{12} \) and \( H_{12} \) cannot have real parts, which implies an entirely reactive network.)

(7) Equivalent circuit (result from mechanical parts).

\[ V_1 = 300 V, \quad X_1 = -0.002. \]

\[ \text{By pole-wise matching, we have} \]

\[ V_1 = 300 V, \quad X = \frac{F(s)}{0.18 + 0.01 + \frac{1}{0.002 s}}. \]

\[ = \frac{10 F(s)}{s^2 + 0.18 s + 1000}. \]

\[ K_{nc} = s^2 \cdot 10.15 + 0.05 = (s + 0.05 + j \tau_07)(s + 0.05 - j \tau_07). \]

\[ = 10 F_s (1 - e^{-57}) \]

\[ = 10 F_s \left( \frac{1}{s^2 + 0.18 s + 1000} \right) \]
\[ X_i(S) = \frac{10 \cdot F_i(1 - e^{-S})}{(5 + 0.05 + j0.7) \cdot (5 + 0.05 - j0.7)} \]

\[ = \frac{j \cdot F_2(0.02007 \cdot e^{-2S})}{5 + 0.05 + j0.7} - \frac{j \cdot F_2(0.02007 \cdot e^{-2S})}{5 + 0.05 - j0.7} \]

\[ - \frac{j \cdot F_2(0.02007 \cdot e^{-2S})}{5 + 0.05 + j0.7} + j \cdot F_2(0.02007 \cdot e^{-2S}) \cdot \frac{e^{j0.7S}}{5 + 0.05 - j0.7} \]

\[ v_i(t) = -500 V_i(S) - (0.0147) F_2 e^{-0.054t} \cdot \sin(0.0257t) + (0.1411) F_2 e^{-0.054(4-t)} \cdot \sin(0.0257(4-t)) \]

\[ v_i(t) = 0.1283 \cdot 10^{-4} F_2 \left[ e^{-0.054t} \cdot \sin(0.0257t) - e^{-0.054(4-t)} \cdot \sin(0.0257(4-t)) \right] \]

\[ v_i(t) \text{ is a slightly damped oscillation with } \omega = 0.257. \]

\[(8)\]

Using the same method as in (7),

\[ X_i(S) = \frac{1}{0.15 + 0.01 + \frac{1}{0.0025}} \cdot (-500 V_i(S)) \]

\[ = \frac{-0.02 V_i(S)}{S^3 + 1000S^2 + 5000S + 5000000} \]

\[ = \frac{-0.02 V_i(S)}{S^3 + 1000S^2 + 5000S + 5000000} \]

\[ v_i(t) = 0.02 V_i(S) \cdot \frac{1}{S^3 + 1000S^2 + 5000S + 5000000}. \]
\( \chi(s) = \frac{-0.02 \cdot E_i (1-e^{-1T})}{s (s+100)(s+70.7)(s+70.7)} \)

where

\( \chi(t) = -2 \cdot 10^{-8} E_i \left\{ \left[ 2 - 0.67 e^{-100t^2} - 2 \sqrt{\frac{\alpha}{3}} \, \cos \left( \frac{70.7t^2}{2} \right) \right] + \right. \\
- \left. \left[ 2 - 0.67 e^{-100(t-0.7)^2} - 2 \sqrt{\frac{\alpha}{3}} \, \cos \left( \frac{70.7(t-0.7)^2}{2} \right) \right] \right\} \)

It is a small matter to verify that \( \chi(0) = 0 \), and this is because if we write the current at \( t = 0 \) as a step in time of \( 1 \) of this voltage with a step in time of \( 0 \) at \( t = 1 \) (the left-hand part) the entire voltage drop initially will be across the 2.5 V battery and, therefore, the \( \chi(t) \) at \( t = 2 \) at the initial instant.

An important observation to make regarding graphs (1) and (2) is that the same device can be used as a device to withstand (generator) and withstand (motor) transients respectively, as a withstand for winter (winter) transients.
PROBLEM: A SHUNT MOTOR WITH THE FOLLOWING CONSTANTS
WITH AN A-C VOLTAGE: 340 MIN 377 \text{V}. FIND THE
AVERAGE TORQUE. REPEAT, ASSUMING D-C. EXCITATION
@ 240 \text{V (RMS OF 340 V)}
\( R_f = 50, L_{ff} = 25, k_f = 0.9, R_a = 0.1; L_q = 0.01, J_m = 10 \text{ kg - m}^2 \)

EQUIVALENT CIRCUIT:

\[ \begin{align*}
V_q &= V_f = V, \quad \omega_m = 0 \\
D.C. \quad I_f &= \frac{V_f}{R_f}; \quad I_q = \frac{V_q}{R_a} = \frac{240}{50} = 4.8 \text{ A (RMS)} \\
&= \frac{80}{g} = 7.99 \text{ A (RMS)} \\
T &= k_f i_f i_a = (0.9)(4.8)2400 = 1.83 \times 10^4 \text{ n.t.-m.} \\
V_q \\
A.C. \quad V_q &= i_q (R_a + j \omega L_a) \Rightarrow i_q = \frac{R_a + j \omega L_a}{240} \\
&= \frac{R_a}{L_a} + \frac{j \omega L_a}{240} \\
&= \frac{240}{3.77} = 62.4 \text{ A (RMS)} \\
\Rightarrow \quad \omega_a &= k_f i_q i_f = \frac{1.03 \times 10^4}{62.4} \text{ n.t.-m.} \\
&= 1.64 \text{ N-m} \\
\end{align*} \]
CIRCUITS USED

a) SELF-EXCITED

b) SEPERATELY EXCITED

c) SERIES

d) CUMMULATIVE COMPOUND GENERATOR

e) DIFFERENTIALLY COMPOUND
2) FOR A SEPARATELY EXCITED GENERATOR:

\[ V_q = E_q - R_q I_q \]  
\[ 5E_q \]  
\[ \text{VOLTAGE FROM CONSTANT FIELD EXCITATION.} \]

This however is an idealized equation. The data from lab yields a "curved" line, as opposed to the proposed ideal straight line. The curve is due to saturation, which reduces pole flux, reducing the mutual inductance \( M_{ij} \) with an increase on load.

The regulation, defined as:

\[ \text{Reg} (\%) = \left( \frac{V_{\text{no load}} - V_{\text{full load}}}{V_{\text{FL}}} \right) \times 100\% \]

Can be computed as:

\[ \text{Reg} (\%) = \left( \frac{17.5 - 16.5}{16.5} \right) 100\% \]

\[ = 6.1\% \]

Saturation causes the vertical part of the slope, armature reaction is caused causes the linear resistance drop, from when the brushes and flux are in line, the "Back Swing" is a result of both saturation and overload.

b) STEADY STATE EQUATION FOR SHUNT GENERATOR:

\[ I_f = \frac{V}{R_f} \]

\[ E_q = K_q I_q \]

\[ W_m = V_q + R_q I_q \]

\[ V = V_q = V_f \]

\[ \Rightarrow V_q \uparrow \rightarrow I_q \downarrow ; I_f \downarrow \rightarrow V_q \downarrow , E_q \downarrow \rightarrow I_f \downarrow \]

Thus causing dropop in \( V_q-I_q \) plot, than with separately excited generator.
\[ V_q = k_0 \omega \left( \frac{V_a}{V_p} \right) - R_a \left( \frac{V_p + V_q}{R_L} \right) \]

Upon solving for \( V_q-I_q \) relation, instability occurs, as caused by \( R_L \), \( V_q-I_q \) is thus non-linear, for \( \omega \) constant. Residual magnetism is needed for starting, thus the curve does not begin at the origin (5.5 V residual magnetism). Because of this, there is a bend in the \( V-I \) curve near the origin.

C) Compound Generators have both shunt and series windings. When both fluxes are in the same direction, it is commutatively compound. The latter system increases excitation, counteracting the knee of the self excited gen.

For this system,

\[ E = k_1 \omega m I_f + k_2 \omega m I_q \]
\[ V = E - I_q R_q = k_1 \omega m I_f + (k_2 \omega m + k_0) I_q \]

In the differentially compound gen., flux fields of the shunt and series coils back each other again causes a droop in the plot, larger than before.

\[ V = k_1 \omega m I_f - (k_2 \omega m + R_q) I_q \]

For shunt droop, \( E = k_1 \omega m I_f \)

Dif. Comp. Droop: \( E = k_1 \omega m I_f - k_2 \omega m I_q \)
3) Commutating windings reduce arcing and brush, commutator, and coil heating, as well as reducing commutator pitting caused by high voltages and currents in the circuit.

Induced voltage: \( E = L \frac{dI}{dt} \)

Adding a set of commutating poles to the quad axis introduces a new set of coils, such that armature currents flow in such a manner as to introduce an equal and opposite voltage creating a net voltage of 0.

4) A good voltage source generator would be the separately excited shunt, for it has the best voltage regulation. Care, however, must be taken not to overload. Conversely, the differentially compounded machine acts as a fair current source over a certain load range. The cumulative compounded motor is a good voltage source. The separately excited DC motor's torque may require a hazardous speed. It would be good, however
For starting high inertial loads, do to its high torque capabilities, the shunt motor has similar problems. (High speeds @ low field currents), the series motor has a high starting torque that levels out with speed, this is nice for variable speeds, but dangerous under light loads because of ill torque. The differential compounded motor’s field becomes weak when loaded, and may thus run away (a bummer).

5) When the brushes moved, the generator strained under load as the machine went into quadrature. I_d went up and I_q went down, since the fluxes opposed. Thus, the output dropped.
DATA AND PLOTS

EXTRAVAGANT POINTS WERE ADDED
WITH YOU AND YOU FOR SCALING.
PURPOSES OR BY ERROR, AND
ARE ACCORDINGLY SNIPPED.
20 FORMAT('7DATA C',/,'5X','I','19X','V')
21 FORMAT('7DATA U',/,'5X','I','19X','V')
22 FORMAT('7DATA F',/,'5X','I','19X','V')
23 FORMAT('7DATA G',/,'5X','I','19X','V')
24 FORMAT('7DATA G',/,'5X','I','19X','V')
25 FORMAT('6X,F10.6,5X,F10.6')
END

PROGRAM LENGTH INCLUDING I/L BUFFERS
003072

UNUSED COMPILER SPACE
002500
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<th>V</th>
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**Shunt**  
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**SHUNT**

*(LOADED)*
### Data C

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SELF EXCITED
(WITH LOAD)
SELF EXCITED
(NO LOAD)
### Differentially Compounded

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*CUMULATIVELY COMPOUNDED*
EE 353. ENERGY CONVERSION

LAB PROJECT NO. 1

1. Sketch the circuit used in the demonstration for the open circuit test.

2. Repeat (1) for the short circuit test. Be sure to label voltage and current values.

3. Calculate all parameters of the transformer using the data obtained in the demonstration.

4. Draw the equivalent circuit of the transformer as viewed from the high tension side.

5. Repeat (4) for the low tension side.

6. Draw the approximate equivalent circuit.

7. If the transformer is supplying full load at 120 volts and 0.8 power factor lagging, what is the applied voltage?

8. What is the % regulation?

9. What is the efficiency?

10. Assume the load to vary but that its power factor remains constant; at what load does maximum efficiency occur and what is its value?
FOR S.E. TEST

\[ Z = \frac{3V}{17A} = 1.7647 \Omega \quad P.F. = \frac{40}{\theta_3} = 7.84313 \]

\[ \text{Re} = Z \cdot (\text{P.F.}) = 1.38407 \quad (\alpha = 66.92\text{rad}, \text{radians} = 38.31247\text{°}) \]

\[ X_{eq} = \frac{Z \cdot \sin (\text{tan}^{-1} \text{P.F.})}{\text{P.F.}} = 1.10976 \]

FROM O.C. TEST

\[ P.F. = 10.94 \quad \Theta_1 = 1.357 \quad \text{RAD} = 77.77° \]

\[ P.F. = 1.83 \quad \Theta_2 = 1.424 \quad \text{RAD} = 81.39° \]

\[ P.F. = 4.25 \quad \Theta_3 = 1.136 \quad \text{RAD} = 65.11° \]

\[ P.F. = 4.0 \quad \Theta_4 = 1.22356 \quad \text{RAD} = 70.10° \]

\[ I_{he} = 1.0 \quad P.F. = 1.19 \quad P.E. = 2.521 \quad \Gamma_i = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 1.163 \text{A} \]

\[ I_{he} = 0.9 \quad P.E. = 2.257 \quad I_{qh} = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 0.9778 \text{K} \]

\[ I_{he} = 0.54 \quad P.E. = 2.275 \quad I_{qh} = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 0.4839 \text{A} \]

\[ I_{he} = 0.64 \quad P.E. = 2.246 \quad I_{qh} = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 0.6269 \text{A} \]

\[ X_{m1} = \frac{\sin (\text{tan}^{-1} \text{P.F.})}{\text{P.F.}} = 1.02322 \Omega \quad X_{m2} = X_{m1} \left( \frac{360}{2} \right)^2 = 1.43890 \Omega \]

\[ X_{m3} = \frac{\sin (\text{tan}^{-1} \text{P.F.})}{\text{P.F.}} = 0.76427 \Omega \quad X_{m4} = X_{m3} \left( \frac{360}{2} \right)^2 = 0.99038 \Omega \]

\[ X_{m5} = \frac{\sin (\text{tan}^{-1} \text{P.F.})}{\text{P.F.}} = 1.76427 \Omega \quad X_{m6} = X_{m5} \left( \frac{360}{2} \right)^2 = 2.39232 \Omega \]

\[ X_{m7} = \frac{\sin (\text{tan}^{-1} \text{P.F.})}{\text{P.F.}} = 4.72032 \Omega \quad X_{m8} = X_{m7} \left( \frac{360}{2} \right)^2 = 6.53794 \Omega \]

\[ R_{he1} = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 4.72032 \Omega \quad R_{he2} = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 4.72032 \Omega \]

\[ R_{he3} = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 3.36966 \Omega \quad R_{he4} = \frac{1}{\text{tan}^{-1} \text{P.F.}} = 4.16298 \Omega \]

HIGH TENSION SIDE

\[ \frac{1}{2} R_{eq} = 0.0692 \Omega \quad \frac{1}{2} X_{eq} = 0.0542 \Omega \quad 450V \rightarrow \frac{1}{2} X_{eq} = 0.0542 \Omega \quad \frac{1}{2} R_{eq} = 0.0692 \Omega \]

\[ R_{n,he1} = 0.0692 \Omega \quad X_{m1} = 0.0542 \Omega \quad 450V \rightarrow 1.0625 \Omega \]

LOW TENSION SIDE

\[ 0.0692 \Omega \quad 0.0542 \Omega \quad 0.0692 \Omega \quad 0.0692 \Omega \]

\[ R_{n,he} \quad R_{m} \quad 0.0542 \Omega \]

\[ 120V \]

\[ 0.071111 \text{V} \]

\[ R_{he} \quad X_{m} \]

\[ \text{SEE} \quad \text{SEE} \]
XFMN 450/120; 2 KVA

O.C. TEST

<table>
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<th>V (v)</th>
<th>I (A)</th>
<th>P (w)</th>
</tr>
</thead>
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<td>1.19</td>
<td>30</td>
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<tr>
<td>133</td>
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<tr>
<td>93.5</td>
<td>0.64</td>
<td>24</td>
</tr>
</tbody>
</table>

S.C. TEST

| 3     | 17    | 40    |

CIRCUIT FOR SC TEST:
\[ \frac{170}{200} \]
Did you know a 100-watt bulb gives 50 percent more light than four 25-watt bulbs.

EDUCATION

You should have education only so that you won't know is to talk to people; and then more education so that you will be wise enough not to look down on people.

— Max L. Brewer
The Art of Living Successfully
2. Prove (i) for the maximum efficiency problem, where $F_{T_k} = 1$.

3. (a) Prove the differential equation $F_{T_k} = \frac{d^2}{dt^2} + \frac{\gamma}{\mu} \frac{d}{dt} + \alpha$, subject to a given (fixed) angle of rotation $\alpha$.

4. (b) Prove the boundary conditions yielding, under the same conditions as in (a), the displacement $\alpha$, for given torque losses, $\gamma$.

5. In both (a) and (b) determine the hysteresis losses, under the same instance, $L_{h} \neq 0$ and (b) $L_{h} = 0$.

6. (a) The notation $\Pi$ corresponding to (21) for use and is denoted $\Pi$,

   $\Pi = \frac{1}{\mu} \frac{d^2}{dt^2} + \frac{\gamma}{\mu} \frac{d}{dt} + \alpha$

   coefficients $a, b, c$ are

   $\gamma, \beta, \alpha$ are Lagrange multipliers.
(4) (a) Problem 15-2, text.
    (b) "  15-3, text.

(5) Starting with (1-226) for $F_y$ (p. 74) of the notes on linear induction machines, develop a mechanical "small-signal" model for vertical motion of the linear induction machine. Include gravity, levitation force and inertial reaction (in the vertical direction). Discuss the small displacement motion about a quiescent gap length, $s_0$, for two values of slip:

(a) $s < s_0$  and  (b) $s > s_0$.

Hint: Let $g = G + \delta$, where $\delta$ is the small a.c. signal, and expand $F_y$ (equ. 1-226) in a Taylor series keeping the d.c. and linear (in $\delta$) terms. Solve for the d.c. operating point. Use a spring-mass-gravity model. What is the spring constant? Also assume electrical voltages to be fixed (do not worry about the electrical transients).

The figure shows the model of an infinitely long linear induction motor with a double sided stator configuration. The rotor is symmetrically placed in the air-gap.

(a) Set up vector-potential equations in the air gap and rotor and solve for the vector potential (assume an electrically thin rotor, i.e., $|x/A| \ll 1$).

Hint: Use the symmetry of the model to simplify your calculations.
6. (b) Determine the air-gap flux densities.

(c) Calculate the net time-average surface traction on both sides of the rotor using the Maxwell stress tensor. Use an area of \( \mathcal{L} A \) in \( z \)- and \( x \)-directions.

**Answer:**

\[
F_x = \frac{4}{\mathcal{L} A \mu_0} \frac{\int |V|}{V_{sym}} \left[ \frac{\sinh \frac{h_2}{2} \cosh \frac{h_2}{2}}{(\cosh \frac{h_2}{2})^2 + (\frac{\sinh h_2}{2})^2} \right] \left( \frac{SRm}{2} \right)
\]

\[
F_y = 0.
\]

(d) Show that there is a normal force acting on each stator face, tending to push the stator apart, of amount

\[
F_y = \frac{1}{\mathcal{L} A \mu_0} \frac{\int |V|}{V_{sym}} \left[ \frac{1 - \left( \frac{SRm}{2} \right)}{\cosh \frac{h_2}{2}^2 + \left( \frac{\sinh h_2}{2} \right)^2} \right]
\]

(e) Compare (c) with the single-sided stator discussed in class. Compare the results of (c) and (d), relative to the \( y \)-directed forces, and explain the differences.

7. Lab demonstration problem.
SUGGESTION FOR PROB. 4 (Part 2 of Take-Home Final).

For \( f = 220 \text{ Hz} \),

\[
\frac{V_{yn}}{\nu_{yn}} = 300 \text{ m/hr} = 134.1 \text{ m/sec}.
\]

\[
\frac{\gamma}{\delta} = \frac{5/8 \text{ inch}}{1.59 \text{ cm}}.
\]

- Conductivity of copper \( \sigma = 5.9 \times 10^7 \text{ mho/meter} \),
  - Conductivity of aluminum \( \sigma = 3.72 \times 10^7 \text{ mho/meter} \),

\[
M_{0} = 4\pi \times 10^{-7} \text{ H/m}.
\]

- Gap \( g = 11/3 \text{ inch} = 3.5 \text{ cm} \). (The total stator-stator gap \( g + \delta = 2\gamma \)).

\[
\delta = \text{wavelength} = 24 \text{ inches} = 0.61 \text{ meter}.
\]

Make any comments regarding peak force, starting torque, etc., to you think will impress me. (The usual rules about proficiency remain in force).
I) An analog for obtaining the H function for use in Euler's equation for optimality is obtaining the parameter to be optimised \( F \) in the form \( F = \int_{0}^{T} \frac{\dot{\lambda}}{\lambda} dt \). The restrictions on \( F \) are also put into similar form:

\[
\dot{\xi}_{n} = \int_{0}^{T} \lambda_{n} G_{n}(t) dt.
\]

The equation becomes:

\[
H = F(t) + \sum_{n=1}^{m} \lambda_{n} G_{n},
\]

where \( F \) and \( G_{n} \) are functions of time and the optimum parameter to be found, \( \lambda_{n} \) is the number of restrictive equations, and \( \lambda_{n} \) is a Lagrange multiplier.

In the problem, \( T \) is to be optimised,

\[
T = \int_{0}^{T} dt \Rightarrow F = 1.
\]

The restrictions are \( \alpha \) and \( \beta \):

\[
\alpha = \int_{0}^{T} \omega(t) dt \Rightarrow G_{\alpha} = -\omega(t).
\]

\[
G_{\beta} = P_{\beta} \int_{0}^{T} \lambda^{2}(t) dt \Rightarrow G_{\beta} = \lambda^{2}(t).
\]

\[
G_{1} = \frac{\dot{J}_{3}^{2}(t)}{K_{3} L_{3}} \left( \frac{d\omega}{dt} + J_{0} \right)^{2} + \text{from } K_{3} J_{3} \omega(t) = \int \frac{d\omega}{dt} + J_{0}.
\]

\[
\Rightarrow H = F + \lambda_{1} G_{\alpha} + \lambda_{2} G_{\beta}.
\]

\[
= 1 + \lambda_{1} \left( \frac{J_{3}^{2}}{K_{3} L_{3}} \right) \left( \frac{d\omega}{dt} + J_{0} \right)^{2} + \lambda_{2} \omega \checkmark
\]

The problems such as those worked in notes 8 and 9 in optimum control and problems 3 and 7 in this text.
\[ H = 1 + \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 + \lambda_2 \omega \]

**Euler's Equation:***

\[ \begin{align*}
\frac{dH}{d\gamma} - \frac{dH}{d\gamma} (\lambda_1) &= 0 \\
\frac{dH}{d\gamma} &= \frac{dH}{d\gamma} \left[ 1 + \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 + \lambda_2 \omega \right] \\
&= \lambda_2 + \frac{dH}{d\gamma} \left[ \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 \right] \\
&= \frac{\lambda_2}{\gamma} \left[ \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 \right] \\
\frac{dH}{d\gamma} &= 0 \quad \text{or Notes on Optimist Control}
\end{align*} \]

\[ \begin{align*}
\frac{dH}{d\gamma} &= \lambda_2 \\
\frac{dH}{d\gamma} &= \frac{dH}{d\gamma} \left[ 1 + \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 + \lambda_2 \omega \right] \\
&= \lambda_2 + \frac{dH}{d\gamma} \left[ \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 \right] \\
&= \lambda_2 \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 \\
\frac{dH}{d\gamma} &= \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2
\end{align*} \]

**Equation of Motion:**

\[ \begin{align*}
\frac{d^2 \phi}{d\gamma^2} &= \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 + \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 + \lambda_2 \omega \]
\end{align*} \]

\[ \begin{align*}
\frac{d^2 \phi}{d\gamma^2} &= \lambda_2 \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 \\
\frac{d^2 \phi}{d\gamma^2} &= \lambda_1 \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2
\end{align*} \]

**Partial Integration Yields:**

\[ \begin{align*}
\phi(t) &= \frac{1}{\kappa^2 \alpha^2} \left( \lambda_2 \lambda_1 \right) \gamma^2 + C_1 \gamma + C_2 \\
\phi(t) &= \left( \lambda_2 \lambda_1 \right) \frac{1}{\kappa^2 \alpha^2} \left( \sigma^2 + \frac{\sigma}{\gamma} \right)^2 + C_1 \gamma + C_2
\end{align*} \]

**Employing Boundary Conditions:**

\[ \begin{align*}
\phi(0) &= 0 = C_2 \\
\phi(T) &= 0 = \frac{1}{\kappa^2 \alpha^2} \lambda_2 \lambda_1 \gamma^2 + C_1 T \\
\Rightarrow C_2 &= \frac{1}{\kappa^2 \alpha^2} \lambda_2 \lambda_1 \gamma^2 + C_1 T \\
\phi(T) &= \frac{1}{\kappa^2 \alpha^2} \lambda_2 \lambda_1 \gamma^2 + C_1 T
\end{align*} \]

**Hence:**

\[ \begin{align*}
\phi(T) &= \frac{1}{\kappa^2 \alpha^2} \lambda_2 \lambda_1 \gamma^2 + C_1 T
\end{align*} \]

**Now:**

\[ \begin{align*}
\phi &= \int_0^T \phi(t) \, dt \\
&= \left[ \int_0^T \frac{1}{\kappa^2 \alpha^2} \lambda_2 \lambda_1 \gamma^2 + C_1 T \, dt \right] \\
&= \left[ \int_0^T \frac{1}{\kappa^2 \alpha^2} \lambda_2 \lambda_1 \gamma^2 + C_1 T \, dt \right] \\
&= \left[ \frac{1}{\kappa^2 \alpha^2} \lambda_2 \lambda_1 \left( \frac{1}{2} \gamma^2 + \frac{1}{2} T \gamma \right) \right]_0^T
\end{align*} \]
\[
\omega(t) = -\frac{K_L^2 I_f^2}{\frac{R_a}{2} + J^2 R_a} \left( \frac{\lambda_2}{\lambda_1} \right) T^3
\]
\[
\Rightarrow \left( \frac{\lambda_2}{\lambda_1} \right) = -\frac{24 \alpha J^2 R_a}{K_L^2 T^2 T^3}
\]
\[
\Rightarrow \omega(t) = -\frac{K_L^2 I_f^2}{\frac{R_a}{2} + J^2 R_a} \left( \frac{24 \alpha J^2 R_a}{K_L^2 T^2 T^3} \right) (T^2 - T)f
\]
\[
\Rightarrow \omega(t) = -\frac{6\alpha J}{T^2} (T^2 - T)f
\]

Now,
\[
K_L I_f J_a(t) = \int \frac{d\omega}{dt} + T_o
\]
\[
\Rightarrow J_a(t) = K_L I_f \left( \int \frac{d\omega}{dt} + T_o \right)
\]
\[
= \frac{K_L I_f}{1} \left[ \frac{6\alpha J}{T^2} \left( T^2 - T \right) + T_o \right]
\]

The expressions above for \( J_a(t) \) and \( \omega(t) \) for minimum \( T \) are the same equations derived for the minimum loss problem; thus, the system may be designed for concurrent speed and loss optimality.

An expression for the optimum value of \( T \) can be derived as follows:

\[
Q = R_a \int_0^T J_a^2(t) dt
\]
\[
= \left( \frac{R_a}{2} + J^2 R_a \right) \left[ \frac{6\alpha J}{T^2} (T^2 - T) + T_o \right] dt
\]
\[
= \left( \frac{R_a}{2} + J^2 R_a \right) \left[ \left( \frac{6\alpha J}{T^2} (T^2 - T) + T_o \right) + \frac{12\alpha J}{T^2} \left( T^2 - T \right) + T_o \right] dt
\]
\[
= \left( \frac{R_a}{2} + J^2 R_a \right) \left[ \left( \frac{6\alpha J}{T^2} (3T^2 - 2T) + T_o \right) + \frac{12\alpha J}{T^2} (T^2 - T) + T_o \right]
\]
\[
= \left( \frac{R_a}{2} + J^2 R_a \right) \left[ \left( \frac{6\alpha J}{T^2} \right)^2 + T_o^2 T \right]
\]
\[
= \left( \frac{R_a}{2} + J^2 R_a \right) \left[ \left( \frac{6\alpha J}{T^2} \right)^2 + T_o^2 T \right]
\]
\[
= \frac{R_a}{1} \left[ \left( \frac{6\alpha J}{T^2} \right)^2 + T_o^2 T \right]
\]
\[
= \frac{R_a}{1} \left[ \left( \frac{6\alpha J}{T^2} \right)^2 + T_o^2 T \right]
\]
\[
\Rightarrow T_o^2 R_a T = Q \left( K_L I_f \right)^2 T^3 + 108 \alpha R_a^2 J^2 = 0
\]

The above fourth-order polynomial can be solved, yielding four values of \( T \). The smallest positive real of \( T \) is the optimum.
2) Using analogy from problem 1

\[ d = \int_{0}^{t} w(t) \, dt \Rightarrow f = w(t) \]

\[ Q = R_{o} \int_{0}^{T} j^{2} (t) \, dt = \int_{0}^{T} \frac{3^{2} R_{o} (d \Omega_{y} + T_{o})^{2}}{K_{T}^{2} \epsilon_{c}} \, dt \]

\[ \Rightarrow G_{o} = \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} (d \Omega_{y} + T_{o})^{2} \]

Hence:

\[ H = \omega \cdot \lambda_{2} \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} (d \Omega_{y} + T_{o})^{2} \]

Euler's equation:

\[ \frac{d}{dt} \left( \frac{\partial H}{\partial \dot{w}} \right) - \frac{\partial H}{\partial w} = 0 \]

\[ \frac{dH}{dw} = \frac{\dot{\omega}}{\lambda_{2} \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} (d \Omega_{y} + T_{o})^{2}} \]

\[ = \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} (d \Omega_{y} + T_{o})^{2} \]

As in notation or contend, \( \lambda_{2} \).

\[ \frac{d}{dt} \left( \frac{\partial H}{\partial \dot{w}} \right) = \lambda_{2} \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} \left( \frac{d \Omega_{y}}{dt} + T_{o} \right) \]

\[ \Rightarrow \frac{1}{\dot{\omega}} \frac{d}{dt} \left( \frac{\partial H}{\partial \dot{w}} \right) = \lambda_{2} \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} \left( \frac{d \Omega_{y}}{dt} + T_{o} \right) \]

\[ \Rightarrow \frac{d}{dt} \left( \frac{\partial H}{\partial \dot{w}} \right) = \lambda_{2} \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} \left( \frac{d \Omega_{y}}{dt} + T_{o} \right) \]

\[ \Rightarrow \lambda_{2} \]

From Euler:

\[ \frac{d}{dt} \left( \frac{\partial H}{\partial \dot{w}} \right) = \lambda_{2} \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} \left( \frac{d \Omega_{y}}{dt} + T_{o} \right) \]

\[ \Rightarrow \frac{d}{dt} \left( \frac{\partial H}{\partial \dot{w}} \right) = \lambda_{2} \frac{J^{2} R_{o}}{K_{T}^{2} \epsilon_{c}} \left( \frac{d \Omega_{y}}{dt} + T_{o} \right) \]

Partial integration yields:

\[ \omega(t) = \frac{K_{T}^{2} \epsilon_{c}}{J^{2} R_{o} \lambda_{2}} C_{1} T + C_{2} = 0 \]

Employing boundary conditions:

\[ \omega(0) = 0 = C_{2} \]

\[ \omega(T) = 0 = \frac{K_{T}^{2} \epsilon_{c}}{J^{2} R_{o} \lambda_{2}} T + C_{1} T \]

\[ \Rightarrow C_{1} = 0, K_{T}^{2} \epsilon_{c} = T + C_{2} \]

\[ \Rightarrow \omega(t) = \frac{K_{T}^{2} \epsilon_{c}}{J^{2} R_{o} \lambda_{2}} (t^{2} - T t) \]
SOLVING FOR $\lambda_3$ IS NEXT IN ORDER, FROM RESTRICTION

\[ Q - R_0 \int_0^T \dot{J}_a(t) \, dt \]

\[ \dot{J}_a(t) \] CAN BE FOUND FROM EXPRESSION

FOR $\omega(t)$:

\[ \frac{d\omega}{dt} = \frac{k_t^2 J_f}{k_f^{1/3} R_0 \lambda_3} (2t - T) \]

\[ \Rightarrow \dot{J}_a(t) = \frac{1}{J_f K_f} \left[ \frac{k_t^2 J_f}{k_f^{1/3} R_0 \lambda_3} (2t - T) + T_0 \right] \]

\[ = \frac{k_t^2 J_f}{k_f^{1/3} R_0 \lambda_3} (2t - T) + T_0 \]

\[ \Rightarrow \int_0^T \frac{1}{J_f K_f} \left[ \frac{k_t^2 J_f}{k_f^{1/3} R_0 \lambda_3} (2t - T) + T_0 \right] \, dt \]

\[ = \int_0^T \frac{k_t^2 J_f}{k_f^{1/3} R_0 \lambda_3} \left( \frac{T_0}{2} (2t - T) + T_0 \right) \, dt \]

\[ = \frac{T_0^2}{2} \left( \frac{k_t^2 J_f}{k_f^{1/3} R_0 \lambda_3} \right) + R_0 \left( \frac{T_0}{J_f K_f} \right)^2 \]

\[ \Rightarrow \frac{T_0^2}{2} \left( \frac{k_t^2 J_f}{k_f^{1/3} R_0 \lambda_3} \right) = Q - R_0 \left( \frac{T_0}{J_f K_f} \right)^2 \]

\[ \Rightarrow \frac{1}{\lambda_3} = \frac{-4 J}{k_t J_f} \left[ 3 R_0 \left( \frac{T_0}{J_f K_f} \right)^2 \right] \]

\[ \Rightarrow \omega(t) = \left( \frac{k_t^2 J_f}{k_f^{1/3} R_0} \right) \left[ \frac{4 J}{k_t J_f} \right] \left[ 3 R_0 \left( \frac{T_0}{J_f K_f} \right)^2 \right] \left( \frac{T_0}{J_f K_f} \right)^2 (t^2 - T^2) \]

\[ = \frac{-k_t^2 J_f}{J_f K_f} \left[ \frac{3 R_0}{J_f K_f} \left( \frac{T_0}{J_f K_f} \right)^2 \right] \frac{1}{2} (t^2 - T^2) \]

AND

\[ \dot{J}_a(t) = \left( \frac{k_t^2 J_f}{k_f^{1/3} R_0} \right) \left[ \frac{4 J}{k_t J_f} \right] \left[ 3 R_0 \left( \frac{T_0}{J_f K_f} \right)^2 \right] \frac{1}{2} (2t - T) + \frac{T_0}{J_f K_f} \]

\[ = \frac{-1}{T} \left[ \frac{3 R_0}{J_f K_f} \left( \frac{T_0}{J_f K_f} \right)^2 \right] \frac{1}{2} (2t - T) + \frac{T_0}{J_f K_f} \]
POUNTS OF INTEREST

\[ \omega_m = \frac{K_a T_3}{t_d} \sqrt{\frac{B}{J \cdot T}} | Q \cdot P_a (T_0 \frac{K_a}{T_3 K_d})^{-T} | \]

\[ I_m = \frac{T_0}{T_3 K_d} + \sqrt{\frac{B}{P_a}} | Q \cdot P_a (T_0 \frac{K_a}{T_3 K_d})^{-T} | \]

\[ I_c = \frac{T_0}{T_3 K_d} \]

\[ \omega(T) = \omega(0) \cdot 0 \]

\[ t_d \]

\[ t_3 \]

\[ t_1 \]

\[ t_f \]

\[ t_p \]
a) For optimum $T$ (note $K_f = K_f$)

Case $\delta = 0, \ \Delta = 0$

\[ V_a = R_a I_a(t) + K_f I_f \omega(t) \]

\[ = \left[ R_a I_a \left\{ \frac{k_a}{k_I} \left[ T_a \left( T_a - 2 \delta + T_0 \right) \right] + \frac{k_a K_f^T}{k_I} \left( T_a + T_0 \right) \right\} \right] \omega(t) - \mu(t) \]

\[ = \left[ \frac{3a k_f}{k_f} I_f \left( K_f I_f - \frac{2a \mu}{k_f} \right) \right] \omega(t) + \left[ R_a \left\{ \frac{k_a}{k_f} \left( T_a + T_0 \right) \right\} \right] \omega(t) - \mu(t) \]
CASE II - OPTIMUM A. WITH $L_a = c$

AGAIN : $V(t) = R_a i_a(t) + K_p I_p (t(t))$

Let $\psi = (Q - P_a (\frac{I_f}{L_a} + K_p) T)^\frac{1}{2} (\frac{3}{P_a})^\frac{1}{2}$

Then $V(t) = \left[ K_a \frac{\psi}{(T-2t)} + \frac{T_0}{T} \psi + \frac{K_p^2 I_f^2}{jT} \psi \left( T - T_i - T_f \right) \right]$

$= \left[ -\frac{K_p^2 I_f^2}{jT} \psi T_i^2 + \psi \left( \frac{K_p^2 I_f^2}{jT} = \frac{2R_a}{T} \right) t + (R_a \psi + \frac{T_0}{T} \psi) \right]$

$= \left[ \psi \left( \frac{K_p^2 I_f^2}{jT} = \frac{2R_a}{T} \right) t + (R_a \psi + \frac{T_0}{T} \psi) \right]$
with \( L_a \neq 0 \)

\[
V_{q2} = R_a I_a + L_a \frac{d}{dt} i_a + k_F I_F L
\]

\[
= I_a \left[ \frac{1}{R_a} \int_{t_a}^{t} \left( \frac{E_{q1}}{R_a} \left( t - t_a \right) + T_a \right) \left( \mu(t) - \mu(t - T) \right) \right] + \frac{L_a}{R_a} \left( \frac{E_{q1}}{R_a} \left( t - t_a \right) + T_a \right) \left( \mu(t) - \mu(t - T) \right)
\]

\[
= V_{q1} + L_a \left[ \frac{E_{q1}}{R_a} \left( \mu(t) - \mu(t - T) \right) \right] + \frac{E_{q1} + T_a}{R_a} \left( \mu(t) - \mu(t - T) \right) - \frac{E_{q1} - T_a}{R_a} \delta(t - T)
\]

The voltage with \( L_a \) is \( V_{q1} \) with a d.c. value and impulses at \( 0 \) and \( T \), which allow for the instantaneous change of \( I_a \) with \( L_a \).
CASE 3b: OPTIMUM $\alpha$, $L_0 \neq 0$

Again: \[ V_{a2}(t) = \left[ I_n \omega(t) + k_T \int \omega(t) + L_0 \frac{d}{dt} \left( I_a(t) \right) \right] \]

\[ = V_{a1}(t) + I_a \frac{d}{dt} \left[ \left( \frac{T_0}{T} + 2\tau \right) \left( \frac{T_0}{T} + 1 \right) \right] \]

\[ = V_{a1}(t) - 2I_a \psi \left[ u(t) - u(t-T) \right] \]

\[ + \left[ \frac{T_0}{T} \right] \left( \frac{I_a}{T} + \frac{k_T}{T} \right) \left[ s(t) - s(t-T) \right] \]

\[ = V_{a1}(t) - 2I_a \psi \left[ u(t) - u(t-T) \right] \]

\[ + \left[ \frac{T_0}{T} \right] \left( \frac{I_a}{T} + \frac{k_T}{T} \right) \left[ s(t) - s(t-T) \right] \]

\[ \times \left[ \frac{T_0}{T} - I_a \right] s(t-T) \]

Again, the voltage $V_{a2}$ (for case b) is $V_{a1}$, with an added D.C. component and voltage

\[ V_{a2} = \frac{2I_a \psi}{T} \left( \frac{T_0}{T} - I_a \right) s(t-T) \]

\[ \times \left[ \frac{T_0}{T} - I_a \right] s(t-T) \]

\[ \times \left[ \frac{T_0}{T} - I_a \right] s(t-T) \]
CASE 2.b: optimum $\alpha_2, L_2 \neq 0$

Again: $V_{a_2'}(t) = [P_{a_2}(t) + k_4 I_{T_0} \omega(t) + L_2 \frac{d^2}{dt^2} (U_a(t))]$

\[
= V_{a_1}(t) + L_2 \frac{d^2}{dt^2} \left[ \left( \frac{T_0}{T} (T - \Delta t) + \frac{T_0}{T} \right) \right] \left( u(t) - u(t - T) \right)
\]

\[
= V_{a_1}(t) - \frac{2L_2}{T} \left[ \mu (t) - \mu(t - T) \right]
\]

\[
+ \frac{2L_2}{T} \left[ \psi(t) - \psi(T) \right]
\]

\[
= V_{a_1}(t) - \frac{2L_2}{T} \left[ \mu (t) - \mu(t - T) \right]
\]

\[
+ \left[ L_2 \Psi + \frac{T_0}{T} \right] \left( \psi(t) + \frac{T_0}{T} \psi(T) - L_2 \Psi \right) \cdot \delta(t - T)
\]

\[
\sim V_{a_1} = \frac{2L_2}{T} \left[ \mu (t) - \mu(t - T) \right]
\]

Again, the voltage $V_{a_2}$ (for $\cos \theta$) is $V_{a_1}$, with an added d.c. component and variables $\Delta t - \delta$ and $T - \Delta$ to allow initial phase shift.
\[ \text{Diagram: XYZ system, with axes X, Y, Z.} \]

\[ \text{Equation: } \sum F = 0 \]

\[ \text{Solution: } \]

\[ \text{Further calculations:} \]

\[ \text{Conclusion:} \]
\( A' = (A + B) \)

Also, \( A = \Lambda_1 \) and \( B = \Lambda_2 \).

Furthermore, \( A + B = \Lambda' \) and \( \Lambda' = \Lambda_1 + \Lambda_2 \).

\[ L = A' \]

The expression becomes:

\[ \begin{bmatrix} \cosh k R \sinh \frac{ka}{2} & 0 \\ 0 & \cosh k R \sinh \frac{ka}{2} \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ 0 \end{bmatrix} = 0 \]

Solution of \( \Lambda_1 \):

\[ \Lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3.33 \end{bmatrix} = \begin{bmatrix} 0 & 3.33 \end{bmatrix} \]

\[ \Lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3.33 \end{bmatrix} = \begin{bmatrix} 0 & 3.33 \end{bmatrix} \]
SOLUTION OF $\lambda_1$

$$\lambda_1 = \frac{1}{d} \left( \frac{N}{N_0} \right) \left( y \sinh \frac{KA}{2} - \cosh \frac{KA}{2} \right)$$

Now, $\alpha, m, k_1, k_2, \ldots$ and $[\lambda_1] = 1$

FOR WHICH $N$ THIS EQUATION IS SATISFIED

THUS $\lambda_1, \lambda_2, \ldots$

$$\lambda_n = \frac{1}{d} \left( \frac{N}{N_0} \right) \left( y \sinh \frac{nKA}{2} - \cosh \frac{nKA}{2} \right)$$

Thus the solution is

$$\lambda_n = \frac{1}{d} \left( \frac{N}{N_0} \right) \left( y \sinh \frac{nKA}{2} - \cosh \frac{nKA}{2} \right)$$
\[ N_{K} = \frac{B_{0}}{B_{1}} = \frac{N_{w}^{2} \mu_{w} P_{w}}{\lambda_{w}} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{w}^{2} \tau_{w}^{2} \alpha_{w}}{\lambda_{w}} \right] \]

\[ = \frac{N_{w}^{2} \mu_{w} \lambda_{w}^{2}}{2} \left[ \frac{\xi^{2} \pi^{2} \phi_{w}^{2} \tau_{w}^{2} \alpha_{w}}{2} + 1 \right] \]

\[ \Rightarrow \langle \rho \rangle = N_{v}^{2} \frac{\mu_{v} P_{v}}{\lambda_{v}} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{v}^{2} \tau_{v}^{2} \alpha_{v}}{\lambda_{v}} \right] \quad \text{for} \quad \gamma > 0 \]

\[ \langle \rho \rangle \quad \text{for} \quad \gamma < 0 \quad (\gamma > C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{v}^{2} \tau_{v}^{2} \alpha_{v}) \quad \text{if} \quad \mu_{v} \rightarrow 0 \]

\[ \langle \rho \rangle \quad N_{v}^{2} \frac{\mu_{v} P_{v}}{\lambda_{v}} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{v}^{2} \tau_{v}^{2} \alpha_{v}}{\lambda_{v}} \right] \]

Now, total force = \langle \rho \rangle \cdot \text{AREA} = \lambda_{v} \langle \rho \rangle \cdot \text{AREA}.

\[ F_{\text{up}} = N_{v}^{2} \frac{\mu_{v} P_{v}}{\lambda_{v}} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{v}^{2} \tau_{v}^{2} \alpha_{v}}{\lambda_{v}} \right] \]

\[ F_{\text{down}} = N_{v}^{2} \frac{\mu_{v} P_{v}}{\lambda_{v}} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{v}^{2} \tau_{v}^{2} \alpha_{v}}{\lambda_{v}} \right] \]

\[ F_{\text{tot}} = F_{\text{up}} + F_{\text{down}} = 2N_{v}^{2} \frac{\mu_{v} P_{v}}{\lambda_{v}} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{v}^{2} \tau_{v}^{2} \alpha_{v}}{\lambda_{v}} \right] \]

\[ = \frac{N_{v}^{2} \mu_{v} P_{v}}{\lambda_{v}} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{v}^{2} \tau_{v}^{2} \alpha_{v}}{\lambda_{v}} \right] \]

Now, \( N_{v} \mu_{v} \lambda_{v} = \frac{V_{w}}{N_{w} \mu_{w} \lambda_{w}} \)

\[ \Rightarrow F_{\text{tot}} = \frac{1}{2} N_{w}^{2} \mu_{w} \lambda_{w} \left[ \frac{C_{0} \xi^{2} \sigma_{0}^{2} \pi^{2} \phi_{w}^{2} \tau_{w}^{2} \alpha_{w}}{\lambda_{w}} \right] \]

\[ \frac{1}{20} \]
c) The connecting wires can be formed by an equal force from the two stators. Act in a clockwise fashion, both stators act on the rotor in such a manner as to cancel each other. The stator forces are balanced, producing the y component of force. And the x component of forces.

A note should be made that all rotor in the system.

Configuration is stable equilibrium, for a stable differential pump, in center, there is no x component of force.
The peak force is, as expected, independent of the conductivity, such is obviously not true concerning the speed at which the peak force occurs; the larger the conductivity, the greater the speed at which maximum force is generated.
\[
\left[ \mathbf{V} \right]' = \left[ P \right] \mathbf{I} + \int \left[ \mathbf{H} \right] d \mathbf{I} + \frac{d}{dt} \left[ \mathbf{E} \right]
\]

\[
\dot{\mathbf{I}}_a = \mathbf{I}_m \sin \left( P \mathbf{x}_m - \phi \right) - \mathbf{I}_m \sin \left( P \mathbf{x}_m - \phi \right)
\]

\[
\Rightarrow \mathbf{V}_c = \mathbf{R}_q \mathbf{I}_c + \mathbf{L}_{qa} \frac{d}{dt} \mathbf{I}_a + \mathbf{L}_{fa} \frac{d}{dt} \mathbf{I}_a + \mathbf{L}_{fe} \frac{d}{dt} \mathbf{I}_e - P \omega_m \mathbf{M}_a \mathbf{I}_c \sin \left( P \mathbf{x}_m - \phi \right)
\]

\[
= \mathbf{R}_f \mathbf{I}_f - P \omega_m \mathbf{M}_a \mathbf{I}_c \sin \left( P \mathbf{x}_m - \phi \right)
\]

\[
\mathbf{V}_a = - \left( L_{ba} \frac{d}{dt} \mathbf{I}_a - P \omega_m \mathbf{M}_a \mathbf{I}_c \sin \left( P \mathbf{x}_m - \phi \right) \right)
\]

\[
= - \left( L_{ba} \mathbf{I}_a \left( \omega + \phi \right) - \mathbf{M}_a \mathbf{I}_c \sin \left( \omega t + \frac{\omega}{2} \right) \right)
\]

\[
\mathbf{V}_b = - \left( L_{cb} \frac{d}{dt} \mathbf{I}_c - P \omega_m \mathbf{M}_a \mathbf{I}_c \sin \left( P \mathbf{x}_m - \phi \right) \right)
\]

\[
= - \left( L_{cb} \mathbf{I}_c \left( \omega + \phi \right) - \mathbf{M}_a \mathbf{I}_c \sin \left( \omega t + \frac{\omega}{2} \right) \right)
\]

\[
\mathbf{V}_c = - \left( L_{ca} \frac{d}{dt} \mathbf{I}_a - P \omega_m \mathbf{M}_a \mathbf{I}_c \sin \left( P \mathbf{x}_m - \phi \right) \right)
\]

\[
= L_{ca} \mathbf{I}_a \left( \omega + \phi \right) - \mathbf{M}_a \mathbf{I}_c \sin \left( \omega t + \frac{\omega}{2} \right)
\]
\[ \int_{0}^{T} \left( \frac{\partial \rho}{\partial t} + \rho \frac{\partial f}{\partial x} \right) \, dt \]

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \]

\[ Q = \int_{0}^{T} \left( \frac{\partial w}{\partial t} + T_{eff} \right)^{2} \, dt \]

where we have used

\[ k_{eff} \frac{\partial w}{\partial x}(t) = \int \frac{\partial u}{\partial x} + T_{0}. \]

\[ \frac{\partial w}{\partial t} - \frac{1}{\rho_{0}} \left( \frac{\partial \rho}{\partial x} \right) = \frac{1}{\rho_{0}} \left( \frac{\partial \rho}{\partial x} \right) \]

\[ \frac{d^{2}w}{dt^{2}} + \frac{d^{2}w}{dt^{2}} \left( \frac{d^{2}w}{dt^{2}} \right) = \frac{d^{2}w}{dt^{2}} \left( \frac{d^{2}w}{dt^{2}} \right) \]

Finally, \( \frac{\partial w}{\partial x} < 2 \). Hence, if \( \rho_{0} > 0 \), then

we have a minimum point, according to (29). If \( \rho_{0} < 0 \), we have maximum point.
\[
\lambda = \frac{\frac{1}{T} \sqrt{\frac{\alpha}{\lambda T}}}{T}
\]

\[
\delta = \left( \frac{1}{T} \right) \frac{A}{2\pi R_a}
\]

\[
\left( \frac{R_T}{R_a} \right)^2 - \left( \frac{T}{T_0} \right)^2 = \frac{A}{2\pi R_a}
\]

\[
\delta = \left( \frac{1}{T} \right) \frac{A}{2\pi R_a}
\]

\[
\lambda = \frac{\frac{1}{T} \sqrt{\frac{\alpha}{\lambda T}}}{T}
\]

\[
\frac{1}{T} \sqrt{\frac{\alpha}{\lambda T}} = 1
\]
\[
\frac{d}{dt} \left( \frac{1}{K_p T_p} \right) \frac{T^3}{T_p^3} + \frac{12 \pi^2 T_p}{K_p T_p^2} = 0.
\]

This equation can be solved in order to determine \( T_p \), given \( K_p \) and other known constants.

\[
\left( \frac{12 \pi^2 T_p}{K_p T_p^2} \right) = \left( \frac{12 \pi^2 \rho c}{K_p T_p^2} \right)^{1/2}
\]

\text{Required:}\begin{align*}
\text{The motor speed} (N) & = \frac{V}{2} \\
\text{The motor torque} (T) & = \frac{V}{2}
\end{align*}

\text{The platform must have polariser (for more interesting results).}

\text{The platform must have polariser (for more interesting results).}
\( \kappa = \int_0^T \omega(t) \, dt \),

\( \beta = \kappa \int_0^T \kappa \, dt \),

\( \phi = \kappa \int_0^T \kappa \, dt \),

\( \phi = \frac{\kappa J^2}{K_0^2 I_0^2} \int_0^T \left( \frac{d\omega}{dt} + \frac{\tau_0}{J} \right)^2 \, dt \).

\[ F = \frac{N + \frac{\kappa J^2}{K_0^2 I_0^2} \left( \frac{d\omega}{dt} + \frac{\tau_0}{J} \right)^2}{K_0^2 I_0^2} \]

\( \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \)

\( \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \)

\[ \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \]

This is the usual equation, with the usual solution (mention T is known).

\[ \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \]

\( \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \)

\[ \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \]

\( \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \)

\[ \frac{\partial^2 W}{\partial \omega^2} - \frac{\partial (\partial W)}{\partial \omega} = 1 - 27 \beta \frac{\kappa J^2}{K_0^2 I_0^2}, \quad \frac{\partial^2 W}{\partial \omega} = 0 \]
\[
\frac{\partial^2 H_3}{\partial \omega^2} = \frac{2\alpha \kappa^2 J^2}{k^2 L_p^4} = \frac{\kappa J^2}{(k_1 k_2)^2} \left( -\frac{(k_1 k_2)^2 T^2}{12 J^2 k_0 \omega} \right)
\]

\[
= -\frac{T^3}{12\kappa} < 0
\]

Because the second derivative is negative, we have indeed minimized the Hamiltonian of interest. The point of these two problems (other than to learn about control of DC motors) is that the proof will (will) never be furnished optimal solution for these control problems involving both maximization or minimization.

(1) \( (\frac{dT}{dt}) = \frac{k}{L} \frac{\Delta I}{t} + k + k_L \Delta I \)

(2) \( \frac{dT}{dt} = k + k_L \Delta I \)

Any feedback from an existing current,ток, will have the above form, and it will exhibit some type of "spike" or asymmetry. If no other control is

(3) \( \frac{dT}{dt} = k + k_L \Delta I \)

In conclusion, the entire control system described above will exhibit the desired characteristics.
Hand in your project proposals no later than March 25, 1972. You may consult with me prior to that time about possibilities for your project.

The proposal should be brief and to the point. It should tell exactly what you propose to do in detail. It should be typed on 8-1/2 x 11 plain white paper. This proposal when approved will constitute a contract between student and instructor. The possibility of getting an A in the course depends entirely upon your achieving everything you set out to do as described in the proposal.

You may renegotiate the contract during the first four weeks of the course in order to decrease the scope of the project with a accompanying decrease in possible grade. Anyone who does not finish his project and turn in the formal report will receive an incomplete in the course.

We will have one formal class meeting each week which all students enrolled in EE498 are expected to attend. It is not required that you be in the lab working on your project during the scheduled lab period Friday morning.

The proposal should give some motivation from your choice of project. It should give a bibliography if possible and a short resume of work which you have already done prior to this quarter. It should list the name of an EE staff member than the course instructor may consult with on your project (if not JHD).

We will have oral presentations of your proposals given on March 31 during the lab period (attendance required). An interim report (hand written) is to be turned in no later than April 20, 1972. This report will be a quick summary of your progress on the project.
Two-wire transmission line

Induction pickup

Ferrite strip

Conducting ground strip

(a) Schematic view of transmission line with induction pickup

(b) Cross section view of (a) Typical dimensions (cm) $a = 40, b = 25, c = 120, d = 10, e = 1, f = 100, g = 1, h = 12, k = 40, 2e$ = length $= 2000, k = 40$

FIGURE 6-1 - Typical Structure for Induction Pickup

FIGURE 6-2 - Circuit model for transmission line supplying power to a vehicle by means of an induction pickup. For a given pickup magnetizing inductance, $L$, with $wL \gg R$, maximum power in $R$ occurs for $R = wL$. $P_{\text{max}} = \left(\frac{V_o}{R_o}\right)^2 \frac{wL}{2}$. 

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6.0 A LINEAR INDUCTION POWER-TRANSFER SYSTEM FOR VEHICLES

6.1 Introduction

The usual method of transferring electric power to a moving vehicle is by means of a sliding contact. For a high speed vehicle this technique appears to have problems, both because of the speed, and because of the large power required. Moreover, the normal use of dc or 60 eps power poses problems because the high voltage which is required for efficient power transmission is not convenient for propulsion, so heavy on-board voltage reducing equipment may be required.

This report considers the possibility of increasing the power frequency and transferring power by inductive coupling instead of by sliding contact. Some of the important design parameters are considered and rough calculations for a typical system are presented.

6.2 Pickup Design Considerations

There are several ways in which an induction pickup could extract power from a power line. Perhaps the simplest would be a large ferrite rod antenna inserted between the wires in a two wire line. Clearly, this type of structure will not be very efficient at 60 hertz but at a frequency of several hundred kilo-hertz, a substantial power could be transformed. Unfortunately, the use of too high a frequency creates problems because of the relatively short wavelength and because of the small skin depth for currents in the transmission line wires. Moreover, it is easier to generate, control, and utilize electric power if the frequency is in a range suitable for high power semiconductor devices.

Hence, it is desirable that steps be taken to increase the magnetic coupling between the power line and the pickup coil in order that the transmission frequency can be kept as low as possible.

Figure 6-1 shows a tentative design for an induction pickup in which an "E" shaped ferrite core is used with the pickup coil wound around the center leg. A thin ferrite strip is placed under the transmission line, so that the flux encircling the transmission line wires sees a relatively small air gap between the pickup and this strip. The transmission line and pickup coil form a transformer with the "E" core and ferrite strip analogous to the E-1 laminations used in conventional transformers.

The pickup could, in principle, be operated with a constant voltage or a constant current in the primary. However, if we assume that the transmission line is many wavelengths long, it appears that the only practical mode of operation is with a constant amplitude travelling wave on the line. For example, the line could be excited at one end and terminated in a matched load at the other end. An equivalent circuit for this mode is shown in Fig. 6-2 where $R_0$ is the characteristic impedance of the line, $L$ is the primary magnetizing inductance of the transformer, and $X$ is the lead referred to a one turn pickup coil.

Clearly, the larger the magnetizing inductance, the larger the power which can be delivered to the load. As a practical matter, the inductive resistance of the pickup, $L$, will be small compared with $R_0$, so the transformer will effectively operate with a constant current in the primary. If several vehicles are coupled to the line at the same time, there will be some reduction in current, but this does not appear to be significant for a multiply excited system of the type to be described.
6.5 Conclusions

On the basis of relatively crude calculations, it appears to be feasible to transmit power from a transmission line to a moving vehicle by means of an induction pickup. For a typical example, it is possible to transfer a power of 2.5 megawatts to a pickup 20 meters long, assuming a transmission frequency of 18 khz, a current of 400 amperes, and a voltage of 100 kv. The weight of the pickup is on the order of 2 kilograms per kilowatt of power output capability, but this could be reduced by using a smaller clearance between the pickup and the ferrite ground plane, or by using a higher line frequency. The assumed clearance was 1 cm with an allowance of ± 10 cm for side to side motion of the pickup with respect to the transmission line. The system efficiency is between 69% and 98%, depending on the economics of transmission line construction, and assuming 100 vehicles consuming 2 megawatts each on a 1000 km two-way track.

It is anticipated that two major problems are generating high powers at 18 khz, and maintaining a constant voltage on a transmission line which is many wavelengths long. Another problem is the rather large weight and size of the pickup, but since the pickup serves as a transformer, on-board transformers can be eliminated. The possible feasibility of an induction pickup suggests that further study is warranted in order to arrive at more accurate design criteria and performance expectations. Experiments on simple scale models could be used to verify the design relations.

REFERENCES

would then be 30 kv which does not appear to be too high considering the minimum spacing between pickup and line is 8 cm.

The excitation frequency for the transmission line should be as low as possible consistent with transfer of the desired amount of power. As a typical example, if the frequency is selected to be 18 khz, if the pickup has the dimensions shown in Fig. 6-1, and if the line current is 400 amperes, then the maximum power which can be delivered to the vehicle is about 2.3 megawatts. The choice of 18 khz appears to be reasonable because it can be converted to dc or lower frequency as by means of high power solid state devices. Also, 18 khz is above the normal audio range so that objectionable noise is minimized.

The choice of line frequency is also governed by wavelength considerations. Typically power might be fed into the line at intervals on the order of 30 to 100 km, and it is desirable to minimize the number of wavelengths between any two feed points, if a reasonably low standing wave ratio is to be maintained in spite of the presence of a number of vehicles on the line. An excitation frequency of 18 khz will produce a wavelength of about 14 kilometers, which is probably on the lower edge of practicality. It would thus be desirable to avoid using a frequency much above this value.

6.4 Efficiency of the Induction Power Transmission System

The power losses on the transmission line will be primarily $R^2$ loss caused by currents in the line. If the wires were aluminum tubes of diameter $d$ and with wall thickness greater than the skin depth $b$, then the resistance per unit length of a two wire line is about $\frac{2\rho}{\pi db}$

where $\rho$ is the resistivity. For aluminum $\rho = 2.8 \times 10^{-8} \text{ohm m}$. Thus for dimensions shown in Fig. 6-1, it is about 0.28 ohm per kilometer. With $I = 400$ amperes, the power loss would be 45 kw/km. This amount of loss is high and probably not practical.

There are, however, a number of ways of reducing this loss. For example, a number of insulated wires could be twisted in such a way that the current is shared between the various wires. As a lower limit on loss we could calculate the resistance assuming that the entire cross section of the line is carrying current. The resistance is then less than 0.01 ohms per kilometer so the power loss is reduced to less than 2 kw/km or 2 megawatts for the entire 1000 km line. Other losses in the line appear to be negligible, with, for example, ferrite losses amounting to less than 10 watts per kilometer.

For purpose of example, assume that there are 100 evenly spaced vehicles on a 1000 km two-way track. Assume further that the vehicle speed is 480 km/hr (300 mph), and each vehicle consumes 2 megawatts of power. These assumptions imply a vehicle spacing of 20 km (12 miles) or 40 seconds. If the power-feeding stations are located every 50 km (31 miles), then each station will supply an average of 2 1/2 vehicles on each track, for a total power of 10 megawatts per station or 200 megawatts overall. The $R^2$ loss for the simple aluminum tube wires would be 90 kilowatts, giving a transmission efficiency of 90%. The lower limit on loss for stranded wires would be 4 megawatts, giving a transmission efficiency of 95%. The economically optimum solution probably lies somewhere between these extremes.
transmission lines can be tied together at quarter wavelength intervals so that any quarter wavelength section can be de-energized without interrupting operation on the remaining sections.

Figure 6-3 shows a typical two-way system with quarter wavelength tie points and a constant voltage generator connected across the lines every 3/4 wavelength. The generators are so phased that power flows in opposite directions in the two lines and thus the rms voltage and current on the lines are nearly independent of position. It is assumed that only one or two vehicles will be between any two generators, so the vehicles will not cause more than about a 5% variation in line voltage.

In order for the transmission line to operate in the travelling wave mode, it is necessary that the ratio of line-to-line voltage to line current be equal to the characteristic impedance of the line. There is relatively little design freedom in the choice of the line impedance because it depends on the logarithm of the ratio of various cross sectional dimensions. For any choice that appears at all reasonable for use with an induction pickup, this impedance is in the range of 200 to 300 ohms. It appears that the impedance of the magnetizing inductance of the pickup is about an order of magnitude less than this, so it is desirable to use the lowest possible line impedance. In other words, the power transferred to the pickup depends almost entirely on the line current, and we would like to minimize the voltage required to produce this current. As a typical example, the transmission line shown in Fig. 6-1 is estimated to have a characteristic impedance on the order of 250 ohms and typical operating conditions might be 100 kV line-to-line voltage and 400 amperes-line current. The line to ground voltage

![Diagram](attachment:image.png)
The pickup design shown in Fig. 6-1 has a magnetic field air gap with a relatively large area and short length. The flux density in the air gap is quite low so the ferrite magnetic circuit has less than 1% of the area of the air gap. A wide spacing is allowed between the transmission line and the pickup in order that a high voltage line can be used. Also, a considerable latitude is allowed for sideways motion of the pickup with respect to the transmission line. The height of the transmission line above ground is made large enough to minimize power losses in the ferrite strip when the pickup is not present, and to minimize the characteristic impedance of the line. A conductor under the ferrite provides a low loss ground plane and minimizes the external field of the line. An aluminum frame over the pickup provides both structural strength for the pickup and shielding for the vehicle. It is envisioned that the pickup would be an integral part of the vehicle and would extend almost the full length of the vehicle.

Typical dimensions are shown in Fig. 6-1. Perhaps the most important dimension is the clearance between the pickup and the ferrite ground strip (i.e., "g" in Fig. 6-2). This distance is assumed to be 1 cm but the power output could be doubled by reducing this distance to 1/2 cm. A minimum clearance of about 4 cm is assumed between the pickup and the transmission line, with an allowable side-to-side motion for the pickup of about ±10 cm with respect to the transmission line. The shoes on the bottom of the pickup are made as long as practical without interfering with the transmission line supports. The length of the pickup is assumed to be 20 meters based on a vehicle length of about 30 meters. The magnetic material in the pickup is assumed to be ferrite, but laminated steel could equally well be used. The mass of the pickup is estimated at about 4,000 kg.

Since the pickup behaves like a transformer, almost any output voltage can be developed by using an appropriate turns ratio. If desired, the pickup could be divided into two or more sections with each section behaving almost as an independent pickup. This mode might be desirable if there were two or more separate motors used for propulsion.

6.1 Transmission Line Design Considerations

The transmission line can be operated in several possible modes, and the best choice can only be made after consideration of a number of economic factors. For example, one possibility is to divide the line into a number of short sections with each section excited only when a vehicle is present. Such a system would minimize line losses, but at an increase in the cost of control equipment.

For purpose of example, it is assumed that it is desirable to excite the entire line, and that the vehicles are nearly evenly spaced. Power is assumed to be fed into the line at periodic intervals in order to maintain a nearly constant voltage on the line. The entire line can be thought of as a resonant structure, except that power is flowing from one end of the line to the other.

Instead of actually transmitting power from one end of the line to a dummy load at the other end, a two-way track system can be used with power flowing in opposite directions in the two parallel transmission lines. The two lines are then chosen to be an integer number of quarter wavelengths long and are joined at each end. In addition, the two
FIGURE 6-1 - Typical Structure for Induction Pickup

(a) Schematic view of transmission line with induction pickup

(b) Cross section view of (a) Typical dimensions (cm) a = 40, b = 25, c = 120, d = 10, e = 1, f = 100, g = 1, h = 12, p = length = 2000, k = 40

FIGURE 6-2 - Circuit model for transmission line supplying power to a vehicle by means of an induction pickup. For a given pickup, magnetizing inductance, \( L \), with \( \omega L \gg R_0 \), maximum power in \( R \) occurs for \( R = \omega L \). \( P_{\text{max}} = \left( \frac{V_0}{R_0} \right)^2 \omega L/2 \).