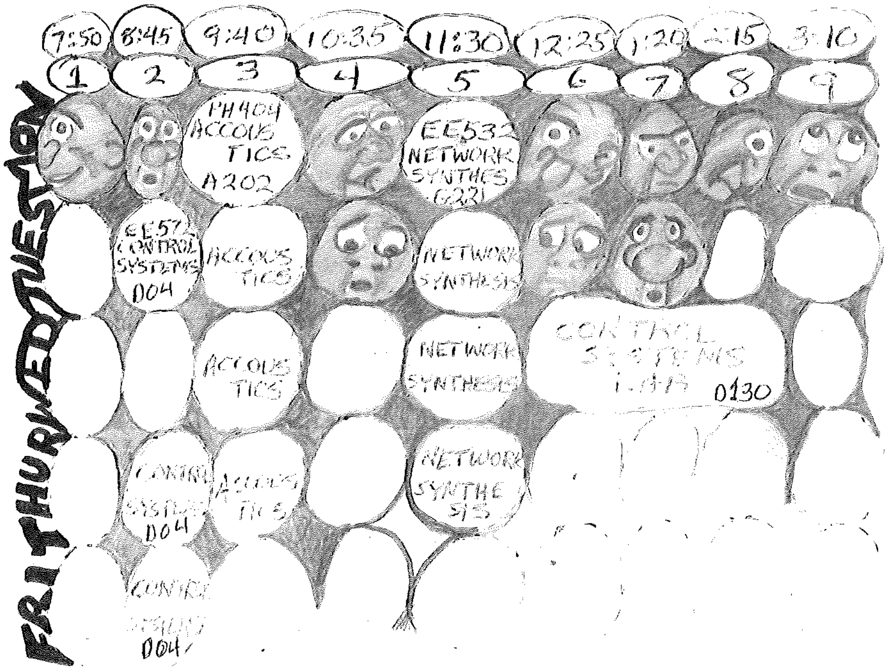
Network Synthesis

R.J. Marks II Class Notes

Rose-Hulman Institute of Technology (1970)

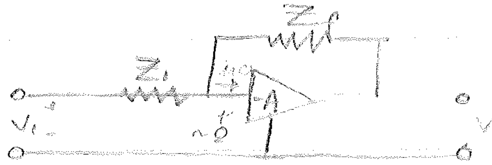
NETWORK SYNTHESIS



T T+DT

Jan 1970

9-11-72



$Z_1 = V_1/I_1$

$V_o/-I_2 = Z_f$

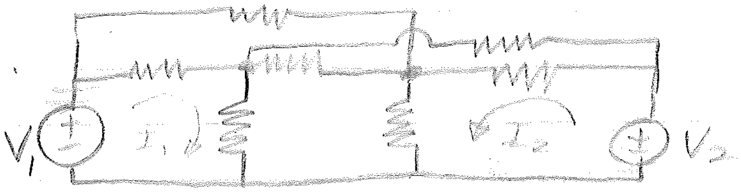
$V_o = -\frac{Z_f}{Z_1} V_i$

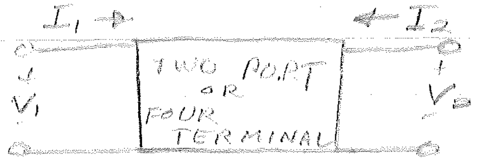
$\frac{V_1}{Z_1} = -\frac{V_o}{Z_f}$

ASSIGNMENT: 1,5,7 EXAM ON SEPT 26 !!

9-12-72

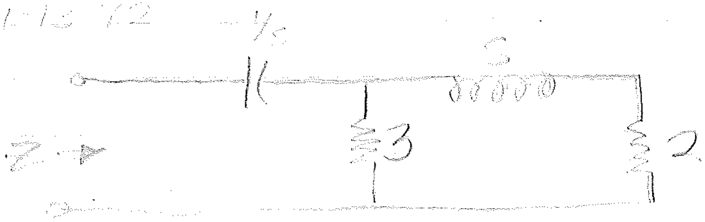
NON PLANAR NETWORK SOLUTION



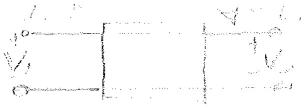


ASSIGNMENT: 1-9

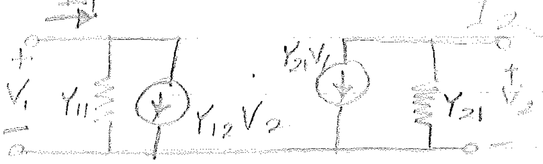
9-13-72



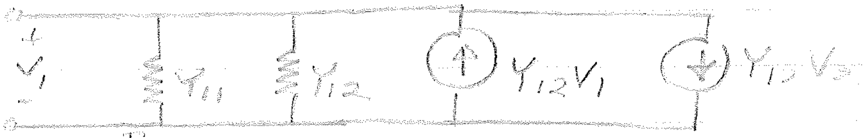
$$Z_s = \frac{1}{s} + \frac{1}{\frac{1}{3} + \frac{1}{s+2}} = \frac{1}{s} + \frac{3s+6}{s+5} = \frac{3s^2+7s+5}{s(s+5)}$$



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



FOR A PASSIVE NETWORK Y_{12} Y_{21}



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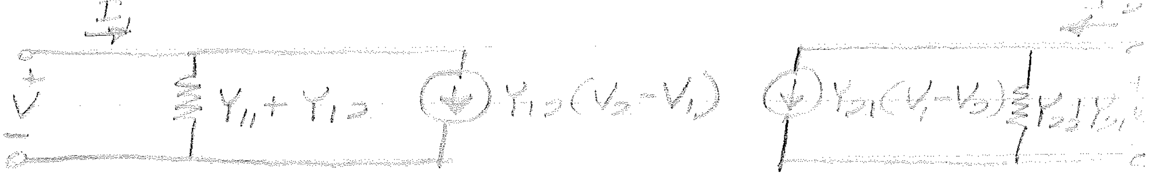

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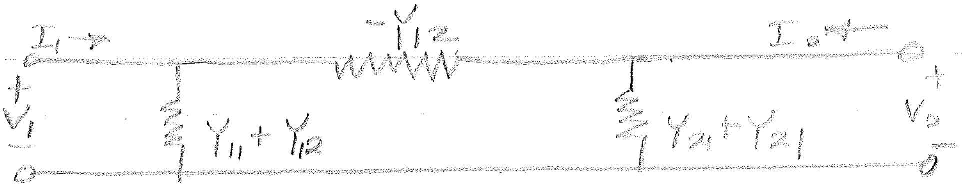


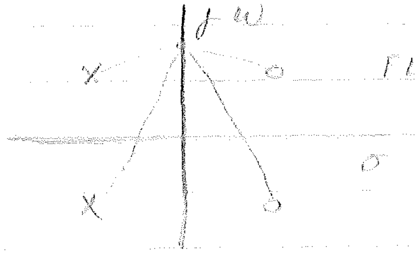
IF $Y_{12} = Y_{21}$

(TRUE FOR PASSIVE NETWORKS)

2.2. (HOMEWORK)

9-14-72
1-9 d)

$$\frac{S^2 - S + 1}{S^2 + S + 1}$$



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} I_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} I_2 = \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} V_1 + \begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix} V_2 \Updownarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_2 = \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} I_1 + \begin{bmatrix} Z_{12} \\ Z_{22} \end{bmatrix} I_2$$

OR $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Y^{-1} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

$$\Delta Y = Y_{11} Y_{12} - Y_{12} Y_{21}$$

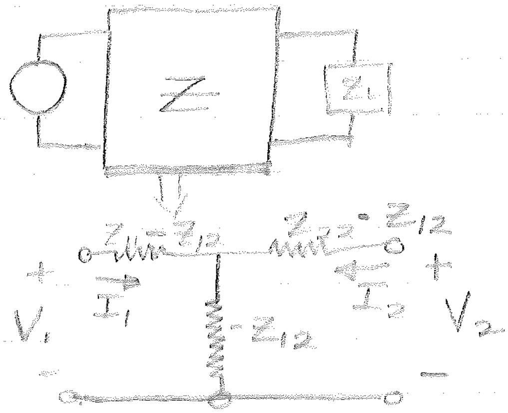
$$Y^{-1} = \begin{bmatrix} +Y_{22}/\Delta_Y & -Y_{12}/\Delta_Y \\ -Y_{21}/\Delta_Y & Y_{11}/\Delta_Y \end{bmatrix}$$

h PARAMETERS

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} V_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_2 = \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix} I_1 + \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} \cancel{V_1}$$

$$\Rightarrow \begin{bmatrix} 1 & -h_{12} \\ 0 & -h_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} h_{11} & 0 \\ h_{21} & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} h_{11} - \frac{h_{12} h_{21}}{h_{22}} & \frac{h_{12}}{h_{22}} \\ -h_{12}/h_{22} & 1/h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$



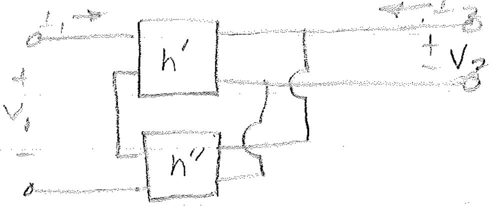
$$V_2 = 0 \Rightarrow G_{12} = \frac{I_1 Z_{12}}{I_1(Z_{11} - \cancel{Z_{12}} + Z_{12})} = \frac{Z_{12}}{Z_{11}}$$

(2-10)

9-18 ; (MON)

HYBRID:

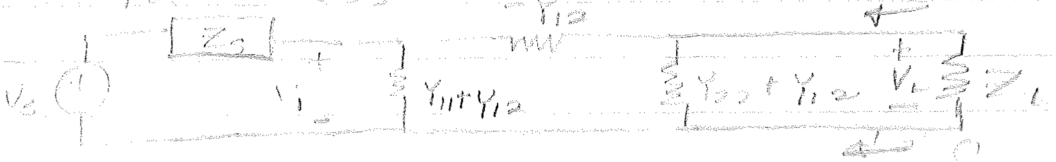
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

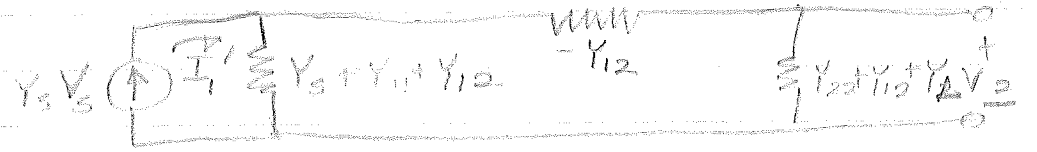


$h = h' + h''$

2-20 (pg 58)

7-11-72 (HW 5)





$$\begin{bmatrix} I_1' \\ 0 \end{bmatrix} = \begin{bmatrix} Y_S V_S \\ 0 \end{bmatrix} = \begin{bmatrix} Y_S + Y_{11} & y_{12} \\ y_{12} & y_{22}+Y_L \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

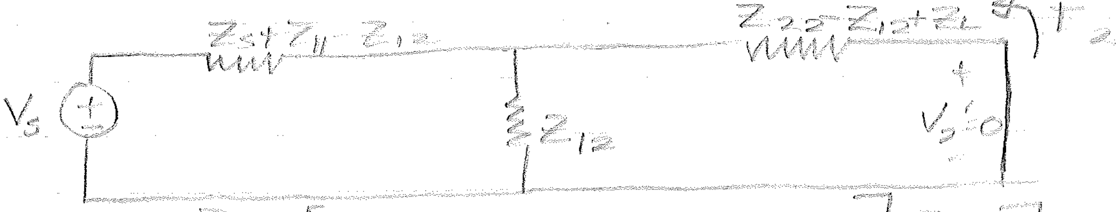
$$V_1 = Y_S V_S (Y_{22}+Y_L)/\Delta$$

$$V_2 = -Y_{12} Y_S V_S/\Delta$$

$$\Rightarrow \frac{V_2}{V_S} = \frac{-Y_{12} Y_S}{\Delta_Y + Y_S y_{22} + Y_L y_{11} + Y_S Y_L}$$

$$Z_L \to \infty \Rightarrow Y_L \to 0$$

$$\left.\frac{V_2}{V_S}\right|_{Y_L=0} = \frac{-y_{12} Y_S}{\Delta_Y + Y_S Y_{22}}$$

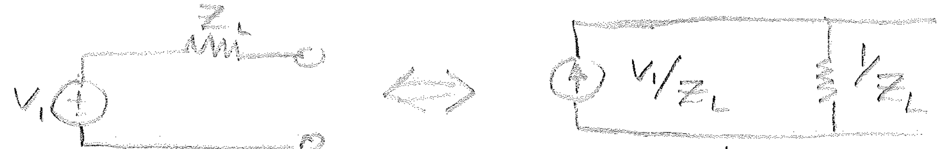


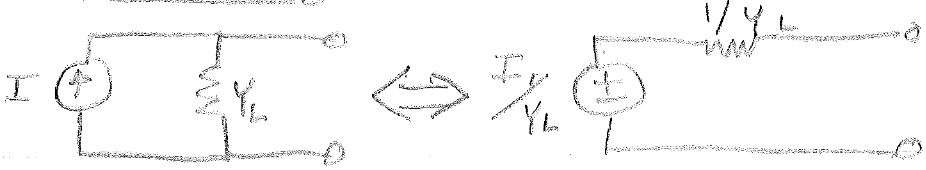
$$\begin{bmatrix} V_S \\ 0 \end{bmatrix} = \begin{bmatrix} Z_S + Z_{11} & Z_{12} \\ Z_{12} & Z_{22}+Z_L \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

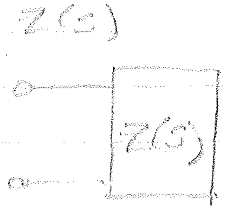
$$\Rightarrow I_2 = -Z_{12} V_S/\Delta$$

$$V_2 = I_L Z_L$$

$$\Rightarrow \frac{V_2}{V_S} = \frac{Z_{12} Z_L}{\Delta_Z + Z_S Z_{22} + Z_L Z_{11} + Z_S Z_L}$$







$$\Rightarrow Z(s) = \frac{a_0 s^n + a_1 s^{n-1} + \dots + a_n}{b_0 s^m + b_1 s^{m-1} + \dots + b_m}$$

$$P(s)/q(s)$$

$$= \left(\frac{a_0}{b_0}\right) \frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

MUST! $\Rightarrow m = n$; $m = n+1$; $m = n-1$

$Re\{Y(s)\} \geq 0$ if $Re\{s\} \geq 0$

$Y(s)$ is Real for s real (3. P.)

9·20·77

$$Z(s) = \frac{a_0 s^n + \dots + a_n}{b_0 s^m + \dots + b_m}$$

$$= \frac{a_0}{b_0} \frac{(s-z_1)(s-z_2)+\dots(s-z_n)}{(s-p_1)(s-p_2)+\dots(s-p_m)} \quad ; a_i, b_i \geq 0$$

$R_e[Y(s)] \geq 0$ FOR $R_e(s) \geq 0$

$Y(s)$ is Real FOR s REAL

$\angle[Y(s)] \leq |\angle s|$ FOR $|\angle s| < \pi/2$

$Re[Y(j\omega)] \geq 0$

9-2-1-72

$$Y(s) = \frac{(s+2)(s+3)}{(s+1)(s^2+3)} = \frac{A}{s+1} + \frac{B}{s+j\sqrt{3}} + \frac{B^*}{s-j\sqrt{3}}$$

IF $Y(s)$ IS POS. REAL,

$$Re(Y(s)) \geq 0 \quad \text{FOR } Re(s) \geq 0$$

Pg. 85

Pg. 86 HURWITZ CRITERIAN

$$p(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4$$

$$= m(s) + n(s)$$

$$\psi = \frac{m(s)}{n(s)}$$ (GREATEST PWR IN NUMERATOR)

$$= \frac{s^6 + 8s^4 + 17s^2 + 4}{3s^5 + 15s^3 + 12s}$$

$$\begin{array}{r|l} & \tfrac{1}{3}s \\ \hline 3s^5+15s^3+12s & s^6+8s^4+17s^2+4 \\ & \underline{s^6+5s^4+4s^2} \\ & 3s^4+13s^2+4 \end{array}$$

$$\begin{array}{r|l} & s \\ \hline 3s^4+13s^2+4 & 3s^5+15s^3+12s \\ & \underline{3s^5+13s^3+4s} \\ & 2s^3+8s \end{array}$$

$$\begin{array}{r|l} & \tfrac{3}{2}s \\ \hline 2s^4+8s & 3s^4+13s^2+4 \\ & \underline{3s^4+12s^2+} \\ & s^2+4 \end{array}$$

$$\begin{array}{r|l} & 2s \\ \hline s^2+4 & 2s^3+8s \\ & 2s^3+8s \end{array}$$

$$\Rightarrow \psi(s) = \tfrac{1}{3}s + \cfrac{1}{s + \cfrac{1}{\tfrac{3}{2}s + \cfrac{1}{2s}}}$$

IF ALL TERMS ARE POSITIVE & THERE ARE n TERMS (s^n), THERE ARE NO RIGHT HAND POLES

BUT IN THIS CASE, BOTH $m(s) \div N$, AND THUS $p(s)$ HAS COMMON TERMS OF s^2+4

AN EASIER WAY IS:

s	$n(s) = 3s^5+15s^3+12s$	$s^6+8s^4+12s^2+4 = m(s)$	$\frac{1}{3}s$
	$3s^5+13s^3+4s$	$s^6+5s^4+4s^2$	
$2s$	$2s^3+8s$	$3s^4+13s^2+4$	$\frac{3}{2}s$
	$2s^3+4s$	$3s^4+12s^2$	
		s^2+4	

AGAIN; A COMMON FACTOR $p(s)$

LET $p(s) = s\,W(s)\,p_1(s)$; $W(s) = s^2+4$

$$\text{LET } \psi = \frac{m(s)}{n(s)} = \frac{\frac{1}{2}[p(s)+p(-s)]}{\frac{1}{2}[p(s)-p(-s)]}$$

$$= \frac{p(s)/p(-s) + 1}{p(s)/p(-s) - 1}$$

IF $P(s)$ IS HERWITZ, $\psi(s)$ IS TOO, AS IS $1/p(s)$

$$\text{LET } \psi(s) = a_1 s + \frac{1}{\psi_1(s)}$$

9-25-72

(4-4) $p(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$

$n(s) = a_0 s^3 + a_2 s$

$m(s) = a_1 s^2 + a_3$

$\psi(s) = \frac{n(s)}{m(s)}$

$\frac{a_0}{a_1} > 0$ ~~$a_1 a_2 > a_0 a_3$~~

$a_1 a_2 - a_0 a_3 > 0 \Rightarrow a_1 a_2 > a_0 a_3$

~~$a_1 a_2 - a_0 a_2 > a_1^2$ (MORE RESTRICTIVE)~~

USING ROUTH

$s^3 \quad a_0 \quad a_2$

$s^2 \quad a_1 \quad a_3$

$s^1 \quad \frac{a_2 a_1 - a_0 a_3}{a_1}$

$s^0 \quad a_3$

$\therefore a_2 a_1 - a_0 a_3 > 0$

(4-5)

$\psi(s) = \frac{s^5 + 24 s^3 + 23 s}{8 s^4 + 22 s^2 + 6}$

1) 4-12, 4-19

4-26-72

$Y(S)$ MUST BE POSITIVE REAL FOR RLC SYNTHESIS

NECESARY AND SUFFICIENT CRITERIA

① IF $Y(S)$ IS REAL FOR REAL S AND

$Re\{Y(s)\} \geq 0$ FOR $Re(s) \geq 0$

→ SOLVED BY HURWITZ OR ROUTH ALGORYTHM

② IF POLES OF $Y(S)$ AREN'T IN THE RIGHT HAND PLANE AND $j\omega$ AXIS POLES ARE SIMPLE, AND HAVE POSITIVE REAL RESIDUES

→ SOLVED BY HURWITZ AND ROUTH

③ IF $Re\, Y(j\omega) \geq$ FOR $0 \leq \omega \leq \infty$

LET $A(\omega^2) = [m_1 m_2 - n_1 n_2]_{s=j\omega}$

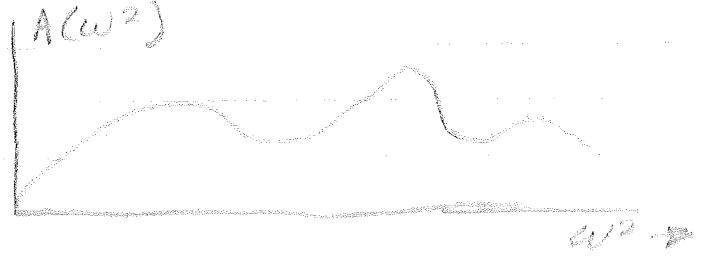
$\ni$ m_1 = EVEN PART OF NUMERATOR

n_1 = ODD " " "

m_2 = EVEN " " DENOMINATOR

n_2 = ODD " " "

EMPLOY STURMS ALGORITHM;



4-3) a) LET $F(s_1) = Re[F(s_1)] + Im[F(s_1)]$

$= m(s_1) + n(s_1)$

$F(s_1^*) = m(s_1^*) + n(s^*)$

$= m(s) - n(s_1)$

WRONG

b) $A(s_1^*)\ B(s_1^*) = \overline{A(s_1)\ B(s_1)}$

LET $F(s_1) = A(s_1)\underline{B(s_1)}$

$\Rightarrow F(s^*) = F(s)$

$\Rightarrow A(\bar{s}_1)\ B(\bar{s}) = AB$

4-12) ALL COEFF ARE > 0

$A(\omega)^2 = \omega^6 - 3\omega^4 - \omega^2 + 3$

$= (\omega^4 - 1)(\omega^2 - 3)$

$= (\omega^2 + 1)(\omega^2 - 1)(\omega^2 - 3)$

CAN'T SYNTHYSIZE

STURM

$P_0(x) = x^3 - 3x^2 - x + 3$

$P_1(x) = 3x^2 - 6x - 1$

FIND $\frac{P_0(x)}{P_1(x)} =$

TEST ON MONDAY

9-27-72

4.2) b) $A(\bar{s})B(\bar{s}) = A(s_1)B(\bar{s})$

a)

$$F(s) = \frac{F(s)+F(\bar{s})}{2} + \frac{F(s)-F(\bar{s})}{2}$$

$$= \frac{F(s)+F(\bar{s})}{2} - \frac{\dot{F}(s)-F(\bar{s})}{2}$$

$$= \underline{Re\{F(s_1)\}} - Im\{F(s_1)\}$$

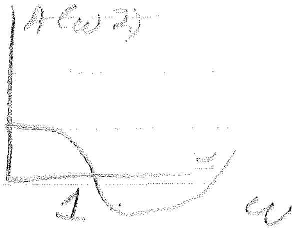
$$= F(s)$$

4-12) a) ALL COEFF ARE POS ∴ $A(\omega^2) > 0$

b) $A(\omega^2) = \omega^6 - 3\omega^4 - \omega^2 + 3$

$= (\omega^4 - 1)(\omega^2 - 3)$

$= (\omega^2 - 1)(\omega^2 + 1)(\omega^2 - 3)$



$P_0(x) = x^3 - 3x^2 - x + 3$

$P_1(x) = 3x^2 - 6x - 1$

$$\frac{P_0(x)}{P_1(x)} = \frac{x^3 - 3x^2 - x + 3}{3x^2 - 6x - 1} = \frac{1}{3}x - \frac{1}{3} - \frac{P_2(x)}{P_1(x)}$$

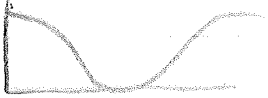
$P_2(x) = -\frac{8}{3}x + 8/3$

$$\frac{P_1(x)}{P_2(x)} = 3x - 3 - \frac{P_3}{P_2}$$

	P_0	P_1	P_2	P_3	V
0	+	−	−	+	2
∞	+	+	+	+	0

⇒ 2 − 0 = 2 REAL ZEROS FOR $0 < \omega^2 < \infty$

CHECK FOR



0	+	−	−	+	2
2	−	−	+	+	1
∞	+	+	+	+	0

c) $A(\omega^2) = \omega^8 + 2\omega^6 - 3\omega^4 - 4\omega^2 + 2$ COULD QUIT

FOR $\omega = 1$, $A(\omega^2) = -2 < 0$

$P_0(X) = X^4 + 2X^3 - 3X^2 - 4X + 2$

	P_0	P_1	P_2	P_3	P_4	
0	+	−	−	+	−	3
1	−	0	+	+	−	2
∞	+	+	+	−	−	1

∴ ROOTS ARE SIMPLE, AND $A(\omega^2) < 0$

d) $A(\omega^2) \geq 0$

4-19) $Y(s) = \frac{s^3 + 6s^2 + 7s + 3}{s^2 + 2s + 1} = \frac{\quad}{(s+1)^2}$

1) NO RT. H-P. POLES

2) WHEN j AXIS POLES ARE SIMPLE w/+ RESIDUS

3) $Re[Y(j\omega)] \geq 0$, $0 \leq \omega \leq \infty$; $A(\omega^2) \geq 0$

① CHECK NUMERATOR

$$\begin{array}{c|cc} s^3 & 1 & 7 \\ s^2 & 6 & 3 \\ s^1 & \frac{39}{6} & \\ s^0 & 3 & \end{array}$$

∴ HURWITZ

$A(\omega^2) = (m_1 m_2 - n_1 n_2)$

$= \left[(6s^2 + 3)(s^2 + 1) - (s^3 + 7s)2s\right]_{s=j\omega}$

$= 4\omega^4 + 5\omega^2 + 3 > 0$

$a_i > 0 \Rightarrow$ POSITIVE REAL

b) $Y(s) = \frac{s(s+3)(s+5)}{(s+1)(s+2)}$

NUM & DEN ARE HURWITZ BY INSPECTION

$A(\omega^2) = 5\omega^4 + 29\omega^2 \geq 0$

c) $Y(s) = \frac{s^3 + s^2 + 1}{s^3 + 2s^2 + s + 2}$

ROUTH

DEN: $\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s & 0^{\epsilon} & \\ s^1 & 2 & \end{array}$ $\rightarrow 2s^2 + 2 = 0$

MODIFIED HURWITZ

NUM: $\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 1 \\ s^1 & 0^{\epsilon} & \\ s^0 & 1 & \end{array}$ $\rightarrow s^2 + s = 0$

$\Rightarrow Y(s) = \frac{s+1}{s+2}$

d) $Y(s) = \frac{s^5 + 5s^3 + 4s}{s^4 + 8s^2 + 15} = \frac{s(s^2+1)(s^2+4)}{(s^2+3)(s^2+5)}$

RESIDUES:

NOTE $Y(s) = s + Y_1(s) \Rightarrow Y_1(s) = -\frac{3s^3 + 11s}{s^4 + 8s^2 + 15}$

$Y_1(s) = \frac{A}{s+j3} + \frac{\bar{A}}{s-j3} + \frac{B}{s-j\sqrt{5}} - \frac{B}{s+j\sqrt{5}}$

RESIDUES ARE REAL, BUT < 0

e) (−) AND POS COEFF ⇒ NOT POSITIVE REAL

f) DEN. IS HURWITZ FROM ROUTH, WITH NO j AXIS POLES, SO IS NUM

$A(\omega^2) = \omega^8 - 4\omega^6 + 6\omega^4 + 4\omega^2 + 1$

USE STURM ⇒ + REAL

g)

	P_0	P_1	P_2	P_3	
0	+	+	−	−	1
∞	+	+	+	−	1

$A(\omega^2) > 0$ & $Y(s)$ IS POS REAL

h) NOT POS. REAL

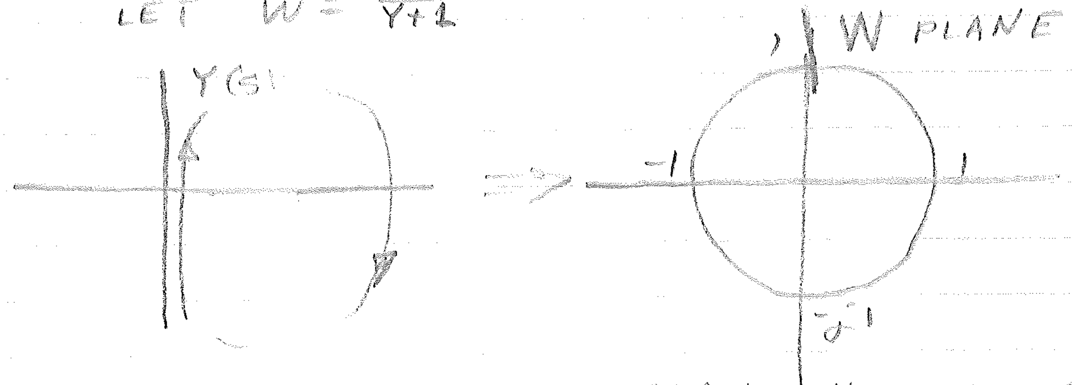
i)

9-30-71

$$Y(S) = \frac{p(S)}{q(S)}$$

LOOK AT $p(S) + q(S)$

LET $W = \frac{Y-1}{Y+1}$



PUTS RIGHT HALF PLANE IN UNIT CIRCLE

$$W = \frac{Y-1}{Y+1} = \frac{\alpha + jB - 1}{\alpha + jB + 1} \Rightarrow \sqrt{\frac{(\alpha-1)^2 + jB}{(\alpha+1)^2 + jB}} = |W|$$

$Y(j\omega) \geq 0 \quad ; \quad 0 \leq \omega \leq \infty$

$$|W(s)| \leq 1 \quad \text{FOR} \quad S \geq 0$$

$Y(S) = \frac{1+W(S)}{1-W(S)} \Rightarrow$ P.r. IF $W(S)$ IS REAL FOR S REAL

$$|W(S)| \leq 1 \quad \text{FOR} \quad \text{Re}\, S \geq 0$$

NOTE THAT POLES OF W ARE ZEROS OF $Y+1$

IF Y IS P.R., THEN $1+Y$ IS P.R.

IS S_1 IS IN RHP, AND $Y(S_1) + 1 = 0$,

THEN $Y(S_1) = -1$

$$W = \frac{Y-1}{Y+1} = \frac{p(S) - q(S)}{p(S) + q(S)}$$

POLE REMOVAL

1) @ ∞
2) @ 0
3) CONJUGATE IMAGINARY
4) PARTIAL POLE
5) CONSTANT (NOT A POLE)

10-2-72

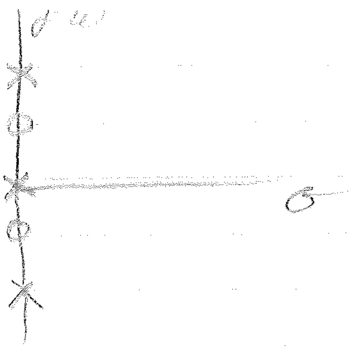
TEST IS OPEN BOOK - OPEN NOTES

10-5-72

(LC)

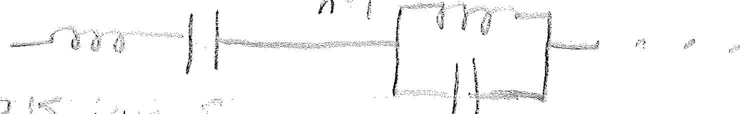
$Z_{LC}(S) = \frac{m_1}{n}$ OR $\frac{n}{m}$

$Z_{LC}(j\omega) = jX(\omega)$ $\frac{dX(\omega)}{d\omega} > 0$



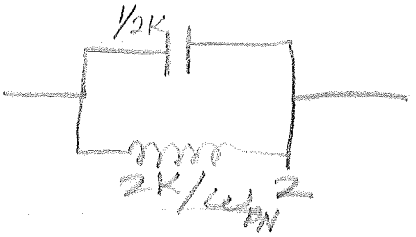
FOSTER'S FIRST EXPANSION

$Z_{LC}(S) = HS + \frac{K}{S} + \sum_{n=1}^{N} 2K_j \omega_{pn}{}^{s} \Big/ \left(S^2 + \omega_{pN}^2\right)$



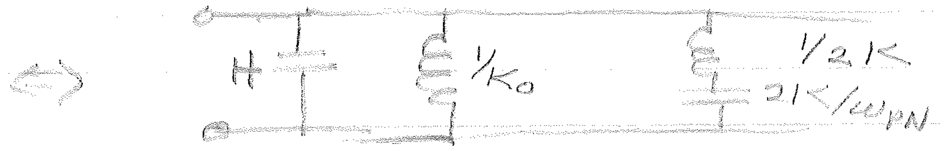
$$\sum_{n=1}^{N} \frac{2K_j\omega_{pn}{}^{s}}{S^2 + \omega_{pN}^2}$$

$$= \sum_{n=1}^{N} \frac{1}{\frac{S}{2K} + 1\Big/\left(2KS/\omega_{pN}^2\right)}$$



FOSTERS 2nd FORM:

$Y_{LC}(s) = \frac{1}{Z(s)} = Hs + \frac{K_0}{s} + \Sigma$



(5-2)

$$Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$$

FIRST FOSTER FORM

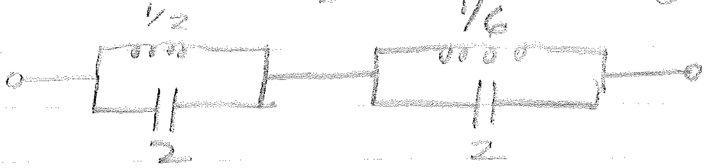
$$Z(s) = \frac{As}{s^2+1} + \frac{Bs}{s^2+3}$$

$A = +1/2$

$B = 1/2$

$$> Z(s) = \frac{\frac{1}{2}s}{s^2+1} + \frac{\frac{1}{2}s}{s^2+3}$$

$$= \frac{1}{2s + \frac{1}{\frac{1}{2}s}} + \frac{1}{2s + \frac{1}{\frac{1}{6}s}}$$



SECOND FOSTER FORM

$$Y(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$

$$= As + \frac{Bs}{s^2+2} + \frac{C}{s}$$

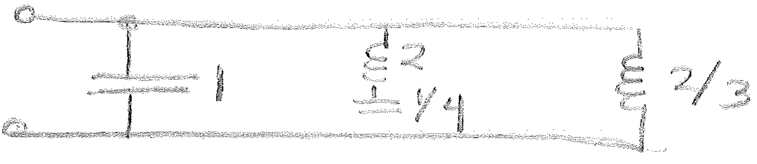
$A = 1$

$B = \frac{1}{2}$

$C\ 3/2$

$$Y(s) = s + \frac{\frac{1}{2}s}{s^2+2} + \frac{3/2}{s}$$

$$= s + \frac{1}{2s + \frac{s}{1/4}} + \frac{1}{\frac{3}{2}s}$$



NEED @ ORIGIN TO ALTERNATE POLES AND ZEROS

$$5\text{-}8)\quad Z(s) = H\frac{s\left(s^2+(2\pi\,2400)^2\right)^2}{\left[s^2+(2\pi 1200)^2\right]\left[s^2+(2\pi 5000)^2\right]}$$

CAUER'S FIRST (REMOVAL OF POLES @ ∞

FROM BOOK

$$Z(s) = \frac{s^4+10s^2+9}{s^3+4s}$$

$$= Z_1 + Z_2$$

$$= s + Z_2$$

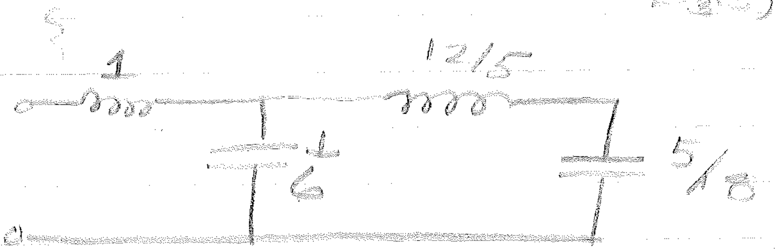
$$\Rightarrow Z_2(s) = Z(s) - s$$

$$= \frac{6s^2+9}{s^3+4s}$$

$$\Rightarrow Y_2(s) = \frac{s^3+4s}{6s^2+9}$$

$\vdots$

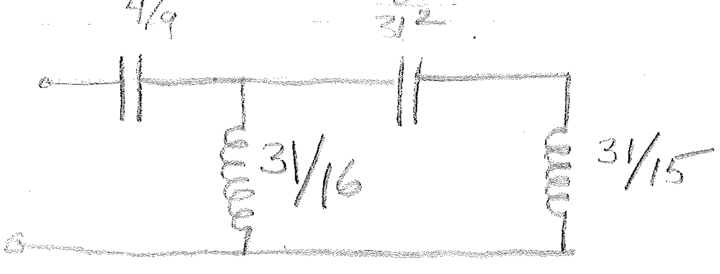
$$\therefore Z(s) = Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s)+\cfrac{1}{Y_4(s)}}}$$



CAUER'S SECOND (REMOVAL OF POLES @ 0)

$$Z(s) = \frac{s^4+10s^2+9}{s^3+4s} = \frac{9+10s^2+s^4}{4s+s^3}$$

Y			Z
$\frac{16}{31s}$	$4s+s^3$	$9+10s^2+s^4$	$9/4s$
	$4s+\frac{16}{31}s^3$	$9+\frac{9}{4}s^2$	
$\frac{15}{31}\frac{1}{s}$	$\frac{15}{31}s^3$	$\frac{31}{4}s^2+s^4$	$\frac{31\cdot 31}{15\cdot 4s}$
	$\frac{15}{31}s^3$	$\frac{31}{4}s^2$	
		s^4	
	0		

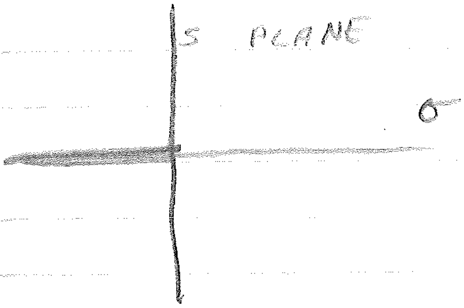


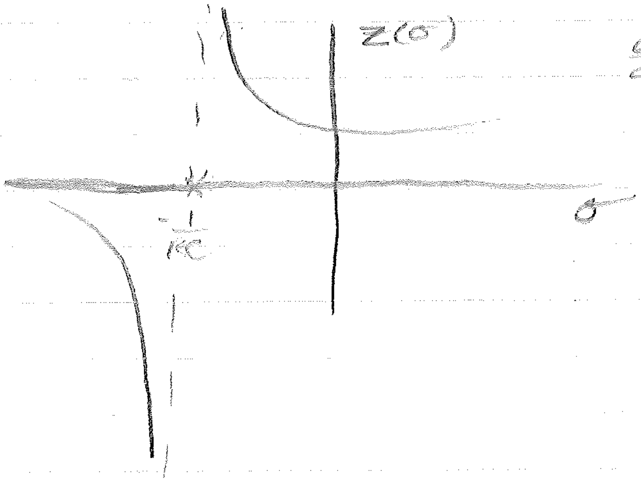
10-5-72

CHAP. 6

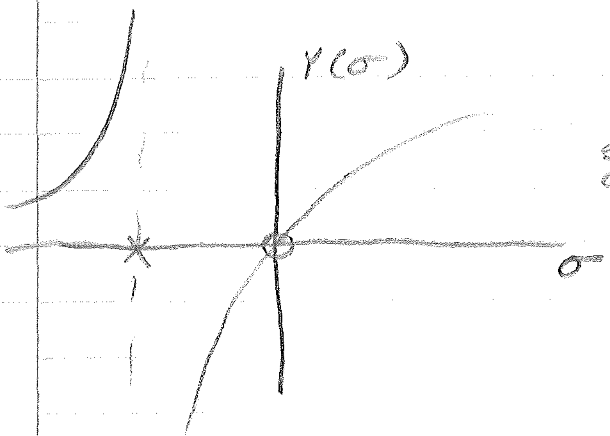
$$Z = \frac{1}{\frac{1}{R} + sC} = \frac{1}{C}\ \frac{1}{s + 1/RC}$$

$$Y = \frac{1}{R + \frac{1}{Cs}}\quad \frac{1}{R}\ \frac{s}{s + 1/RC}$$



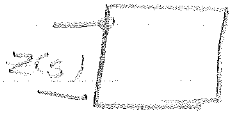


$\frac{dZ}{d\sigma} < 0$



$\frac{dY(\sigma)}{d\sigma} > 0$

$Z_{LC}(s) = \frac{m}{n}$ OR $\frac{n}{m}$



$\left[\quad \right] I = V$ (FROM MESH)

$$I = V_1 \Delta_{11} / \Delta \Rightarrow Z = \frac{V_1}{I} = \frac{\Delta(s)}{\Delta_{11}(s)}$$

IN RC

$$R + \frac{1}{Cs}$$

$$p^2 = s$$

$$\frac{1}{p}\left(Rp + \frac{1}{Cp}\right)$$

SO FOR R-C NETWORK

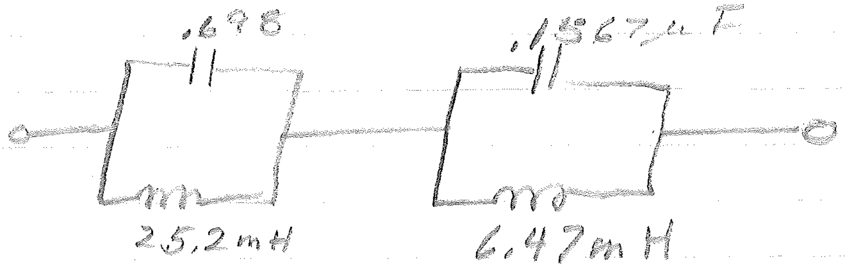
$$Z_{RC} = \frac{\Delta(s)}{\Delta_{11}(s)} = \frac{1}{p} Z_{LC}(p)\Big|_{p^2=s}$$

$$Y_{RC}(s) = p\left[Y_{LC}(p)\right]\Big|_{p^2=s}$$

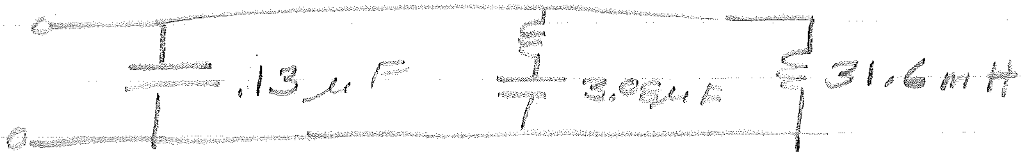
10-13-72

5-8)

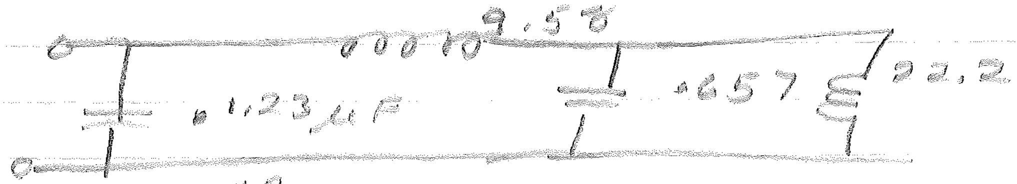
$$Z(s) = \frac{.1834}{s^2+1.2^2} + \frac{.816}{s^2+3^2}$$



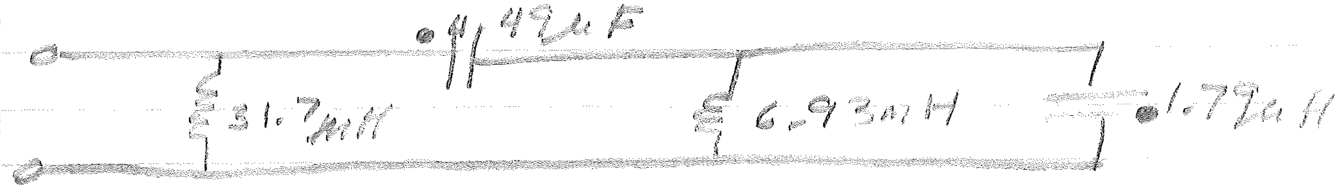
~~2~~ FOSTER'S SECOND FORM



CAUER'S FIRST

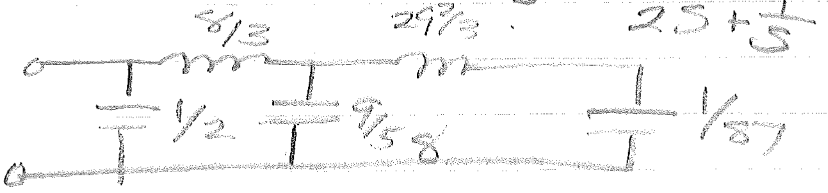


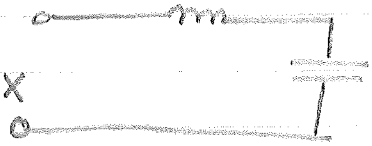
2ND

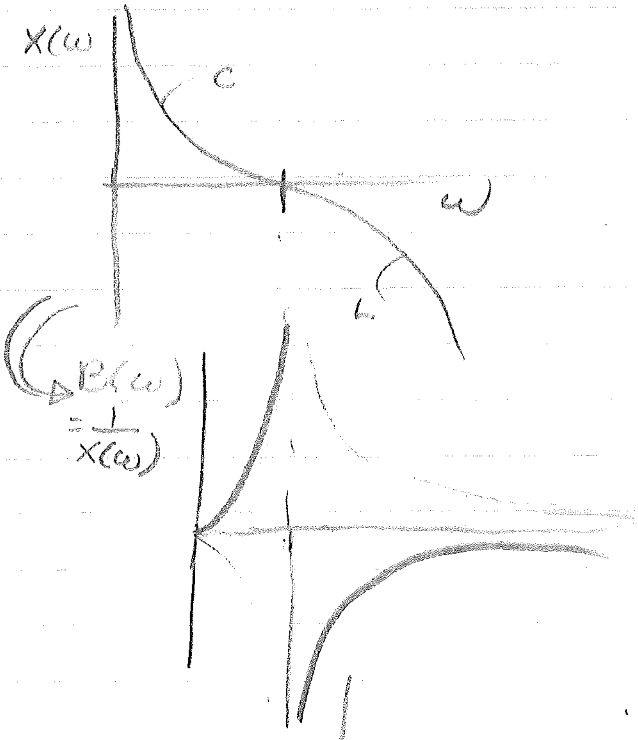


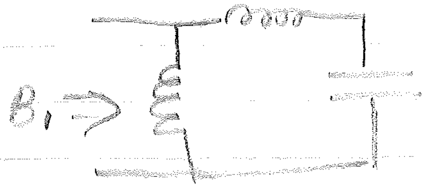
5-12)

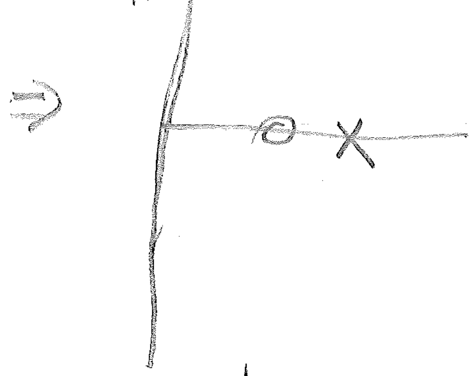
$$Z = \frac{1}{s} + \cfrac{1}{s + \cfrac{1}{\frac{1}{s} + \cfrac{1}{\frac{1}{s} + \cfrac{1}{2s + \frac{1}{s}}}}}$$

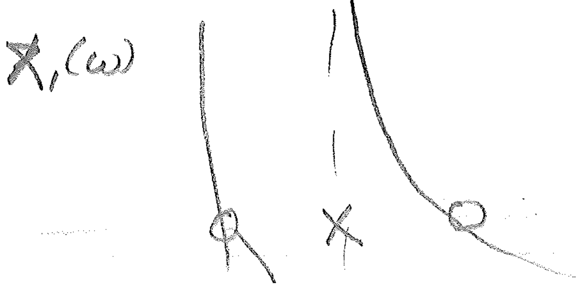




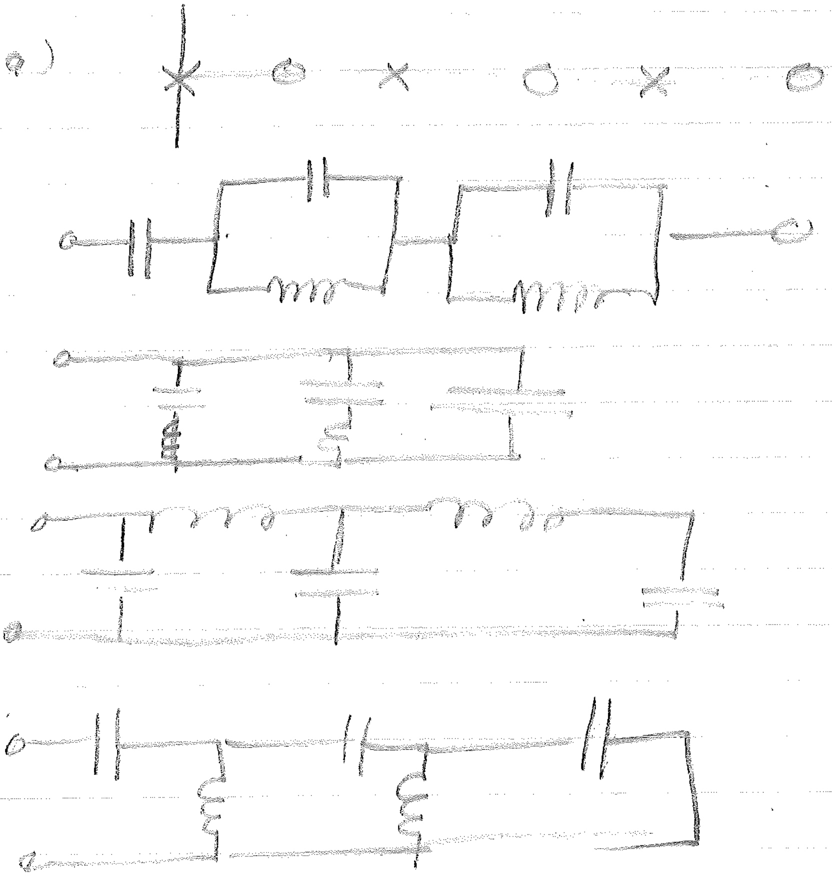




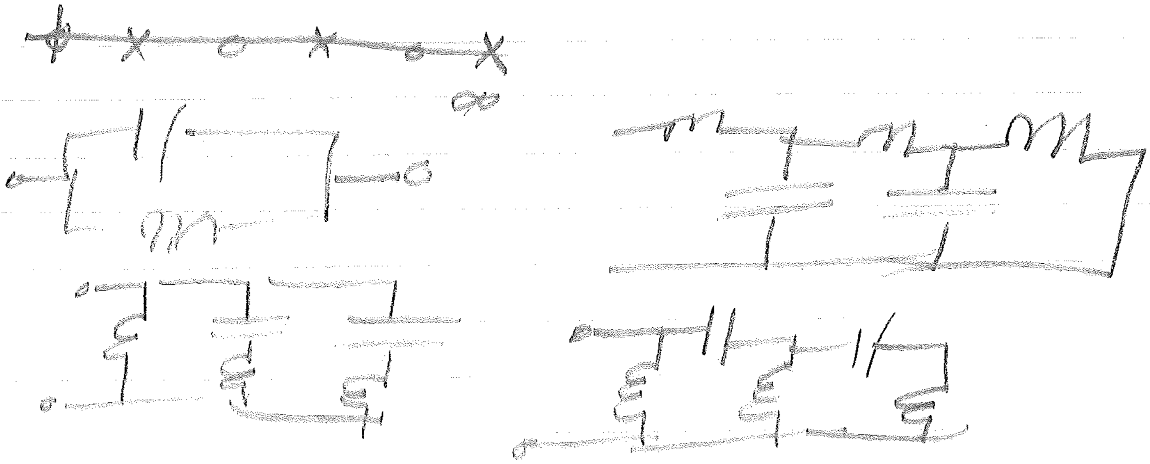




5-14) a)



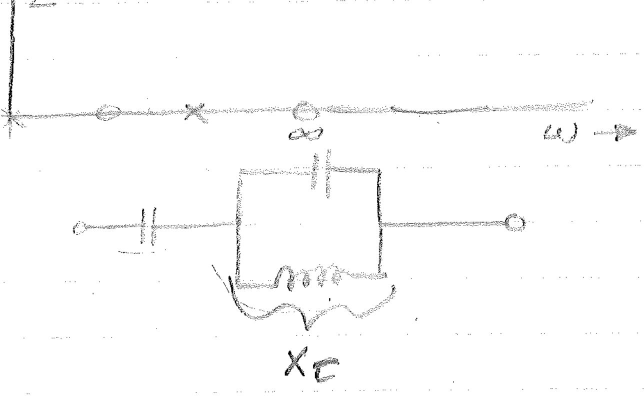
b)

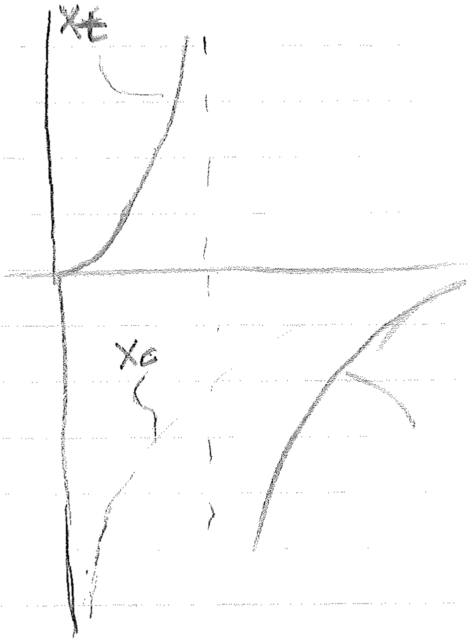


10-10-72

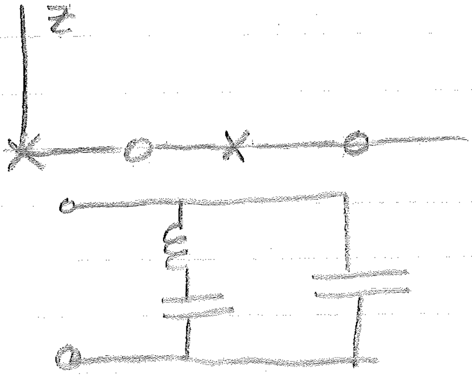
FOSTER'S FIRST

(514)





FOSTER'S SECOND

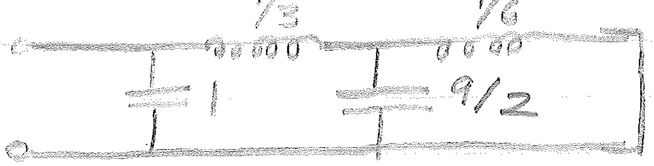


(5-16) $Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+4)} = \frac{s^3+2s}{s^4+5s^2+4}$

Z

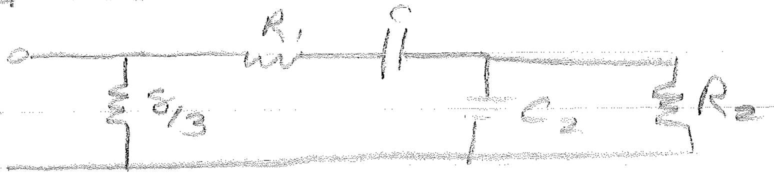
$\frac{1}{3}s$	s^3+2s	s^4+5s^2+4	s
	$s^3+\frac{4}{3}s$	s^4+2s^2	
$\frac{1}{6}s$	$\frac{2}{3}s$	$3s^2+4$	$\frac{9}{2}s$
	$\frac{2}{3}s$	$3s^2$	
	0	4	

FIRST CAUER:



[redacted]

(6-1)



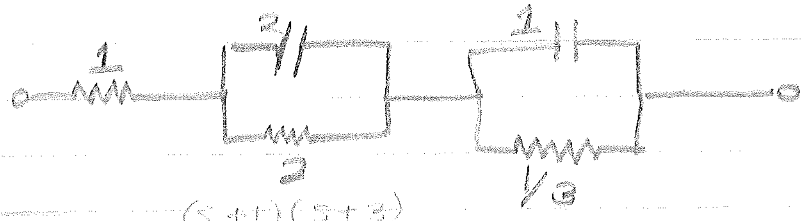
$$Z = \cfrac{1}{\frac{3}{8} + \cfrac{1}{R_1 + \frac{1}{sC} + \cfrac{1}{\frac{1}{R_L} + sC_2}}}$$

6-3) $Z(s) = \frac{(s+2)(s+5)}{(s+1)(s+3)}$

$= 1 + \frac{A}{s+1} + \frac{B}{s+3}$

$A = 2 \; ; B = 1$

$Z(s) = 1 + \frac{1}{\frac{1}{2}s + \frac{1}{2}} + \frac{1}{s+3}$

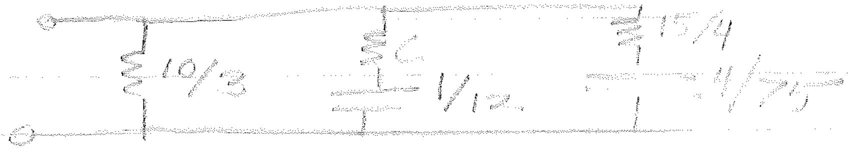


$Y(s)/s = \frac{(s+1)(s+3)}{s(s+2)(s+5)}$

$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5}$

$Y(s) = \frac{3}{10} + \frac{1/6 \, s}{s+2} + \frac{4/15 \, s}{s+5}$

$= \frac{1}{10/3} + \frac{1}{6 + \frac{12}{s}} + \frac{1}{\frac{15}{4} + \frac{75}{4s}}$

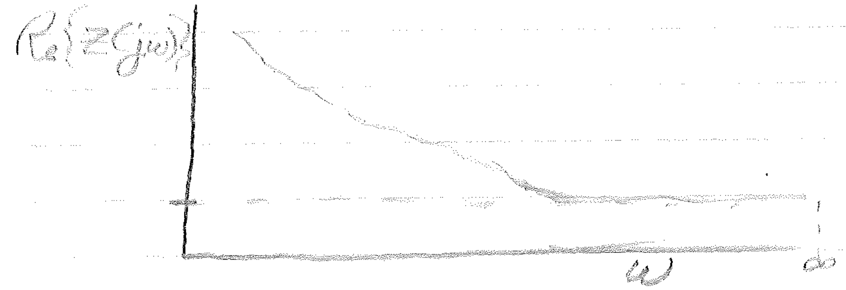


10-12-72

6-8 ; 6-11 ; (2-25 ; 6-15) ; 6-1 ; 6-3

CAUER:

$$Z_{RC}(s) = \frac{K_0}{s} + \sum_i \frac{K}{s+\sigma_i} + K_\infty$$



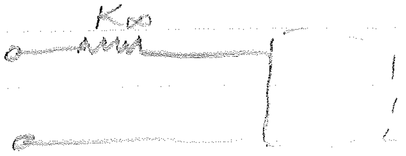
$$Re\{Z_{RC}(s)\} = \frac{Z(s)+Z(-s)}{2}$$

$$= \frac{Z(j\omega)+Z(-j\omega)}{2}$$

$$= \left[\sum_i K_i \frac{\sigma_i}{\omega^2+\sigma_i^2}\right] + K_\infty$$

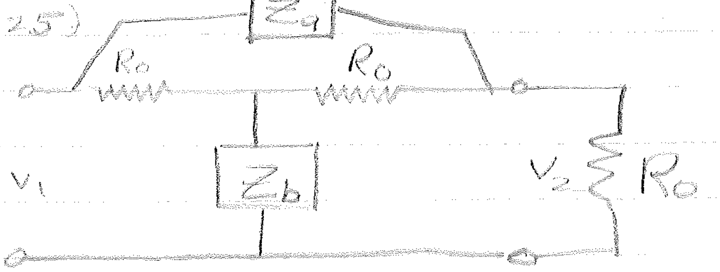
NOW $\sigma_i > 0$

$$\lim_{\omega \to 0}\left[\sum_i K_i \frac{\sigma_i}{\omega^2+\sigma_i^2} + K_\infty\right] = K_\infty$$



2-25-72

(2-25)

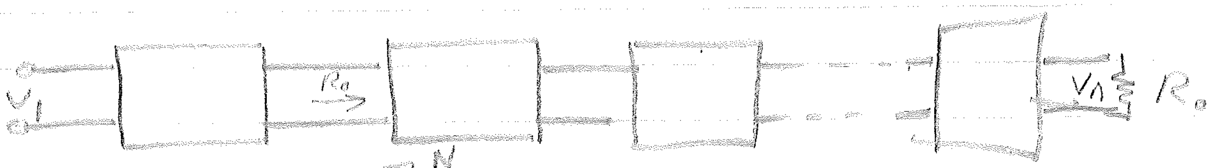


$Z_b Z_a = R_0^2$

$$\frac{v_1}{v_2} = \frac{1}{2}\left(1 + \frac{Z_a}{R_0}\right)$$

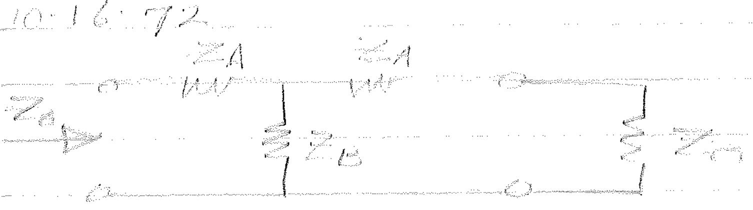
$$\Rightarrow \frac{v_2}{v_1} = \frac{2}{(1 + Z_a/R_0)}$$

(6-15)

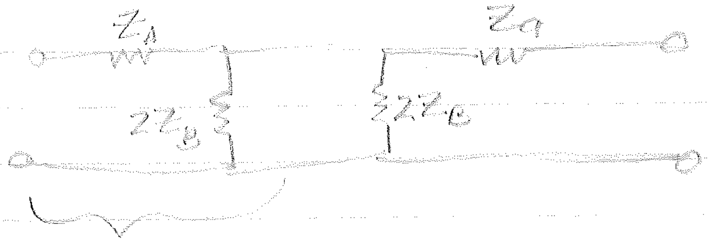


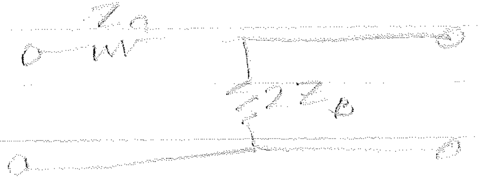
$$\frac{v_N}{v_1} = \frac{2^N}{(1+Z_1)(1+Z_2)\cdots(1+Z_n)}$$

10-16-72



$Z_0 = \sqrt{Z_A^2 + 2Z_A Z_B}$

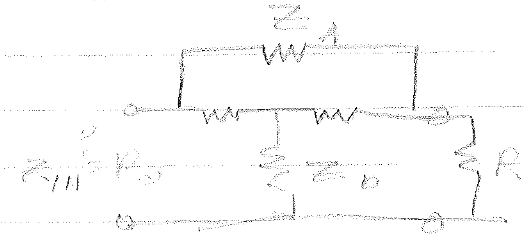




$Z_{oc} = Z_A + 2Z_B$

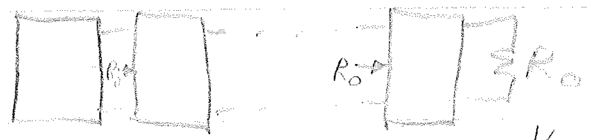
$Z_{sc} = Z_A$

$Z_0 = \sqrt{Z_{oc}Z_{sc}} = \sqrt{Z_A^2 + 2Z_AZ_B}$

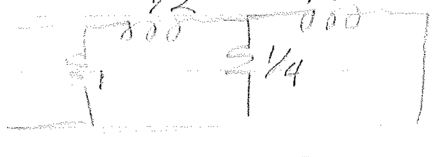


$Z_a Z_b = R_0^2$

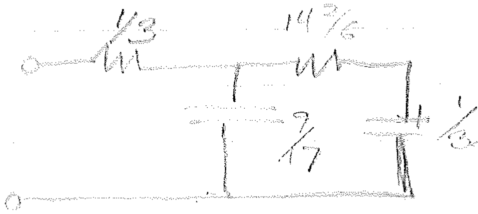
$\rightarrow \frac{V_2}{V_1} = \underline{1} + \overset{2}{\overline{Z_A/R_0}}$

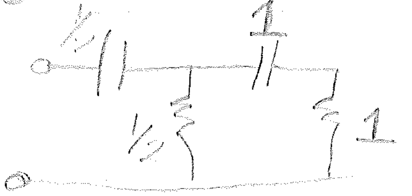


(6-8)(a) Z_{RL}

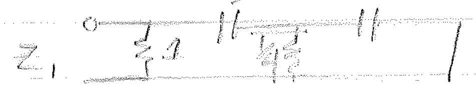


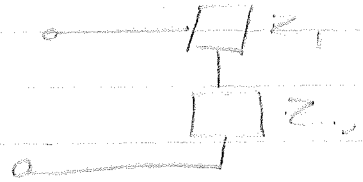
(b) Z_{RC}



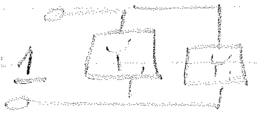


6-11)





$Y_1 + Y_2 = 1$



$Z_1 + Z_2 = 1$

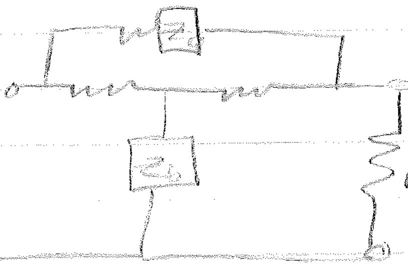
a) $Z_2 = 1 - Z_1$ WILL BE POS. REAL IF $Re(Z_1) \le 1$

$Y_2 = 1 - Y_1$ " " " $Re(Y_1) \le 1$

$$Y_1 = \frac{1}{\frac{1}{2s} + \frac{1}{4+7s}}$$

6-17-72

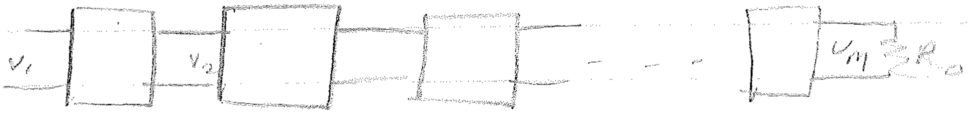
2-25)



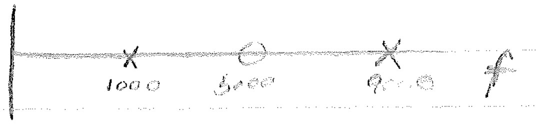
$$\frac{V_2}{V_1} = \frac{2}{1 + Z_a/R_o}$$

IFF $Z_a Z_b = R_o^2$

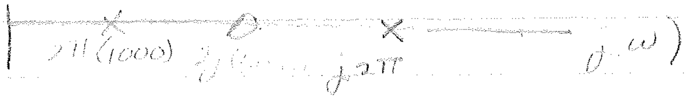
6-15)



$$\frac{V_M}{V_1} = \left(\frac{2}{1 + \frac{Z_{a1}}{R_o}}\right)\left(\frac{2}{1 + \frac{Z_{a2}}{R_o}}\right)\left(\quad\right)\left(\quad\right) \cdots$$



$Z(j100) = 600$



$$Z = \frac{s^2 + 2\pi(5000)^2}{[s^2 + (2\pi 1000)^2][s^2 + (2\pi 9000)^2]}$$

RLC

FOSTER PREAMBLE

P.REAL,

PULL POLES @ 0 (∞)

" " ON j-AXIS

PULL OUT RESISTANCE

EX)

$$Z = \frac{12s^5 + 8s^4 + 32s^3 + 14s^2 + 8s + 2}{4s^5 + 14s^4 + 10s^3 + 9s^2 + 4s + 1} \quad P.R.$$

NO ∞ OR 0 POLES

USE HURWITZ (ROUTH) TO FIND j AXIS POLES

NUM s^5	12	32	8
s^4	8	14	2
s^3	11	5	
s^2	10,36	2	
s^1	2.87		
s^0	2		

DEN s^5	4	10	4
s^4	15	9	1
s^3	$\frac{52}{7}$	$\frac{26}{7}$	
s^2	2	1	
s^1	0		
s^0			

$\rightarrow (2s^2 + 1) = 0$

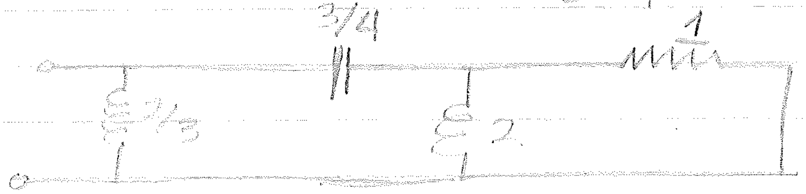
$$\Rightarrow Z(s) = \frac{K_1 s}{s^2 + 1/2} + \sim$$

$$\begin{array}{r} 2s^3 + 7s^2 \\ 2s^2 + 1 \overline{\big) \, 4s^5 + 14s^4 + 10s^3 + 9s^2 + 4s + 1} \\ 4s^5 \qquad\qquad + 10s \end{array}$$

2.15.72

$$Z(s) = s^3 + \frac{4}{3}s^2 + \frac{2}{3}s \Big/ (s^3 + 2s^2 + 2s + 1) \qquad (P.R.)$$

$$\begin{array}{r|r|r|l}
\frac{4}{3s} & \frac{2}{3}s + \frac{4}{3}s^2 + s^3 & 1 + 2s + 2s^2 + s^3 & \frac{3}{2s} \quad Y \\
 & \frac{2}{3}s + \frac{4}{3}s^2 & 1 + 2s + \frac{3}{2}s^2 & \\
\hline
1 & s^3 & \frac{1}{2}s^2 + s^3 & \frac{1}{2s} \\
 & s^3 & \frac{1}{2}s^2 & \\
\hline
 & 0 & s^3 &
\end{array}$$



Ex 3) $Z(s) = \dfrac{6s^2 + 6s + 9}{s^2 + s + 1}$ (P.R.)

a) NO 0 OR ∞ POLES } MINIMUM REACTANCE
b) NO j AXIS POLES } MINIMUS SUSCEPTANCE
→ CAN'T PULL OUT L-C

c) RESISTANCE REMOVAL

$$Ev(Z(s)) = (m_1 m_2 - n_1 n_2) \Big/ m_2^2 - n_2^2$$

$$= \frac{(6s^2+9)(s^2+1) - 6s^2}{(s^2+1)^2 - s^2}$$

$$= \frac{6s^4 + 9s^2 + 9}{s^4 + s^2 + 1}$$

$$\Rightarrow Re(Z(j\omega)) = Ev[Z(s)]_{s=j\omega}$$

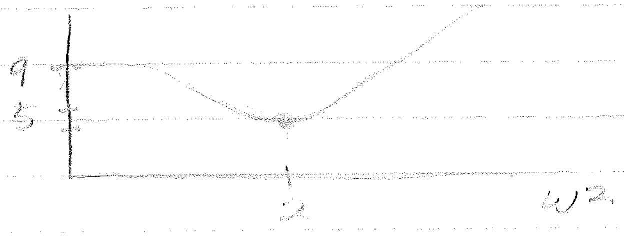
$$= \frac{6\omega^4 - 9\omega^2 + 9}{\omega^2 - \omega^2 + 1}$$

FIND MIN. VALUE

$$\frac{d}{d(\omega^2)}[Re(Z(j\omega)) = \frac{(\omega^4 - \omega^2 + 1)(12\omega^2 - 9) - (6\omega^4 - 9\omega^2 + 9)(2\omega^2}{D^2(\omega^2)}$$

$$= \frac{3\omega^2 - 6\omega^2}{D^2(\omega^2)} = \frac{3\omega^2(\omega^2 - 2)}{D^2(\omega^2)} = 0$$

$$\Rightarrow \omega^2 = 0; \; \omega^2$$



SO REMOVE RESISTANCE OF 5Ω

$$\Rightarrow Z_1(s) = Z(s) - 5 = \frac{6s^2+6s+9-5(s^2+s+1)}{s^2+s+1}$$

$$= \frac{s^2+s+4}{s^2+s+1}$$ MIN X, B

MIN R, G

TO BRUNE'S METHOD:

$$Re[Z(j\sqrt{2})] = 0 \Rightarrow Z(j\sqrt{2}) = X(j\sqrt{2})$$

FIND FUNCTION'S ODD PART;

$$Odd[Z(s)] = (m_2 n_1 - m_1 n_2)/(m_2^2 - n_2^2)$$

$$= \frac{(s^2+1)s-(s^2+4)s}{(s^2+1)^2-s^2}$$

$$= \frac{-3s}{s^4+s+1}$$

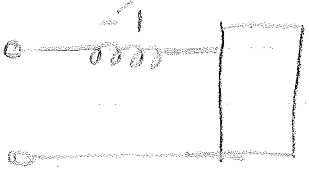
$$Im[Z(j\omega)] = Odd[Z(s)]\Big|_{s=j\omega}$$

$$\Rightarrow j\,Im[Z(j\sqrt{2})] = \frac{-3j\sqrt{2}}{4-2+1} = -j\sqrt{2}$$

$\therefore$ @ $\omega = \sqrt{2}$; $X = -\sqrt{2}$ (NEGATIVE INDUCTANCE)

$L = -1$

$$\Rightarrow Z_1(s) = Z(s) - (-s) = Z(s) + s$$



$$Z_1(s) = \frac{s^2+s+4+s(s^2+s+1)}{s^2+s+1}$$

$$= \frac{s^3+2s^2+2s+4}{s^2+s+1}$$

MUST HAVE COMMON FACTOR OF s^2+2, BECAUSE $R(j\sqrt{2}) = X(j\sqrt{2}) = 0$

$$\begin{array}{r} s+2 \\ s^2+2\,\overline{\big)\, s^3+2s^2+2s+4} \\ \underline{s^3 \qquad\quad +2s} \\ 2s^2 \quad +4 \end{array}$$

$$\Rightarrow Z_1(s) = \frac{(s^2+2)(s+2)}{s^2+s+1}$$

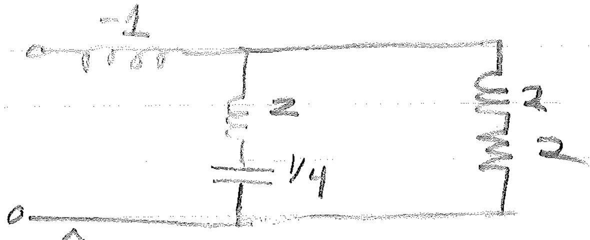
$$Y_1(s) = \frac{s^2+s+1}{(s^2+2)(s+2)} = \frac{As}{s^2+2} + \frac{B}{s+2}$$

$$A = \frac{s^2+1}{s}\left(\frac{s^2+s+1}{(s^2+2)(s+2)}\right)\Bigg|_{\substack{s^2=-2 \\ s=j\sqrt{2}}} = \frac{1}{2}$$

$$B = \frac{s^2+s+1}{s^2+2}\Bigg|_{s=-2} = 3/6 = \frac{1}{2}$$

$$\Rightarrow Y_1(s) = \frac{\frac{1}{2}s}{s^2+2} + \frac{\frac{1}{2}}{s+1}$$

$$= \frac{1}{2s+\frac{1}{4S}} + \frac{1}{2s+2}$$



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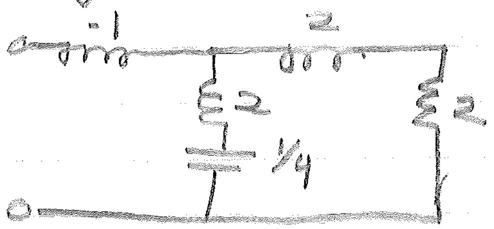

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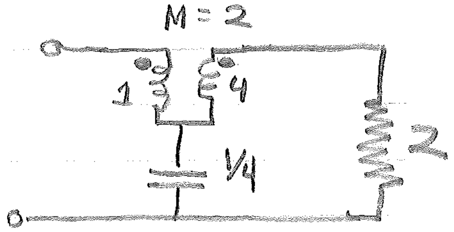
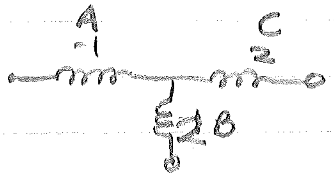
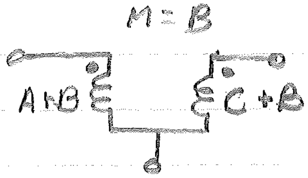
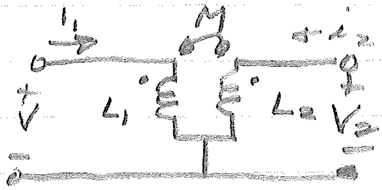
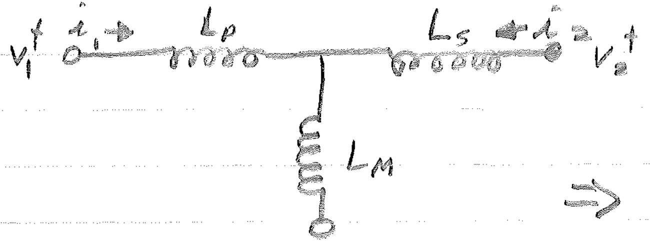








$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$
$$v_2 = +M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$



$$v_1 = (L_p + L_M) \frac{di_1}{dt} + L_M \frac{di_2}{dt}$$
$$V_2 = L_M \frac{di_1}{dt} + (L_s + L_M) \frac{di_2}{dt}$$

$\Rightarrow L_1 = L_p + L_M$

$L_M = M$

$L_2 = L_s + L_M$

10-23-72

(7-1, 7-2, 7-5, 7-11, 7-15)

7-5) $Y(s) = \frac{1}{Y(s)}$

$Y(s) - G = Y_1$

WHAT IS MIN$[Re\{Y(j\omega)\}]$?

$$Re[Y(j\omega)] = \frac{A(\omega^2)}{m_2^2 - n_2^2} \quad ; A(\omega^2) = m_1 m_2 - n_1 n_2$$

$$\frac{d\,Re[Y(j\omega)]}{d(\omega^2)} = 0 = x^4 + a_1 x^3 + a_2 x^2 + a_3 x + k$$

SYNTHETIC DIVISION

$$\begin{array}{r|l} & 3x^2 + 4x + 12 \\ \hline x - 2 & 3x^3 - 2x^2 + 4x - 27 \\ & \underline{3x^3 - 6x^2} \\ & \quad 4x^2 + 4x - 27 \\ & \quad \underline{4x^2 - 8x} \\ & \qquad 12x - 27 \\ & \qquad \underline{12x - 24} \\ & \qquad\qquad -3 \end{array}$$

$$\begin{array}{r|l} & 3 + 4 + 12 \\ \hline 1 - 2 & 3 - 2 + 4 - 27 \\ & \underline{3 - 6} \\ & \quad 4 \\ & \quad \underline{4 - 8} \\ & \qquad 12 \\ & \qquad \underline{12 - 27} \\ & \qquad\qquad -3 \end{array}$$

$$\begin{array}{rrrrr} & 3 & +4 & +12 & \\ \hline 1-2) & 3 & -2 & +4 & -27 \\ & & -6 & -8 & -24 \\ \hline & & 4 & 12 & -3 \end{array}$$

$$\begin{array}{rrrrr} & 3 & +4 & +12 & \\ \hline 2) & 3 & -2 & +4 & -27 \\ & & 6 & 8 & 24 \\ \hline & 3 & 4 & 12 & -3 \end{array}$$

$$\begin{array}{rrrrr} \hline 2) & 3 & -2 & +4 & -27 \\ & & 6 & 8 & 24 \\ \hline & 3 & 4 & 12 & -3 \end{array}$$

$$\begin{array}{rrrr|l} 3 & -2 & 4 & -27 & \underline{2} \\ & 6 & 8 & 24 & \\ \hline 3 & 4 & 12 & -3 & \end{array} \quad \Bigg| \quad \begin{array}{rrrr|l} 3 & -2 & 4 & -27 & \underline{-2} \\ & -6 & +16 & -40 & \\ \hline 3 & -8 & 20 & -67 & \end{array}$$

REMAINDER THEM.

$$p(s) = a_n s^n + \dots + a_0$$

$$\frac{p(s)}{(s-s_1)} = q(s) + \frac{R}{s-s_1}$$

$$p(s) = q(s)[s-s_1] + R$$

$$p(s_1) = R$$

10-23-72

EVALUATION OF RESIDUES

$$Z_{21} = \frac{p}{q} = \frac{s^4 - 4s^3 + 7s^2 - 24s + 36}{(s^2+2s+5)(s^2+4s+13)(s+4)}$$

$$= \frac{K_1}{s+1-j2} + \frac{K_2}{s+2-j3} + (\text{conj.}) + \frac{K_3}{s+4}$$

$$K_3 = \frac{(s+4)\,p(s)}{q(s)}\Big|_{s=-4}$$

LOOKING @ $p(s) / (s+4)$

$$\begin{array}{rrrrrl} 1 & -4 & 7 & -24 & 36 & \underline{\lfloor -4} \\ & -4 & 32 & -156 & 720 & \\ \hline 1 & -8 & 39 & 180 & \underline{\underline{756}} & \end{array}$$

$$q = s^4 + 6s^3 + 26s^2 + 46s + 65$$

$$\begin{array}{rrrrrl} 1 & +6 & 26 & 46 & 65 & \underline{\lfloor -4} \\ & -4 & -8 & -72 & 104 & \\ \hline 1 & 2 & 18 & -28 & \underline{\underline{169}} & \end{array}$$

$$\Rightarrow K(3) = \frac{756}{169}$$

$$K_1 = \frac{(s+1-j2)\,p(s)}{q(s)}\Big|_{s=-1+j2}$$

$$\begin{array}{rrrrrr} & 1 & -6 & 14 & & \\ 1+2+5 \,\big) & 1 & -4 & 7 & -24 & 36 \\ & 1 & 2 & 5 & & \\ \hline & & -6 & 2 & -24 & \\ & & -6 & -12 & -30 & \\ \hline & & & 14 & 6 & 36 \\ & & & 14 & 28 & 70 \\ \hline & & & & -22 & -34 \end{array}$$

REMAINDER: $(-22s - 34)\big|_{s=-1+j2}$

$= 22 - j44 - 34 = -12 - j44$

$\Rightarrow k_1 = \dfrac{-(12+j44)}{d}$

$den = s^3 + 8s^2 + 29s + 52$

$$\begin{array}{llllll}
 & & 1 & 6 & & \\
\hline
1 & 25\,) & 1 & 8 & 24 & 52 \\
 & & \underline{1} & \underline{2} & \underline{5} & \\
 & & & 6 & 14 & 52 \\
 & & & \underline{6} & \underline{12} & \underline{30} \\
 & & & & 12 & 22
\end{array}$$

$(12s + 22)\big|_{s=-1+j2} = -12 + j22 + 22$

$= 10 + j24$

$\Rightarrow k_1 = \dfrac{(s+1+j2)(12s+22)\big|_{s=-1+j2}}{(-96+j40)}$

$n > m+1 \Rightarrow \sum k_j = 0$

$n = m+1 \Rightarrow \sum k_j = \dfrac{a_m}{b_n}$

$\longleftrightarrow$

$f = \dfrac{p(s)}{q(s)} = \dfrac{p(s)}{(s-s)q_1(s)}$

$\Rightarrow q(s) = (s - s)q_1(s)$

$q'(s) = q_1(s) + (s - s_1)q_1'(s)$

$q'(s_1) = q_1(s)$

$\Rightarrow k = \left[\dfrac{p(s)}{q'(s)}\right]_{s=s_1}$

$$K = \left[\frac{d(1/f)}{ds}\right]^{1}_{s=s_1} = \left(\frac{dq}{ds}\right)^{-1}_{s=s_1} \Rightarrow q = 1/f$$

$$\frac{1}{f} = \frac{q(s)}{p(s)}$$

$$\frac{d\left(\frac{1}{f}\right)}{ds} = \left(\frac{pq' - \cancel{qp'}^{\,0}}{p^2}\right)_{s=s'} = \left(\frac{q'}{p}\right)_{s=s_1}$$

$$Y = X^4 + 4x^3 - 8X^2 - 8x + 3 = 0 \qquad \Rightarrow x = \omega^2$$

DESCARTE'S RULE OF SIGN

V = SIGN CHANGES = 2 ⇒ 2 POS RTS

CHANGING SIGNS ON EVEN OR ODD TERMS

$$Y' = X^4 - 4x^3 - 8X^2 + 8x + 3 = 0$$

V = 2 ⇒ 2 NEGATIVE ROOTS

ROOTS TO TRY (3, ⁻3, 1, ⁻1)

$$\begin{array}{rrrrr|l} 1 & 4 & -8 & 8 & 3 & \underline{-1} \\ & -1 & -3 & 11 & -3 & \\ \hline 1 & 3 & -11 & 3 & 0 & \end{array}$$

$Y_1(0) = 3$

$Y_1(1) = -9$ RT

10-25-72

BRUNE'S METHOD

$Z(S)$ IS MINIMUM FUNCTION

a) ⇒ NO 0 OR ∞ POLES

b) NO j AXIS POLES OR 0

c) $R_e[Z(j\omega)] = 0$ @ ω_1

A) DETERMINE $Z(j\omega_1) = jX \Rightarrow X > 0$ OR $X < 0$

B) REMOVE $L_1 = X/\omega_1$

($L < 1 \Rightarrow$ TRANSFORMER)

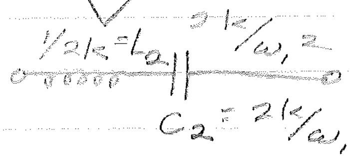
LET $Z_1(s) = Z(s) - sL$

C) SINCE $Z_1(s)$ HAS A ZERO @ ω_1

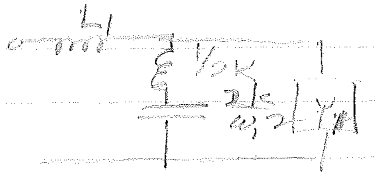
⇒ THE POLE OF $Y_1(s)$ IS NOW REMOVED.

$$Y_2(s) = Y_1(s) - \underbrace{\frac{2ks}{s^2+\omega_1^2}}_{} = \underbrace{\frac{1}{\frac{1}{2k}S + \frac{1}{\frac{2ks}{\omega_1^2}}}}_{}$$

$$\Downarrow$$



TO DATE:



$$k = \left[\frac{d\left(\frac{1}{Y}\right)}{dS}\right]^{-1}_{s=s_1}$$

$$= \left[\frac{d\frac{1}{Y_1}}{dS}\right]^{-1}_{s=j\omega_1} = \left[\frac{dZ_1}{dS}\right]^{-1}_{s=j\omega}$$

$$= \left[\frac{d(Z(s) - L_1 S)}{dS}\right]^{-1}_{s=j\omega_1}$$

$$\frac{1}{L_2} = 2k = 2\left[\frac{d(Z(s) - L_1(s))}{dS}\right]^{-1}_{s=j\omega_1}$$

(MINIMUM FUNCTION : $b_1 a_1 = \left(\sqrt{a_0'} - \sqrt{b_0'}\right)^2$)

$s^2 + a_1 s + a_0$

$s^2 + b_1 s + b_0$

NOTE $C_2 = \frac{1}{L_2 \omega_1^2}$

D) $Z_2 = \frac{1}{Y_2}$ MUST HAVE A POLE @ $s = \infty$

CORRESPONDING TO L_3

$$L_3 = \lim_{s \to \infty} Z_2(s)/s$$

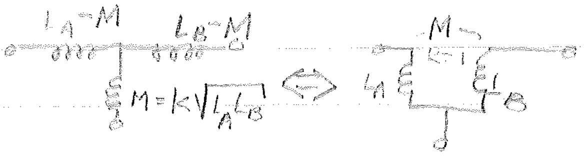
$$= \lim_{s \to \infty} \frac{1}{sY(s)}$$

$$= \lim_{s \to \infty} \frac{1}{s} \left[\frac{1}{Y_1(s) - \frac{2ks}{s^2+\omega^2}} \right]$$

$$= \lim_{s \to \infty} \frac{1}{s} \left[\frac{1}{\frac{1}{Z(s) - sL_1} - \frac{2ks}{s^2+\omega^2}} \right]$$

$$= \frac{-1}{\frac{1}{+L_1} + \frac{1}{L_2}}$$

$$= \frac{-L_1 L_2}{(L_1 + L_2)}$$



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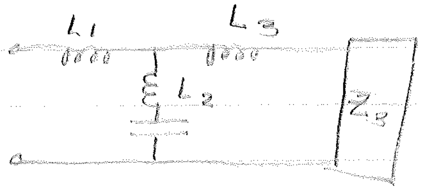


$\Rightarrow Z_3 = Z_2 - sL_3$

Ex) $Z(s) = \dfrac{s^2 + \frac{1}{2}s + \frac{1}{2}}{s^2 + s + 2}$

$Re[Z(j\omega_1)]$ @ $\omega_1 = 1$

$\Rightarrow Z(j\omega_1) = jX$

$[Odd[Z(s)]_{s=j\omega} \Rightarrow Im\, Z(j\omega)]$

$Z(j\omega_1) = Z(j1)$

$= j\frac{1}{2}$

$\Rightarrow L_1 = \frac{1/2}{1} = \frac{1}{2}$

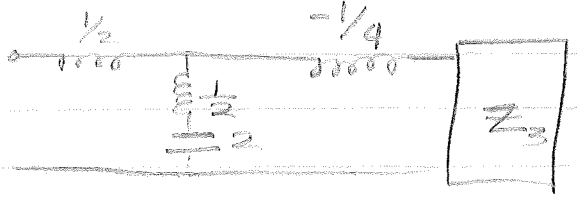
$\Rightarrow Z_1(s) = Z(s) - \frac{1}{2}S$

$L_2 = \frac{1}{2}\left[\frac{d[Z(s) - sL_1]}{ds}\right]$

$= \frac{1}{2}\left[\frac{qp' - pq'}{q^2} - L_1\right]_{s=j\omega_1}$

$= \frac{1}{2}\left[\frac{(s^2+s+2)(2s+\frac{1}{2}) - (s^2+\frac{1}{2}s+\frac{1}{2})(2s+1)}{(s^2+s+2)^2} - \frac{1}{2}\right]_{s=j}$

$= \frac{1}{2}$



$Z_1(s) = Z(s) - L_1 s$

$Y_2 = Y_1 - K_2 s / (s^2 + \omega_1^2)$

$Z_3 = Z_2 - L_3 S$

7-26-72

$Z(s)$

$Z_1(s) \rightarrow$ MINIMUM

$Z_1(j\omega_1) = jX$ $\rightarrow X_1 > 0 \Rightarrow Z$

$\rightarrow X_1 < 0 \Rightarrow Y$

RICHARD'S THEOREM

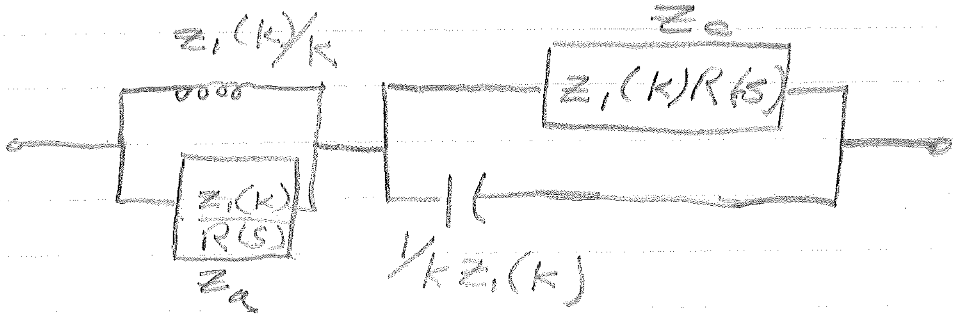
$$R(s) = \frac{kZ_1(s) - sZ_1(k)}{kZ(k) - sZ_1(s)} \qquad \text{P.r}, \ k \geq 0 \quad |k| = k$$

$$\Rightarrow Z_1(s) = \frac{sK_1(K) + kZ_1(K)R(s)}{K + sR(s)}$$

$$= \frac{1}{\frac{R(s)}{Z_1(K)} + \frac{K}{sZ_1(K)}} + \frac{1}{\frac{1}{Z_1(k)R(s)} + \frac{s}{KZ_1(K)}}$$

$$= \frac{1}{Y_a + Y_b} + \frac{1}{Y_c + Y_D}$$

$$Y_b = \frac{K}{Z_1(K)} = \frac{1}{sL_b}$$



$$Z_a Z_c = Z_1^2(k)$$

(7-19 7-21)

10-31-72

7-5) $Z(s) = \frac{2s^3 + s^2 + 4s + 1}{s^3 + 3s^2 + 3s + 1}$ NO POLES @ 0 OR ∞

ROUTH GIVES NO j AXIS POLES

WISH TO REMOVE CONDUCTANCE

$$Y(s) = \frac{1}{Z(s)}$$

$$Ev(Y(s))\Big|_{s=j\omega} = Re(Y(j\omega))$$

$$= \frac{\omega^6 + \dots}{\omega^6 + \dots}$$

LET $x = \omega^2$

FIND $\frac{d\,Re(Yj\omega)}{dx} = 0$

$\Rightarrow x = 1.918 \; ; \; -5.215$ $\quad .3$

1.918 IS THE ONE! OR $.3$

$\Rightarrow \omega^2 = 1.918$ $\quad \omega^2 = .7$

$Re(Y(j\sqrt{1.918})) = 2.22$ $\quad Re(Y(j\sqrt{3})) = .713$

NOW $Re[Y(j0)] = \underline{1}$; $Re[Y(j\infty)] = \frac{1}{2}$

SO TAKE OUT $\frac{1}{2}$

$$2s^3+s^2+4s+1 \left| \begin{array}{c} s^3 + 3s^2 + 3s + 1 \\ \underline{s^3 + \frac{1}{2}s^2 + 2s + \frac{1}{2}} \\ \frac{5}{2}s^2 + s + \frac{1}{2} \end{array} \right| \; 1/2$$

$$\Rightarrow Y(s) = \frac{1}{2} + \frac{\frac{5}{2}s^2 + s + \frac{1}{2}}{2s^3 + s^2 + 4s + 1}$$

$$= \frac{1}{2} + \cfrac{1}{\frac{4}{5}s + \underbrace{\cfrac{\frac{1}{5}s^2 + \frac{18}{5}s + 1}{5/2\,s^2 + 2s + 1/2}}_{Z_2(s)}}$$

$$Ev(Z_2(s))\Big|_{s=j\omega} = Re(Z_2(j\omega)) = \frac{2}{25}\;\frac{x^2 + 2x + 1}{x^2 - \frac{6}{25}x + \frac{1}{25}} \quad \Rightarrow x = \omega^2$$

$$= \frac{p}{q}$$

$$qp' - pq' = 0 \Rightarrow \frac{p}{q} = \frac{p'}{q'}$$

$$x = \begin{cases} 0.631 \\ 0.227 \end{cases}$$

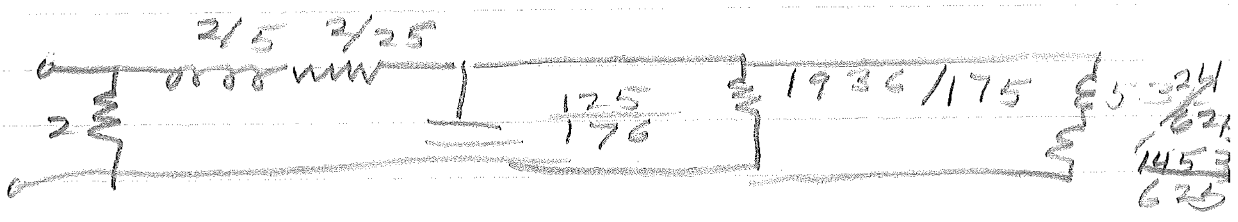
$$Re[Z(j\sqrt{.631})] = \frac{2}{25}\;\frac{2x+2}{2x - 6/25} = .255$$

$$Re[Zj\sqrt{0.227}] = 0.917$$

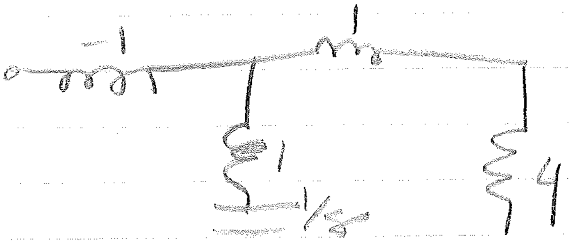
$$Re[Z(j0)] = 2$$

$$Re[Z(j\infty)] = 0.08 \longleftarrow \text{MIN}$$

7-5



7-11)

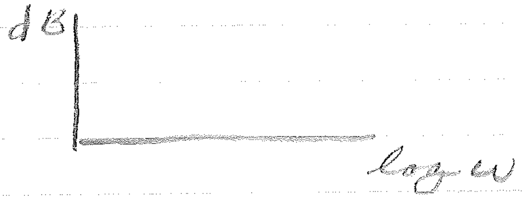


11-1-72

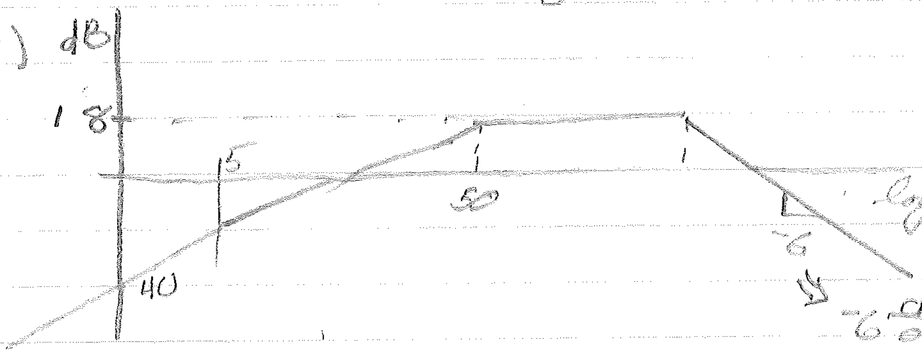
$$Z(s) = \frac{(s-z_1)(s-z_2)\dots}{(s-p_1)(s-p_2)\dots}$$

BODE PLOT

20 dB/DECADE ; 6.02 dB/OCTAVE

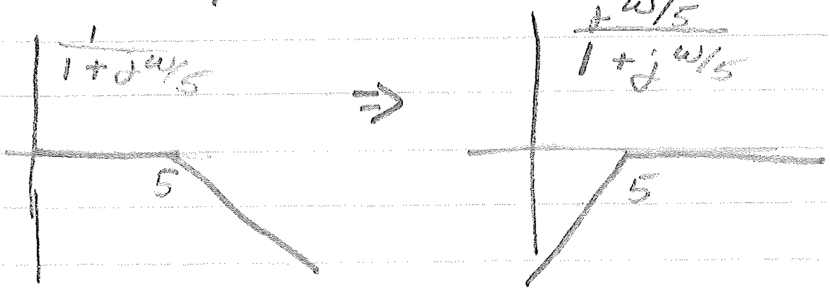


9-19)



⇒ $-6 \frac{dB}{OCTAVE} = -20 \frac{dB}{DECADE}$

$$G = K \frac{(s-z_1)\dots}{(s-p)\dots}$$

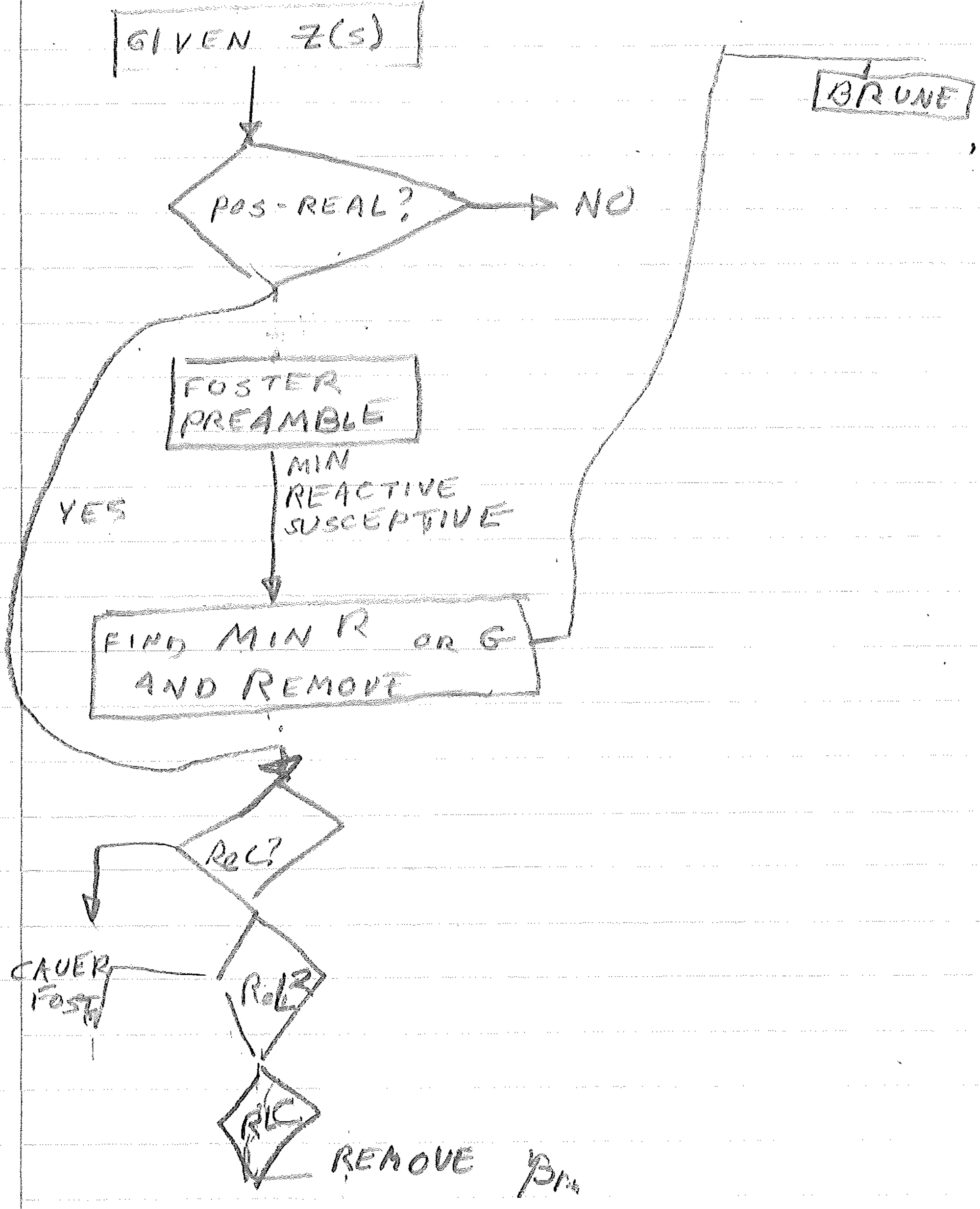


$$G(s) = K \frac{s/5}{1+s/5} \frac{s/50}{1+s/50} \frac{1}{1+s/500}$$

$K_{db} = 18$

$\Rightarrow 20 \log_{10} K = 18 \Rightarrow K \sim 8$

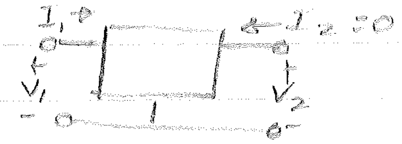
BRUNE METHOD



11-2-72 (TEST DUE MONDAY)

CHAPT. 10

TRANSFER FUNCTIONS (UNTERMINATED) $\Rightarrow I_2 = 0$



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ 0 \end{bmatrix} \quad ; \quad z_{21} = Z_{12}$$

$$G_{12} = \left. \frac{V_2}{V_1} \right|_{I_2 = 0} = \frac{Z_{12}}{Z_{22}}$$

$$\begin{bmatrix} I_1 \\ \cancel{I_2} 0 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$G_{12} = \left. \frac{V_2}{V_1} \right|_{I_2 = 0} = -\frac{Y_{12}}{Y_{22}}$$

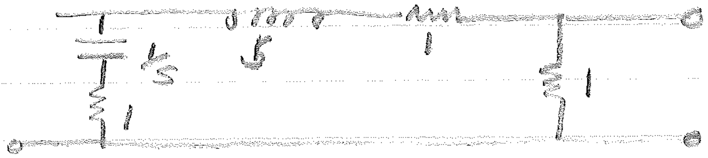
FOR MESH EQUATION

$$[Z][I] - [V]$$

$$Y_{11} = \frac{\Delta_{11}}{\Delta} \quad ; \quad -Y_{12} = \frac{\Delta_{21}}{\Delta}$$

POLES OF $-Y_{12}, Y_{11}, Y_{22}$

EX)

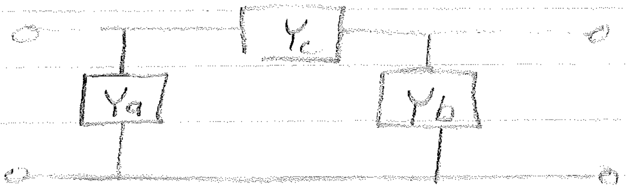


$$Y_{11} = \frac{1}{1 + \frac{1}{s}} + \frac{1}{s} = \frac{s + 1 + 1 + \frac{1}{s}}{1 + \frac{1}{s} + s + \frac{s}{s}} = 1$$

$$-Y_{12} = \frac{1}{1 + s}$$

$$Y_{22} = 1 + \frac{1}{1 + s} = \frac{s + 2}{s + 1}$$

NOTE Y_{11} HAS NO POLES $\Rightarrow$ DIFF. POLES FOR Y_{22} & Y_{12} => CASE 2



$Y_{11} = Y_a + Y_c$

$-Y_{12} = Y_c$

$Y_{22} = Y_b + Y_c$

Y_{11} & Y_{21} POLES NOT FROM Y_c ARE "PRIVATE" POLES

REQUIREMENTS ON COEFFICIENTS OF NUM

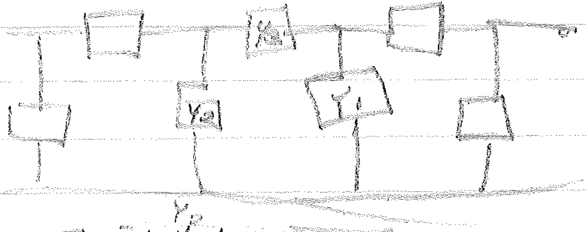
$$-Y_{12} = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{q(s)}$$

$$Y_{11} = \frac{b_0 + b_1 s + \ldots + b_m s^m}{q(s)}$$

$$Y_{22} = \frac{c_0 + c_1 s + \ldots + c_m s^m}{q(s)}$$

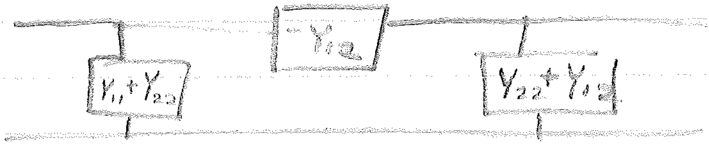
$b_j \geq a_j \geq 0$

$c_j \geq a_j \leq 0$



$Z_b = Y_1 \overset{Y_P}{Y_5} + Y_2 Y_3 + Y_1 Y_3$

MAY EXPAND INTO 2-PORT



$$Y_{11} - Y_{12} = \frac{(b_0 - a_0) + (b_1 - a_1)s + \ldots}{q(s)}$$

$\Rightarrow (b_0 - a_0) \geq 0 \Rightarrow b_0 \geq a_0 \geq 0$

NUM (Y_{12}) degree $\leq$ Num$(\overset{Y_{11}}{Y_{22}})$ degree

Y_{12} MUST HAVE POS. NUMERATOR COEFFICIENTS,
BUT CAN HAVE MISSING TERMS

$$G_{12} = \frac{V_2}{V_1} = \frac{N_{12} D_{22}}{D_{12} N_{22}}$$

EX

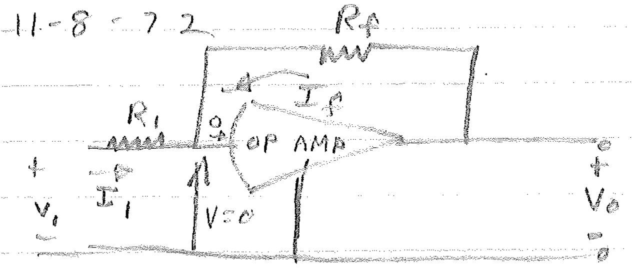
$$-Y_{12} = \frac{S+1.5}{(S+1)(S+3)} \qquad Y_{22} = \frac{(S+2)(S+4)}{(S+1)(S+3)\underbrace{(S+5)}_{\text{PRIVATE POLE}}}$$

$$G_{12} = \frac{-Y_{12}}{Y_{22}} = \frac{(S+1.5)(S+5)}{(S+2)(S+4)}$$

POLES IN LHP, OR SIMPLE j AXIS

FOR LADDER NETWORK, 0'S IN LHP

11-8-72

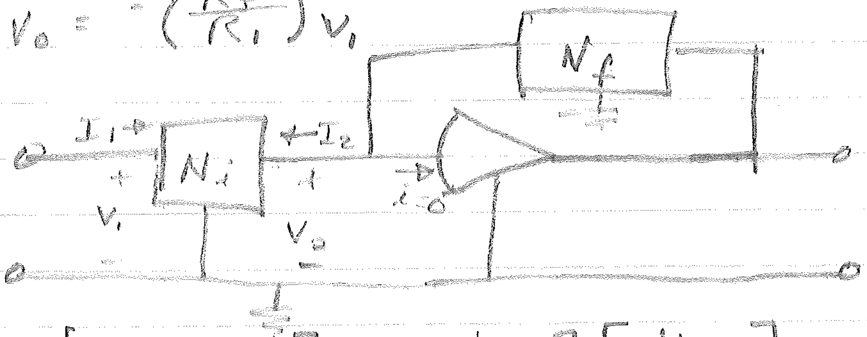


$$I_1 + I_f = 0$$

$$\frac{V_1}{R_1} + \frac{V_0}{R_f} = 0$$

$$\Rightarrow V_0 = -\left(\frac{R_f}{R_1}\right) V_1$$

OR!



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 = 0 \end{bmatrix}$$

$$\Rightarrow -I_i = I_2 = Y_{12i} V_1 \qquad I_f = -Y_{12f} V_0$$

$$\Rightarrow V_0 = -\frac{Y_{12i}}{Y_{12f}} V_i$$

May 5, 1972

FREQUENCY-RESPONSE PROGRAM

The frequency-response (FREQR) program computes data for:

a) the Nyquist plot

$$\text{Re}\left\{GH(j\omega)\right\} \quad \text{vs.} \quad \text{Im}\left\{GH(j\omega)\right\}$$

or

$$\text{Mag}\left\{GH(j\omega)\right\} \quad \text{vs.} \quad \underline{/GH(j\omega)}$$

b) Bodé Plot ("open loop" or "loop" function)

$$\left[G\text{[redacted]}(j\omega)\right]_{dB}, \quad \underline{/G\text{[redacted]}(j\omega)} \quad \text{vs.} \quad \log\omega$$

c) Nichol's chart

$$\left[GH(j\omega)\right]_{dB} \quad \text{vs.} \quad \underline{/GH(j\omega)}$$

d) Closed-loop Bodé Plot

$$\frac{G}{1 + GH}_{dB}, \quad \underline{\Big/ \frac{G}{1 + GH}} \quad \text{vs.} \quad \log\omega$$

The G and H functions must be of the form

$$G(S) = \frac{AG_1 + AG_2 S + \ldots + AG_{NG+1} S^{NG}}{BG_1 + BG_2 S + \ldots + BG_{MG+1} S^{MG}}$$

$$H(S) = \frac{AH_1 + AH_2 A + \ldots + AH_{NH+1} S^{NH}}{BH_1 + BH_2 S + \ldots + BH_{MH+1} S^{MH}}$$

The ω_{start} and "number of decades" must be specified along with the orders of the S functions NG, MG, NH, MH and the coefficients of each term. Note that $AH_1 + BH_1 = 1$ and NH = MH= 0 for H(S) = 1.

EXAMPLE PROBLEM

$$G(S) = \frac{5.895 + 152.936\ S + 140.259\ S^2 + 32.751\ S^3}{0 + 116.103S + 45.423\ S^2 + 33.888S^3 + 10.855\ S^4 + S^5}$$

H(S) = 1

for $\omega_{start} = 1$ and $1 \leq \omega \leq 1000$

$\omega_{start} = 1.0$ Num. of Dec = 3

NG = 3, MG = 5, NH = 0, MH = 0

CALLING PROGRAM

```
$007            239          P D SMITH
41031,STUDENT, T10, P10, CM50000.
FUN(S)
PFILES(GET,FREQR)
REWIND(FREQR)
FUN(G,,,FREQR)
7
8
9
      PROGRAM MAIN(INPUT,OUTPUT,TAPE2=INPUT, TAPE3=OUTPUT)
      CALL FREQR
      CALL EXIT
      END
7
8
9

                    Data Deck

6
7
8
9
```

INPUT DATA

	1	11	21	31	41	51
ω_s NUMDEC	1.0	03				
NG,MG,NH,MH	3500					
AG_j	5.895	152.936	140.259	32.751		
EG_j	0.	116.103	45.423	33.888	0.855	1.0
AH_j	1.					
BH_j	1.					

NET. SYNTH 9-13

1-9) a) $\frac{1}{(S+1)(S+2)} = \frac{1}{2+3S+S^2}$ FOR $10^{-3} \leq \omega \leq 10 = 4$ DECADES

ω_{START} 10^{-3} NUMDEC = 03

NG = 0 ; MG = 2 ; NH = 0 ; MH = 0

$AG_1 = 1$

$BG_1 = 2$; $BG_2 = 3$ $BG_3 = 1$

$AH_1 = 1.$

$BH = 1.$

b) $\frac{S}{(S+1)(S+2)} = \frac{S}{2+3S+S^2}$ FOR $10^{-3} \leq \omega \leq 10 = 4$ DECADES

$\omega_{START} = 10^{-3}$ NUMDEC = 03

NG = 1 ; MG = 2 ; NH = 0 ; MH = 0

$AG_1 = 0$; $AG_2 = 1$

$BG_1 = 2$; $BG_2 = 3$; $BG_3 = 1$

$AH_1 = 1$

$BH_1 = 1$

c) $\frac{5S-5}{S+1}$ FOR $10^{-3} < \omega < 10 = 4$ DECADES

$\omega_{START} = 10^{-3}$

NG = 1 ; MG = 1 ; NH = 0 ; MH = 0

$AG_1 = 5$; $AG_2 = -5$

$BG_1 = 1$; $BG_2 = 1$

$AH_1 = 1$

$BH_1 = 1$

d) $\frac{5(S^2-S+1)}{S^2+S+1}$ FOR $10^{-3} < \omega < 10 = 4$ DECADES

$\omega_{START} = 10^{-3}$

NG = 2 ; MG = 2 ; NH = 0 ; MH = 0

$AG_1 = 5$; $AG_2 = -5$; $AG_3 = 5$

$BH_1 = 1$; $BH_2 = 1$; $BH_3 = 1$

$AH_1 = 1$

$BH_1 = 1$

DATA DECKS FOR FREQR

1	11	21
.001	03	
0200	—	
1.0	—	
2.0	3.0	1.0
1.		
1.		

1	11	21
.001	03	
1200	—	
6.0	1.0	
2.0	3.0	1.0
1.0		
1.0		

1	11	21
.001	03	
1100		
5.0	-5.0	
1.0	1.0	
1.0		

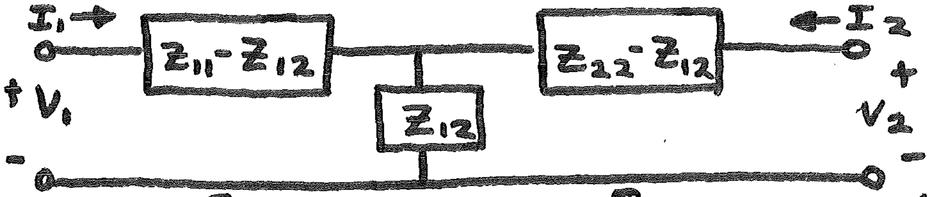
1	11	21
.001	13	
2200	-5.	5.
5.	1.	1.
1.		
1.		
1.		

I) NETWORK ANALYSIS

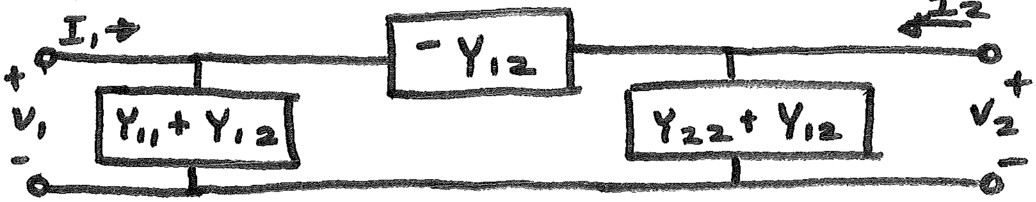
A) PARAMETERS

1) Z:
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{22}/\Delta_Y & -Y_{12}/\Delta_Y \\ -Y_{12}/\Delta_Y & Y_{11}/\Delta_Y \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

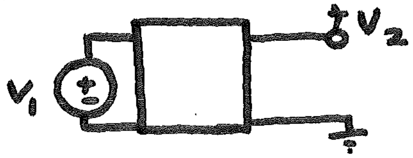
FOR PASSIVE NETWORKS, $Z_{12} = Z_{21}$, AND:



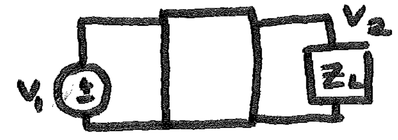
2) Y:
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{22}/\Delta_Z & -Z_{12}/\Delta_Z \\ -Z_{21}/\Delta_Z & Z_{11}/\Delta_Z \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



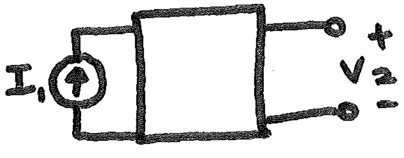
B) NETWORK TRANSFER FUNCTION



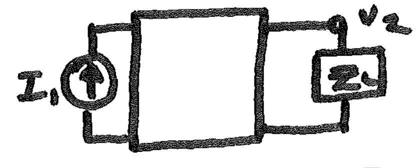
$$G_{12} = \frac{V_2}{V_1} = \frac{-Y_{12}}{Y_{22}} = \frac{Z_{12}}{Z_{11}}$$



$$G_{12} = \frac{V_2}{V_1} = \frac{-Y_{12}}{Y_{22}+Y_L} = \frac{Z_{12}Z_L}{\Delta_Z + Z_{11}Z_L}$$



$$Z_{12} = \frac{V_2}{I_1} = \frac{-Y_{12}}{\Delta_Y} = Z_{12}$$



$$Z_{12} = \frac{V_2}{I_1} = \frac{-Y_{12}}{\Delta_Y + Y_{11}Y_L} = \frac{Z_{12}Z_L}{Z_{22}+Z_L}$$

$$\alpha_{12} = \frac{-I_2}{V_1} = \frac{Z_{12}}{Z_{22}+Z_L}$$

II) TESTING FOR POSITIVE REAL

$$Z(s) = \frac{p(s)}{q(s)} = \frac{m_1(s)+n_1(s)}{m_2(s)+n_2(s)}$$

A) ROUTH TEST ON $p(s)+q(s)$

B) STURM TEST ON $m_1(s)\,m_2(s) - n_1(s)\,n_2(s)$

C) $Z(s)$ IS REAL IF s IS REAL

III) LC SYNTHESIS (INTERLACED POLES & ZERO'S ON $j\omega$ AXIS)

A) CAUER'S
  1) FIRST → REMOVE POLE @ ∞
  2) SECOND → REMOVE POLE @ 0

B) FOSTER'S
  1) TAKE OUT ALL POLES @ 0 & ∞ OF $Z(s)$
     EXPAND THE REST, & FIND RESIDUES
  2) TAKE OUT ALL POLES @ 0 & ∞ OF $Y(s)$
     EXPAND THE REST, & FIND RESIDUES

IV.) RC & LC SYNTHESIS

A) POLES & ZERO'S INTERLACED ON σ AXIS.

1) IF THE CLOSEST SINGULARITY TO $j\omega$ AXIS IS A POLE ⇒ RC (Z(S) POLE)

2) IF THE CLOSEST SINGULARITY TO $j\omega$ AXIS IS A ZERO ⇒ RL (Z(S) ZERO)

RC NETWORKS	RL NETWORKS
$Z(0) > Z(\infty)$ $Y(\infty) > Y(0)$	$Z(\infty) > Z(0)$ $Y(0) > Y(\infty)$
FOSTER:	FOSTER
1^{ST}: EXPAND $Z(S)$	1^{ST}: EXPAND $Z(S)/S$
2^{ND}: EXPAND $Y(S)/S$	2^{nd}: EXPAND $Y(S)$
CAUER	CAUER
1^{ST}: EXPAND AROUND ∞ POLES $[\mathrm{Re}\, Z(j\omega) < \mathrm{Re}\, Z(0)]$	1^{ST}: EXPAND AROUND ∞ POLES $[\mathrm{Re}\, Y(\infty) < \mathrm{Re}\, Y(0)]$
2^{ND}: EXPAND AROUND 0 POLES $[\mathrm{Re}(Y(0)) < \mathrm{Re}(Y(\infty))]$	2^{ND}: EXPAND AROND 0 POLES $[\mathrm{Re}\{Z(0)\} < \mathrm{Re}\{Z(\infty)\}]$

IV) THE FOSTER PREAMBLE

A) CHECK FOR POS. REAL (SEE II)

B) PULL OUT ALL POLES @ 0 OR ∞ [L OR C]

C) PULL OUT ALL jω AXIS POLES [LC TANKS]

D) CHECK FOR MINIMUM RESISTANCE

$$Re\{Z(j\omega)\} = Ev\{Z(s)\}\Big|_{s=j\omega} = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}\Big|_{s=j\omega}$$

$$Re\{Y(j\omega)\} = Ev\{Y(j\omega)\}\Big|_{s=j\omega} = \frac{m_1 m_2 - n_1 n_2}{m_1^2 - n_1^2}\Big|_{s=j\omega}$$

FIND RELATIVE MINIMA, AND CHECK
@ $j\omega = 0$ & $j\omega = \infty$
TAKE OUT THE MINIMUM

THE REMAINING FUNCTION IS
A MINIMUM FUNCTION.

VI) BRUNE'S METHOD

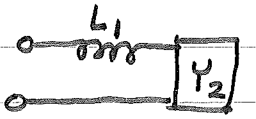
A) $Z(s)$ MUST BE A MINIMUM FUNCTION (FILTERED THRU THE FOSTER PREAMBLE)

B) $Re[Z(j\omega_1)] = 0$

$\Rightarrow Z(j\omega) = jX$

C) REMOVE

$L_1 = \frac{X}{\omega_1}$

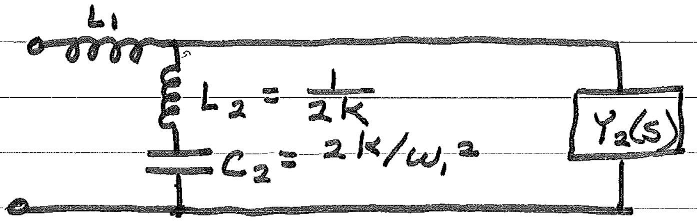


D) LEAVING:

$Z_1(s) = Z(s) - sL_1$

$Y_1(s) = 1/Z_1(s)$ HAS A POLE @ $\pm j\omega_1$

$\Rightarrow Y_1(s) = Y_2(s) + \frac{2ks}{s^2+\omega_1^2}$



ALSO

$L_2 = \frac{1}{2} \frac{d[Z(s) - sL_1]}{ds}\Big|_{s=j\omega_1}$

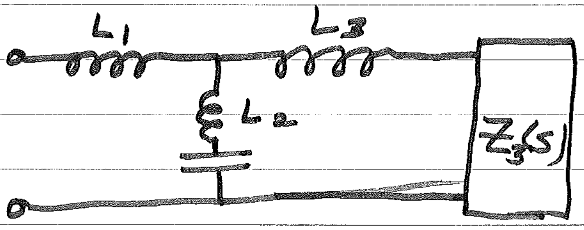
$C_2 = \frac{1}{L_2\omega_1^2}$

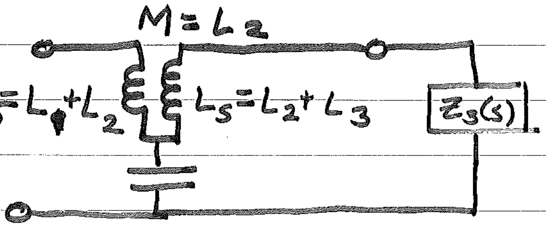
E) $Y_2(s) = Y_1(s) - \frac{2ks}{s^2+\omega_1^2}$

$Z_2(s) = 1/Y_2(s)$ HAS A POLE @ ∞

$L_3 = \frac{-L_1L_2}{L_1+L_2}$

LEAVING





VII) BOTT-DUFFIN

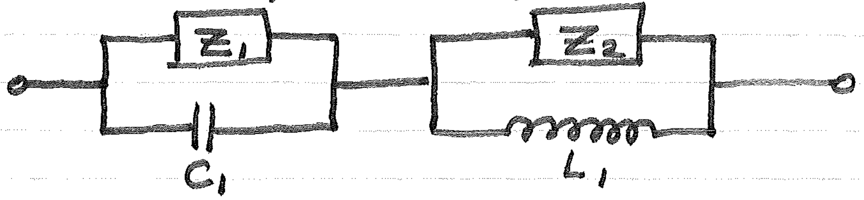
$$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)}$$ (RICHARD'S FUNTION)

OR

$$Z(s) = \frac{R(s)\,k\,Z(k) + s\,Z(k)}{k + sR(s)}$$

$$= \frac{1}{\underset{\uparrow}{\frac{1}{Z(k)R(s)}} + \underset{\uparrow}{\frac{s}{kZ(k)}}} + \frac{1}{\underset{\uparrow}{\frac{k}{sZ(k)}} + \underset{\uparrow}{\frac{R(s)}{Z(k)}}}$$

$$Z_1 = Z(k)R(s)\ ;\ C_1 = \frac{1}{kZ(k)};\ L_1 = \frac{Z(k)}{k}\ ;\ Z_2 = \frac{Z(k)}{R(s)}$$



AGAIN: $Re[Z(j\omega_1)] = 0 \Rightarrow Z(j\omega) = jX$

A) CASE A ; X IS POSITIVE

FORCE $R(s)$ TO HAVE j-AXIS POLES:

$$R(s) = \frac{(s^2+\omega_1)^2(\qquad)}{(\qquad)}$$

$$\Rightarrow R(j\omega_1) = 0 \Rightarrow j\omega_1 Z(k) = kZ(j\omega)$$

OR $L_1 = \frac{X}{\omega_1}$

SOLVE FOR k FROM:

$$Z(k) = kL_1$$

FIND $Z(k)$

SOLVE FOR $C_1 = \frac{1}{kZ(k)}$

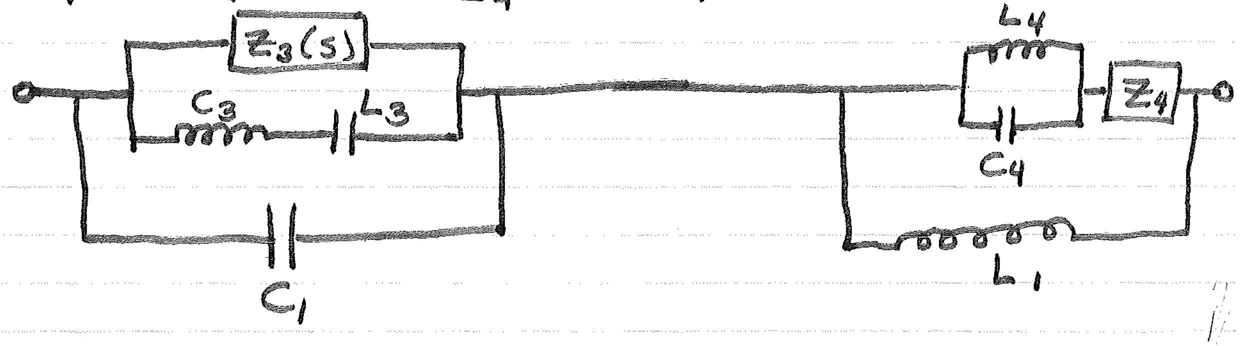
$Z_1(s) = Z(k)R(s)$ HAS A ZERO @ $\pm j\omega_1$
$Z_2(s) = Z(k)/R(s)$ " " POLE @ $\pm j\omega_1$

$$\Rightarrow Y_1(s) = \frac{k_1 s}{s^2+\omega_1^2} + Y_3(s)$$

$$Z_2(s) = \frac{k_2 s}{s^2+\omega_1^2} + Z_4(s)$$

$C_3 = \frac{K_1}{\omega_1^2}$ $\qquad$ $L_3 = 1/K_1$

$C_4 = 1/K_2$ $\qquad$ $L_4 = K_2/\omega_1^2$



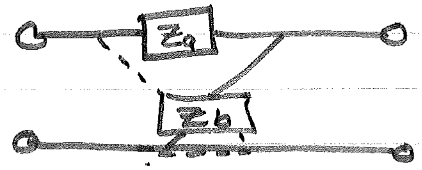
B) CASE B: $X<0$
USE SAME PROCEDURE, BUT
WITH Y

VIII) 2 PORT LADDER DEVELOPMENT

TRANSFER DEFINITIONS : Pg 46
Y_{ij} & Z_{ij} DEFINITIONS : Pg 38
G_{12} PROPERTIES : Pg 258
$-Y_{12}$ PROPERTIES : Pg 260
ZERO SHIFTING : Pg 265

LC LADDER : Pg 260
RC LADDER : Pg 277
GIVEN G_{12} : Pg 287

IX) LATTICE

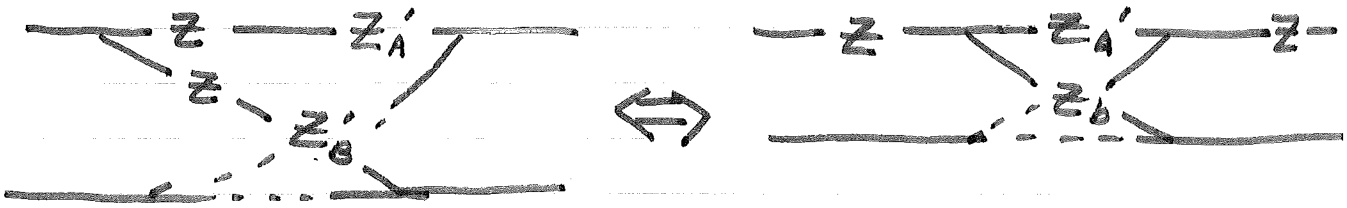


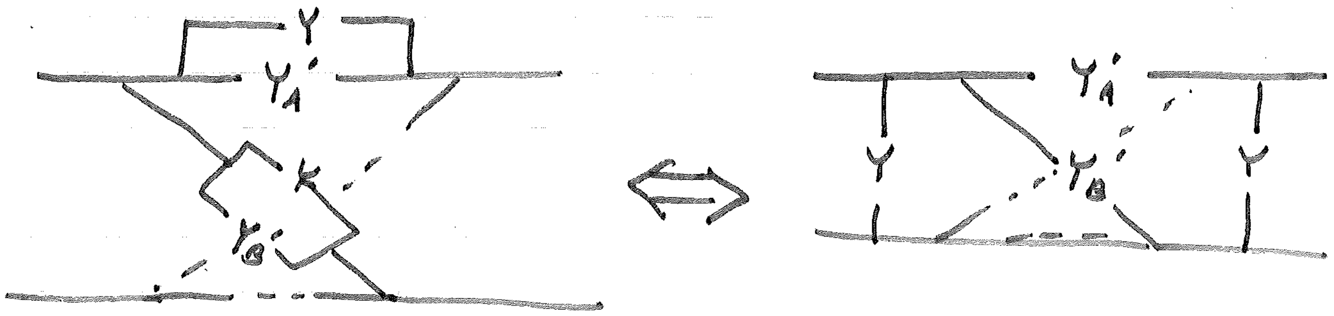
$$Z_{11} = \frac{1}{2}(Z_a + Z_b) = Z_{22}$$
$$Z_{12} = \frac{1}{2}(Z_b - Z_a)$$

$$Z_a = Z_{11} - Z_{12} = Z_{22} - Z_{12}$$
$$Z_b = Z_{11} + Z_{12} = Z_{22} + Z_{12}$$

$$Y_{11} = Y_{22} = \frac{1}{2}(Y_A + Y_b)$$
$$Y_{12} = \frac{1}{2}(Y_B - Y_A)$$

$$Y_A = Y_{11} - Y_{12} = Y_{22} - Y_{12}$$
$$Y_B = Y_{11} + Y_{12} = Y_{22} + Y_{12}$$

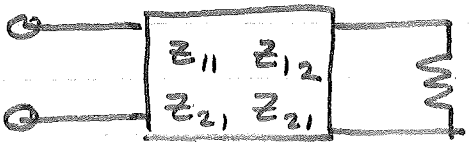




BARTLETT'S BISECTION THEM FOR SYM. NET:

$$Z_b = Z_{1/2}\ (\text{o.c.})$$
$$Z_a = Z_{1/2}\ (\text{s.c.})$$

IX) CONSTANT R



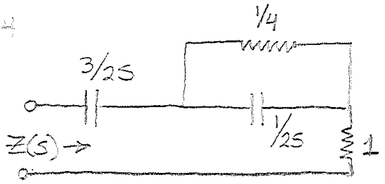
TO SEE R: $Z_A Z_b = R^2$

$$Z_a = \frac{R \cdot Z_{12}}{1 + Z_{12}/R} \quad \& \quad Z_b = R^2/Z_A$$

[illegible]
9-17

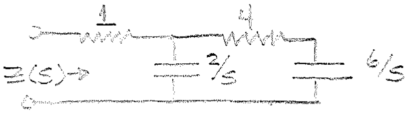
Pg 54

2-2) a)



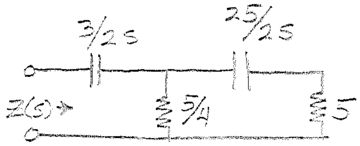
$$Z(s) = \frac{3}{2s} + 1 + \frac{1/8s}{1/4 + \frac{1}{2s}}$$
$$= \frac{3+2s}{2s} + \frac{1}{2s+4}$$
$$= \frac{(3+2s)(s+2)+s}{2s(s+2)}$$
$$= \frac{2s^2+8s+6}{2s(s+2)} = \frac{s^2+4s+3}{s(s+2)}$$
$$= \frac{(s+1)(s+3)}{s(s+2)}$$

b)



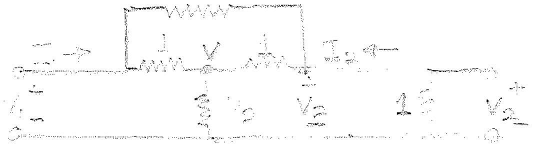
$$Z(s) = 1 + \frac{\frac{2}{s}\left(4+\frac{6}{s}\right)}{\frac{2}{s}+\frac{6}{s}+4}$$
$$= 1 + \frac{2(4s+6)}{8s+4s^2}$$
$$= 1 + \frac{2s+3}{2s+s^2}$$
$$= \frac{s^2+4s+3}{s(s+2)}$$
$$= \frac{(s+1)(s+3)}{s(s+2)}$$

c)



$$Z(s) = \frac{3}{2s} + \frac{\frac{5}{4}\left(\frac{25}{2s}+5\right)}{\frac{5}{4}+5+\frac{25}{2s}}$$
$$= \frac{3}{2s} + \frac{\frac{5}{4}\left(\frac{5}{2s}+1\right)}{5/4 + 5/2s}$$
$$= \frac{3}{2s} + \frac{5/4\,(10+4s)}{5s+10}$$
$$= \frac{3}{2s} + \frac{1/4\,(10+4s)}{s+2}$$
$$= \frac{3(s+2)+s(5+2s)}{2s(s+2)}$$
$$= \frac{3s+6+5s+2s^2}{2s(s+2)}$$
$$= \frac{s^2+4s+3}{s(s+2)}$$
$$= \frac{(s+1)(s+3)}{s(s+2)}$$

2.3)



$(V_1 - V) + (V - V_2) - 2V = 0 \Rightarrow 4V = V_1 + V_2$

a) $I_1 = (V_1 - V) + \frac{1}{2}(V_1 - V_2)$

$= \frac{3}{2}V_1 - \frac{1}{2}V_2 - \frac{1}{4}(V_1 + V_2)$

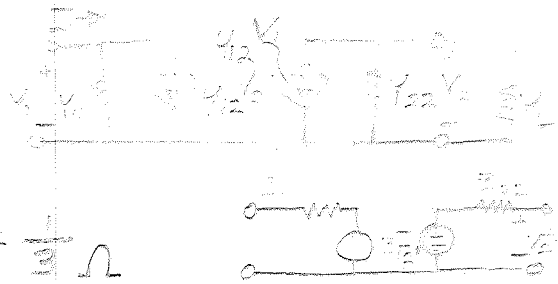
$= \frac{5}{4}V_1 - \frac{3}{4}V_2$

b) $I_2 = \frac{1}{2}(V_2 - V_1) + (V_2 - V)$

$= -\frac{1}{2}V_1 + \frac{3}{2}V_2 - \frac{1}{4}(V_1 + V_2)$

$= -\frac{3}{4}V_1 + \frac{5}{4}V_2$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow$$



$\Delta_Y = 1$

(a) $\frac{V_2}{V_1} = G_{12} = \frac{-Y_{12}}{Y_{22} + Y_L} = \frac{+3/4}{5/4 + 1} = \frac{1}{3}\,\Omega$

(b) $Z_{12} = \frac{V_2}{I_1} = \frac{Z_L Z_{12}}{Z_L + Z_{22}}$

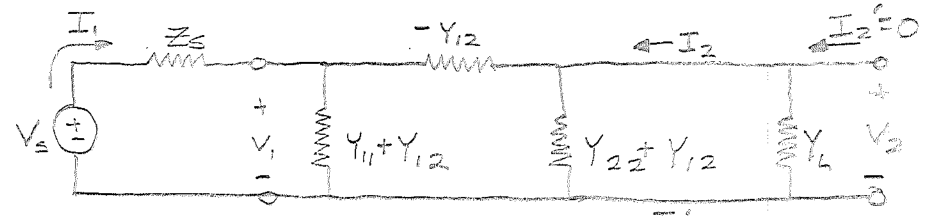
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$\Rightarrow Z_{12} = \frac{V_2}{I_1} = \frac{1}{3}$

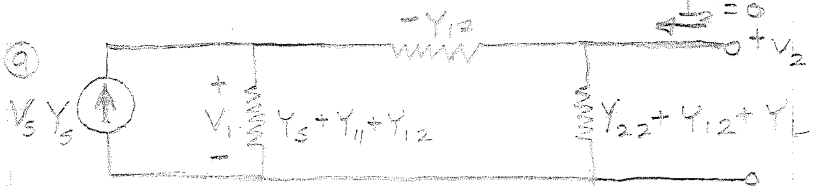
(c) $Y_{12} = \frac{-I_2}{V_1} = \frac{-Y_L Y_{12}}{Y_{22} + Y_L} = \frac{3/4}{9/3} = 1/3$

d) $\alpha_{12} = \frac{-I_2}{I_1} = \frac{-I_2}{V_1}\frac{V_1}{V_2}\frac{V_2}{I_1} = 1/3$

2-20)



ⓐ



$$\begin{bmatrix} V_S Y_S \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11}+Y_S & Y_{12} \\ Y_{12} & Y_{22}+Y_L \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\Delta = (Y_{11}+Y_S)(Y_{22}+Y_L) - Y_{12}^2$$

$$= Y_{22}Y_S + Y_{11}Y_L + Y_S Y_L - Y_{12}^2 + Y_{11}Y_{12}$$

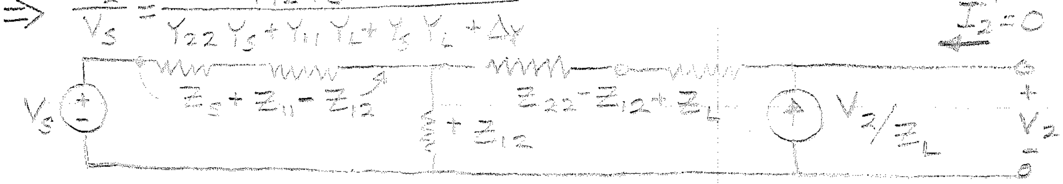
$$= Y_{22}Y_S + Y_{11}Y_L + Y_S Y_L + \Delta_Y$$

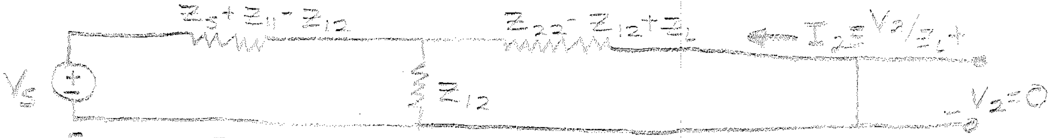
$$V_1 = \frac{V_S(Y_{22}+Y_L)}{\Delta} =$$

$$V_2 = \frac{-Y_{12}V_S Y_S}{\Delta}$$

$$\Rightarrow \frac{V_2}{V_S} = \frac{-Y_{12}Y_S}{Y_{22}Y_S + Y_{11}Y_L + Y_S Y_L + \Delta_Y}$$

ⓑ





$$\begin{bmatrix} V_S \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11}+Z_S & Z_{12} \\ Z_{12} & Z_L+Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\Delta = (Z_{11}+Z_S)(Z_L+Z_{22}) - Z_{12}^2$$

$$= Z_{11}Z_L + Z_S Z_{22} + Z_S Z_L + \Delta_Z$$

$$\frac{V_2}{Z_L} = I_2 = \frac{-V_S Z_{12}}{Z_{11}Z_L + Z_S Z_{22} + Z_S Z_L + \Delta_Z}$$

$$\rightarrow \frac{V_2}{V_L} = \frac{+Z_{12}Z_L}{Z_{11}Z_L + Z_S Z_{22} + Z_S Z_L + \Delta_Z}$$

4-4) $\psi(s) = \dfrac{a_0 s^3 + s a_2}{a_1 s^2 + a_3}$

$$a_1 s^2 + a_3 \overline{\big) a_0 s^3 + s a_2} \quad \Big(\frac{a_0}{a_1} s$$

$$\underline{a_0 s^3 + \frac{a_0 a_3}{a_1} s}$$

$$\left(a_2 - \frac{a_0 a_3}{a_1}\right) s \overline{\big) a_1 s^2 + a_3} \quad \Bigg(\frac{a_1 s}{a_2 - \frac{a_0 a_3}{a_1}}$$

$$\underline{a_1 s^2}$$

$$a_3 \overline{\Big) \left(a_2 - \frac{a_0 a_3}{a_1}\right) s} \quad \Bigg(\frac{\left(a_2 - \frac{a_0 a_3}{a_1}\right) s}{a_3}$$

$$\Rightarrow \quad a_2 - \frac{a_0 a_3}{a_1} > 0$$

$$a_2 > \frac{a_0 a_3}{a_1}$$

$$a_1 a_2 > a_0 a_3$$

EMPLOYING ROUTH CRITERION

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_2 a_1 - a_0 a_3}{a_1} & 0 \\ s^0 & a_3 & \end{array}$$

$$\Rightarrow a_2 a_1 - a_0 a_3 > 0 \Rightarrow a_2 a_1 > a_0 a_3$$

4-5) a) $s^5 + 8s^4 + 24s^3 + 28s^2 + 23s + 6$

$$\psi(s) = \frac{s^5 + 24s^3 + 23s}{8s^4 + 28s^2 + 6}$$

$8s^4 + 28s^2 + 6 \;\big)\; s^5 + 24s^3 + 23s \;\big(\; \frac{1}{8}s$

$\qquad s^5 + \frac{7}{2}s^3 + \frac{3}{4}s$

$\frac{41}{2}s^3 + \frac{89}{4}s \;\big)\; 8s^4 + 28s + 6 \;\big(\; \frac{16}{41}s$

$\qquad 8s^4 + \frac{356}{41}s$

$\frac{792}{41}s^2 - 6 \;\big)\; \frac{41}{2}s^3 + \frac{89}{4}s \;\big(\; \frac{1681}{1584}s$

$\qquad \frac{41}{2}s^3 - \frac{1681}{264}s$

$\frac{3518}{264}s \;\big)\; \frac{792}{41}s^2 - 6 \;\big(\; \frac{264}{3518} \cdot \frac{792}{41}s$

$\qquad \frac{792}{41}s^2$

$-6 \;\big)\; \frac{3518}{264}s \;\big(\; \frac{-264}{3518 \cdot 6}s$

b) $$\psi(s) = \frac{s^5 + 16s^3 + 10s}{6.5s^4 + 18.5s^2 + 2}$$

USING ROUTH CRITERION:

s^5	1	16	10
s^4	$\frac{13}{2}$	$\frac{37}{2}$	2
s^3	$\frac{342}{13}$	$\frac{128}{3}$	0
s^2			
s^1			
s^0			

ech!

4-12) a) YES, OBVIOUSLY

b) $\omega^6 - 3\omega^4 - \omega^2 + 3$

$8 - 12 -$ [illegible]

$P_0(x) = x^3 - 3x^2 - x + 3$

$P_1(x) = 3x^2 - 6x - 1$

$$
\begin{array}{r}
\frac{1}{3}x - \frac{1}{3} \\
3x^2 - 6x - 1 \,\big)\, x^3 - 3x^2 - x + 3 \\
\underline{x^3 - 2x^2 - \frac{1}{3}x} \\
-x^2 - \frac{2}{3}x + 3 \\
\underline{-x^2 + 2x + \frac{1}{3}} \\
-\frac{8}{3}x + \frac{8}{3}
\end{array}
$$

$\Rightarrow P_2(x) = \frac{8}{3}x - \frac{8}{3} = \frac{8}{3}(x - 1)$

$$
\begin{array}{r}
3x - 3 \\
x - 1 \,\big)\, 3x^2 - 6x - 1 \\
\underline{3x^2 - 3x} \\
-3x - 1 \\
\underline{-3x + 3} \\
-4
\end{array}
$$

$\Rightarrow P_3 = 4$

	P_0	P_1	P_2	P_3	
0	+	−	−	+	2
∞	+	+	+	+	0

$|2 - 0| = 2 \Rightarrow 2$ REAL ZEROS $0 < \omega < \infty$

$P_0(1) = 0$

$P_0(2) = -3 \Rightarrow$ NOT + REAL

$1 \quad +2 \quad -3 \quad -4 \quad +2$

c) $A(\omega^2) = \omega^8 + 2\omega^6 - 3\omega^4 - 4\omega^2 + 2$

$A(\omega^2)\big|_{\omega=1} = -2 < 0 \Rightarrow$ NOT + REAL

d) $A(\omega^2) = \omega^8 - \omega^4 - 2\omega^2 + 2$

$P_0(x) = x^4 - x^2 - 2x + 2$

$P_1(x) = 4x^3 - 2x - 2$

$$\begin{array}{r} \frac{1}{2}x - \frac{1}{4} \\ 2x^3 - x - 1 \overline{\big) x^4 - x^2 - 2x + 2} \\ \underline{x^4 - \frac{1}{2}x^2 - \frac{1}{2}x} \\ -\frac{1}{2}x^2 - \frac{3}{2}x + 2 \end{array}$$

(UH-OH!)

@ $x = 1$, $P_0(x) = 0$

$$\begin{array}{r} x^3 + x^2 \\ x - 1 \overline{\big) x^4 - x^2 - 2x + 2} \\ \underline{x^4 - x^3} \\ x^3 - x^2 - 2x + 2 \\ \underline{x^3 - x^2} \\ -2x + 2 = -2(x-1) \end{array}$$

$$\Rightarrow P_0(x) = \left[(x^3 + x^2) + \frac{-2(x-1)}{x-1}\right](x-1)$$

$$= (x^3 + x^2 - 2)(x-1)$$

FOR $P_0(x) > 1$, $x^3 + x^2 - 2 < 0$ FOR $x < 1$

> 0 FOR $x > 1$

WHICH IS ~~THE~~ CASE

$\Rightarrow$ POS. REAL

4-19) a) $\dfrac{s^3+6s^2+7s+3}{s^2+2s+1} = \dfrac{p(s)}{q(s)}$

$p(s)+q(s) = s^3+7s^2+9s+4$

$\dfrac{9 \cdot 7 - 4}{7} = \dfrac{63-4}{7} = \dfrac{59}{7}$

$$\begin{array}{lll} s^3 & 1 & 9 \\ s^2 & 7 & 4 \\ s^1 & \frac{59}{7} & 0 \\ s^0 & 4 & \end{array}$$

$\Rightarrow Y(s)$ IS REAL WHEN s IS REAL

$\Rightarrow \dfrac{p(s)}{q(s)}$ IS STRICTLY HURWITZ

$m_1 = 6s^2+3 = -6\omega^2+3 \; ; \; n_1 = s^3+7s = j(-\omega^3+7\omega)$

$m_2 = s^2+1 = -\omega^2+1 \; ; \; n_2 = 2s = j2\omega$

$A(\omega^2) = m_1 m_2 - n_1 n_2$

$= (-6\omega^2+3)(-\omega^2+1) + 2\omega(-\omega^3+7\omega)$

$= 6\omega^4 - 6\omega^2 - 3\omega^2 + 3 - 2\omega^4 + 14\omega^2$

$= 6\omega^4 + 5\omega^2 + 3$

$P_0(x) = 6x^2+5x+3 > 0$ FOR ALL x

$\Rightarrow$ POSITIVE REAL FUNCTION

5-2) $Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$

a) FIRST FOSTER FORM

$Z(S)$ HAS NO POLES @ 0 OR ∞

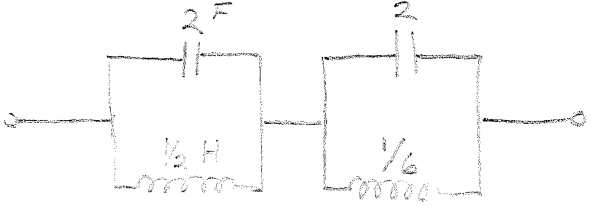
$$\Rightarrow Z(S) = \frac{2AS}{S^2+1} + \frac{2BS}{S^2+3}$$

$$A = \left.\frac{S(S^2+2)}{(S^2+1)(S+j\sqrt{3})}\right|_{S=j\sqrt{3}} = \frac{j\sqrt{3}(-1)}{(-2)(2j\sqrt{3})} = \frac{1}{4}$$

$$B = \left.\frac{S(S^2+2)}{(S+j)(S^2+3)}\right|_{S=j} = \frac{j(1)}{(j2)(2)} = \frac{1}{4}$$

$$\therefore Z(S) = \frac{S}{2(S^2+1)} + \frac{S}{2(S^2+3)}$$

$$= \frac{1}{2S+\frac{1}{2}S} + \frac{1}{2S+\frac{1}{6}S}$$



b) SECOND FOSTER FORM

$$Y(S) = \frac{(S^2+1)(S^2+3)}{S(S^2+2)} = \frac{S^4+4S^2+3}{S^3+2S}$$ POLE @ 0 & ∞

$$S^3+2S \;\big)\; S^4+4S^2+3 \;\big|\; S$$

$$S^4+2S^3$$

$$2S^2+3$$

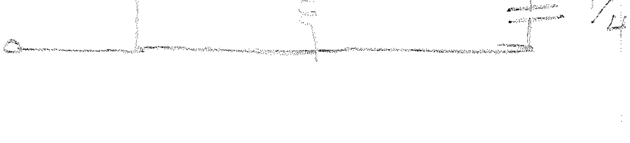
$$\Rightarrow Y(S) = S + \frac{2S^2+3}{S^3+2S}$$

$$2S+S^3 \;\big)\; 3+2S^2 \;\big|\; \frac{3}{2S}$$

$$3+\frac{3}{2}S^2$$

$$\frac{1}{2}S^2$$

$$\Rightarrow Y(S) = S + \frac{3}{2S} + \frac{\frac{1}{2}S^2}{S^3+2S} = S + \frac{3}{2S} + \frac{1}{2S+\frac{1}{4}S}$$



5-3) $Z(s) = 78 \dfrac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}$

$Z'(s) = \dfrac{Z(s)}{78} = \dfrac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)} = \dfrac{s^5+6s^3+8s}{s^4+4s^2+3}$

NO POLES @ 0,

POLE @ ∞

$$\begin{array}{r|l|l} s^4+4s^2+3 & s^5+6s^3+8s & s \\ & s^5+4s^3+3s & \\ \hline & 2s^3+5s & \end{array}$$

$Z'(s) = s + \dfrac{2s^3+5s}{s^4+4s^2+3} = s + \dfrac{s(2s^2+5)}{(s^2+3)(s^2+1)}$

$= s + \dfrac{2As}{(s^2+3)} + \dfrac{2Bs}{(s^2+1)}$

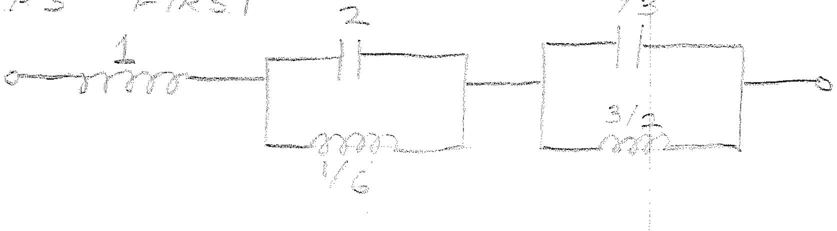
$A = \dfrac{s(2s^2+5)}{(s+j\sqrt{3})(s^2+1)}\Bigg|_{\substack{s=j\sqrt{3}\\ s^2=-3}} = \dfrac{j\sqrt{3}(-1)}{j2\sqrt{3}(-2)} = \dfrac{1}{4}$

$B = \dfrac{s(2s^2+5)}{(s^2+3)(s+j)}\Bigg|_{\substack{s^2=-1\\ s=j}} = \dfrac{j(3)}{(2)j2} = \dfrac{3}{4}$

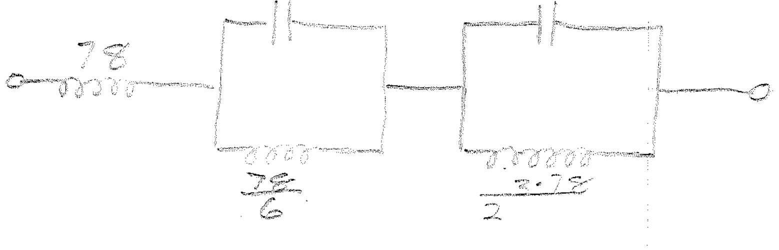
$Z'(s) = s + \dfrac{s}{2s^2+6} + \dfrac{s}{\frac{2}{3}s^2+\frac{2}{3}}$

$= s + \dfrac{1}{2s+\frac{1}{\frac{1}{6}s}} + \dfrac{1}{\frac{2}{3}s+\frac{1}{\frac{3}{2}s}}$

FOSTER'S FIRST



SCALING:

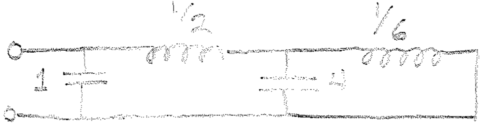


5-4) CAUER'S FIRST

$$Z(S) = \frac{S(S^2+2)}{(S^2+1)(S^2+3)} \Rightarrow \text{NO } \infty \text{ POLES}$$

$$Y(S) = \frac{S^4+4S^2+3}{S^3+2S}$$

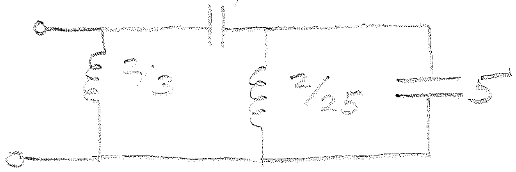
Z $\frac{1}{2}S$	S^3+2S	S^4+4S^2+3	S	Y
	$S^3+\frac{3}{2}S$	S^4+2S^2		
$\frac{1}{6}S$	$\frac{1}{2}S$	$2S^2+3$	$4S$	
	$\frac{1}{2}S$	$2S^2$		
	0	3		



CAUER'S SECOND

$$Y(S) = \frac{3+4S^2+S^4}{2S+S^3}$$

$= \frac{4}{5S}$	$2S+S^3$	$3+4S^2+S^4$	$\frac{3}{2S}$	Y
	$2S+\frac{4}{5}S^3$	$3+\frac{3}{2}S^2$		
$\frac{1}{5S}$	$\frac{1}{5}S^3$	$\frac{5}{2}S^2+S^4$	$\frac{25}{2S}$	
	$\frac{1}{5}S^3$	$\frac{5}{2}S^2$		
	0	S^4		

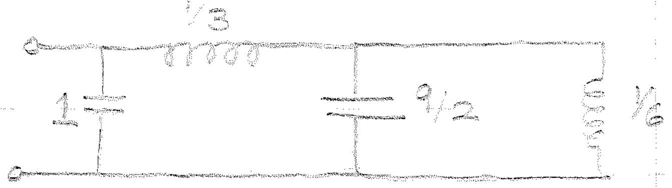


5-16) $Z(s) = \dfrac{S(S^2+2)}{(S^2+1)(S^2+4)}$

a) $Z(s) = \dfrac{S^3+2S}{S^4+5S^2+4} \Rightarrow$ POLE @ 0

$Y(s) = \dfrac{S^4+5S^2+4}{S^3+2S}$

Z			Y
$\frac{1}{3}S$	S^3+2S	S^4+5S^2+4	S
	$S^3+\frac{4}{3}S$	S^4+2S^2	
$\frac{1}{6}S$	$\frac{2}{3}S$	$3S^2+4$	$\frac{9}{2}S$
	$\frac{2}{3}S$	$3S^2$	
	0	4	



5·2) a) $$Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)} = \frac{s^3+2s}{s^4+4s^2+3}$$

$\Rightarrow$ NO POLES @ 0 OR ∞

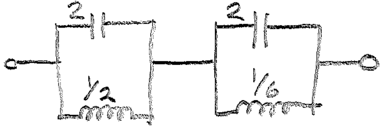
$$Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)} = \frac{2As}{s^2+1} + \frac{2Bs}{s^2+3}$$

$$A = \left.\frac{s(s^2+2)}{(s+j)(s^2+3)}\right|_{\substack{s=j \\ s^2=-1}} = \frac{j(1)}{j\,2(2)} = \frac{1}{4}$$

$$B = \left.\frac{s(s^2+2)}{(s^2+1)(s+j\sqrt{3})}\right|_{\substack{s=j\sqrt{3} \\ s^2=-3}} = \frac{j\sqrt{3}\,[-1]}{(-2)(j2\sqrt{3})} = \frac{1}{4}$$

$$\Rightarrow Z(s) = \frac{s}{2s^2+2} + \frac{s}{2s^2+6}$$

$$= \frac{1}{2s+\frac{1}{\frac{1}{2}s}} + \frac{1}{2s+\frac{1}{6s}}$$



b) $$Y(s) = \frac{s^4+4s^2+3}{s^3+2s} \Rightarrow \text{POLES @ } 0 \;\&\; \infty$$

POLE @ ∞ : $s^3+2s \,\big)\, s^4+4s^2+3 \,\big(\, s$

$$\underline{s^4+2s^2+0}$$

$$2s^2+3$$

$$\Rightarrow Y(s) = s + \frac{2s^2+3}{s^3+2s}$$

$2s+s^3 \,\big)\, 3+2s^2 \,\big(\, \frac{3}{2s}$

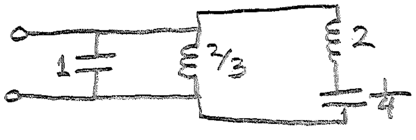
$$\underline{3+\frac{3}{2}s^2}$$

$$\frac{1}{2}s^2$$

$$\Rightarrow Y(s) = s + \frac{3}{2s} + \frac{\frac{1}{2}s^2}{s(s^2+2)}$$

$$= s + \frac{3}{2s} + \frac{s}{2s^2+4}$$

$$= s + \frac{3}{2s} + \frac{1}{2s+\frac{1}{\frac{1}{4}s}}$$

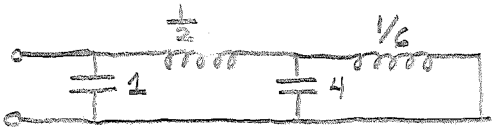


5-4) $Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$

a) CAUER'S FIRST ⇒ POLE @ ∞

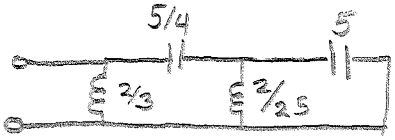
$Y(s) = \frac{s^4+4s^2+3}{s^3+2s}$

Z			Y
$\frac{1}{2}s$	s^3+2s	s^4+4s^2+3	s
	$s^3+\frac{3}{2}s$	s^4+2s^2	
$\frac{1}{6}s$	$\frac{1}{2}s$	$2s^2+3$	$4s$
	$\frac{1}{2}s$	$2s^2$	
	0	3	

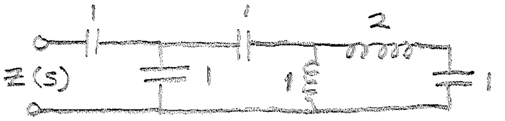


b) CAUER'S SECOND ⇒ POLE @ 0

Z			Y
$\frac{4}{5s}$	$2s+s^3$	$3+4s^2+s^4$	$\frac{3}{2s}$
	$2s+\frac{4}{5}s^3$	$3+\frac{3}{2}s^2$	
$\frac{1}{5s}$	$\frac{1}{5}s^3$	$\frac{5}{2}s^2+s^4$	$\frac{25}{2s}$
	$\frac{1}{5}s^3$	$\frac{5}{2}s^2$	
	0	s^4	



5-12)



$$Z(s) = \frac{1}{s} + \cfrac{1}{s + \cfrac{1}{\frac{1}{s} + \cfrac{1}{\frac{1}{s} + \cfrac{1}{2s + 1/s}}}}$$

$$= \frac{4s^4 + 11s^2 + 3}{s(2s^4 + 7s^2 + 2)}$$

a) FOSTER'S FIRST FORM

$Z(s)$; POLE @ 0 ; NO POLE @ ∞

$$\begin{array}{r|l} & \frac{3}{2s} \\ 2s + 7s^3 + 2s^5 & 3 + 11s^2 + 4s^4 \\ & \underline{3 + \frac{21}{2}s^2 + 3s^4} \\ & \frac{1}{2}s^2 + s^4 \end{array}$$

$$\Rightarrow Z(s) = \frac{3}{2s} + \frac{s^2(1 + s^2)}{4s^5 + 14s^3 + 4s}$$

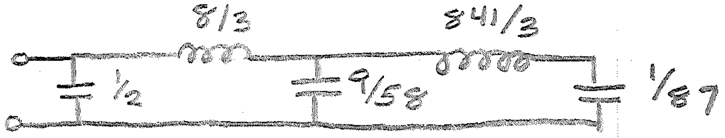
$$= \frac{3}{2s} + \frac{s(1 + s^2)}{2(2s^4 + 7s^2 + 2)}$$

$$= \frac{3}{2s} + \underbrace{\frac{s(s^2 + 1)}{2(2s^2 \quad)(s^2 \quad)}}_{\text{ECH!}}$$

FIRST CAUER FORM

$$Y(s) = \frac{2s^5+7s^3+2s}{4s^4+11s^2+3} \Rightarrow \text{POLE } @ \infty$$

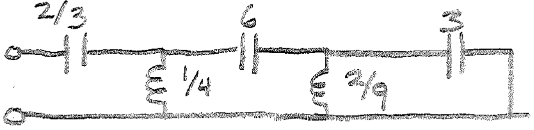
			Y
$\frac{8}{3}s$	$4s^4+11s^2+3$	$2s^5+7s^3+2s$	$\frac{1}{2}s$
	$4s^4+\frac{4}{3}s^2$	$2s^5+\frac{11}{2}s^3+\frac{3}{2}s$	
$\frac{29}{1}\cdot\frac{29}{3}s$	$\frac{29}{3}s^2+3$	$\frac{3}{2}s^3+\frac{1}{2}s$	$\frac{3}{2}\frac{3}{29}s=\frac{9}{58}s$
$=\frac{841}{3}s$	$\frac{29}{3}s^2$	$\frac{3}{2}s^3+\frac{27}{58}s$	
	3	$\frac{1}{29}s$	$\frac{1}{3\cdot29}s$
			$=\frac{1}{87}s$



SECOND CAUER FORM

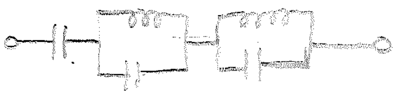
$$Z(s) = \frac{4s^4+11s^2+3}{2s^5+7s^3+2s} \Rightarrow \text{POLE } @ 0$$

			Z
$\frac{4}{s}$	$2s+7s^3+2s^5$	$3+11s^2+4s^4$	$\frac{3}{2s}$
	$2s+4s^3$	$3+\frac{21}{2}s^2+3s^4$	
$\frac{9}{2s}$	$3s^3+2s^5$	$\frac{1}{2}s^2+s^4$	$\frac{1}{6s}$
	$3s^3$	$\frac{1}{2}s^2+\frac{1}{3}s^4$	
	$2s^5$	$\frac{2}{3}s^4$	$\frac{1}{3s}$
		$\frac{2}{3}s^4$	

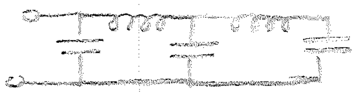


5-14) a)

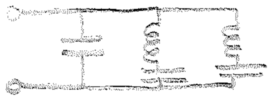
FOSTER FIRST:



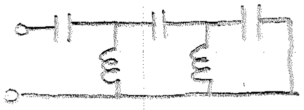
CAUER'S FIRST:



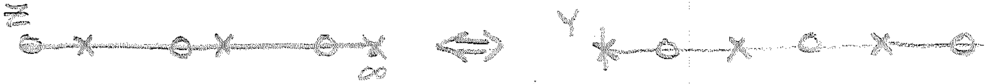
FOSTER'S SECOND:



CAUER'S SECOND:

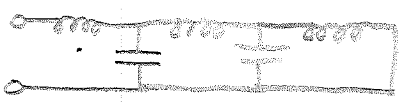


b)



FOSTER'S FIRST:

CAUER'S FIRST:



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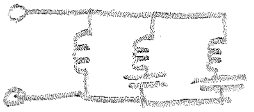

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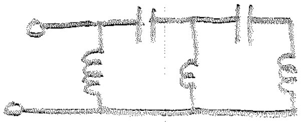
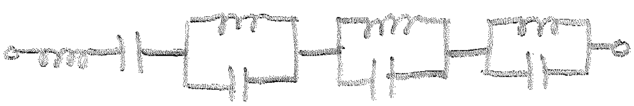
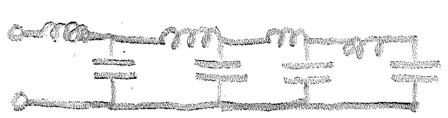


c)

FOSTER'S FIRST



CAUER'S FIRST



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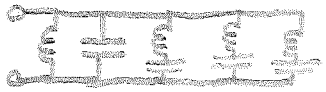

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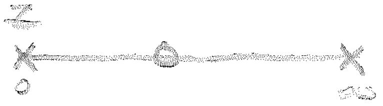
d) 

FOSTER'S FIRST

FOSTER'S SECOND:

CAUER'S FIRST

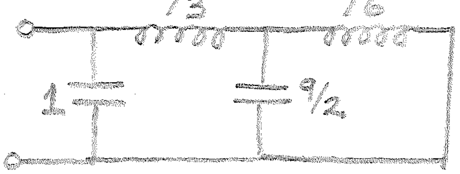
CAUER'S SECOND:

5-16) a) $Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+4)} = \frac{s^3+2s}{s^4+5s^2+4}$

CAUER'S FIRST:

$Y(s) = \frac{s^4+5s^2+4}{s^3+2s} \Rightarrow$ POLE @ ∞

Z			Y
$\frac{1}{3}S$	S^3+2S	S^4+5S^2+4	S
	$S^3+\frac{4}{3}S$	S^4+2S^2	
$\frac{1}{6}S$	$\frac{2}{3}S$	$3S^2+4$	$\frac{9}{2}S$
	$\frac{2}{3}S$	$3S^2$	
	0	4	



6-3) a) $Z(s) = \frac{(s+2)(s+5)}{(s+1)(s+3)}$

LOWEST SINGULARITY A POLE @ $s=-1 \Rightarrow$ RC FUNCTION

FOSTER'S FIRST:

$$Z(s) = \frac{s^2+7s+10}{s^2+4s+3} \quad \Rightarrow Z(\infty) = 1$$

$$\begin{array}{r|l} & \quad 1 \\ \hline s^2+4s+3 & s^2+7s+10 \\ & \underline{s^2+4s+3} \\ & \qquad 3s+7 \end{array}$$

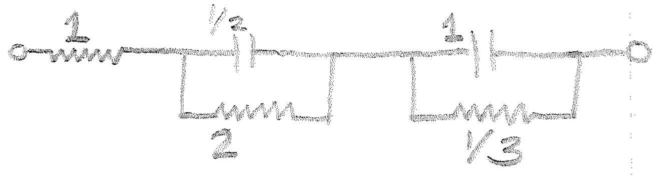
$$\Rightarrow Z(s) = 1 + \frac{3s+7}{(s+1)(s+3)}$$

$$= 1 + \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = \frac{4}{2} = 2 \ ; \quad B = \frac{-2}{-2} = 1$$

$$Z(s) = 1 + \frac{2}{s+1} + \frac{1}{s+3}$$

$$= 1 + \frac{1}{s/2 + 1/2} + \frac{1}{s+3}$$



b) $Y(s) = \frac{(s+1)(s+3)}{(s+2)(s+5)}$

$\frac{Y(s)}{s} = \frac{(s+1)(s+3)}{s(s+2)(s+5)} = \frac{s^2+4s+3}{s^3+7s^2+10s}$

$$\begin{array}{r} \frac{3}{10s} \\ 10s+7s^2+s^3 \overline{\big) \; 3+4s+s^2} \\ \underline{3+\frac{21}{10}s+\frac{3}{10}s^2} \\ \frac{19}{10}s+\frac{7}{10}s^2 \end{array}$$

$\Rightarrow \frac{Y(s)}{s} = \frac{3}{10s} + \frac{19+7s}{10(10+7s+s^2)}$

$= \frac{3}{10s} + \frac{7s+19}{10(s+5)(s+2)}$

$= \frac{3}{10s} + \frac{A}{s+5} + \frac{B}{s+2}$

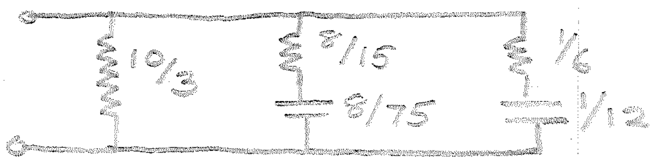
$A = \frac{-16}{10(-3)} = \frac{16}{30} = \frac{8}{15}$

$B = \frac{5}{10(3)} = \frac{1}{6}$

$\Rightarrow Y(s)/s = \frac{3}{10s} + \frac{8}{15}\left(\frac{1}{s+5}\right) + \frac{1}{6}\left(\frac{1}{s+2}\right)$

$Y(s) = \frac{3}{10} + \frac{8}{15}\left(\frac{1}{1+5/s}\right) + \frac{1}{6}\left(\frac{1}{1+2/s}\right)$

$= \frac{3}{10} + \frac{1}{\frac{15}{8}+\frac{75}{8s}} + \frac{1}{6+12/s}$



6-7) $Z(s) = \frac{(s+1)(s+4)}{(s+3)(s+5)} \Rightarrow RL$

a) $\frac{Z(s)}{s} = \frac{(s+1)(s+4)}{s(s+3)(s+5)} = \frac{s^2+5s+4}{s^3+8s^2+15s}$

$15s+8s^2+s^3 \overline{)\,4+5s+s^2\,}\left(\frac{4}{15s}\right.$

$4+\frac{32}{15}s+\frac{4}{15}s^2$

$\frac{43}{15}s+\frac{11}{15}s^2$

$\Rightarrow \frac{Z(s)}{s} = \frac{4}{15s} + \frac{11s+43}{15(s+3)(s+5)}$

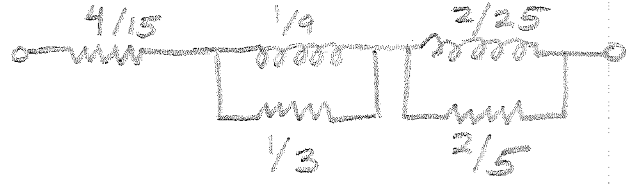
$= \frac{4}{15s} + \frac{A}{s+3} + \frac{B}{s+5}$

$A = \frac{10}{15\cdot 2} = \frac{1}{3}$

$B = \frac{-12}{15(-2)} = \frac{+12}{30} = \frac{2}{5}$

$\Rightarrow \frac{Z(s)}{s} = \frac{4}{15s} + \frac{1}{3s+9} + \frac{2}{5s+25}$

$Z(s) = \frac{4}{15} + \frac{1}{3+9/s} + \frac{1}{\frac{5}{2}+25/2s}$



b) $Y(s) = \dfrac{(s+3)(s+5)}{(s+1)(s+4)}$

$= \dfrac{s^2+8s+15}{s^2+5s+4}$

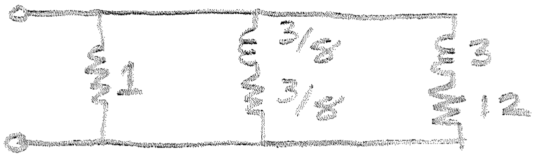
$$\begin{array}{r|l} & \quad 1 \\ \hline s^2+5s+4 & s^2+8s+15 \\ & \underline{s^2+5s+4} \\ & \qquad 3s+11 \end{array}$$

$Y(s) = 1 + \dfrac{3s+11}{(s+1)(s+4)}$

$= 1 + \dfrac{A}{s+1} + \dfrac{B}{s+4}$

$A = \dfrac{8}{3}$, $B = \dfrac{-1}{-3} = \dfrac{1}{3}$

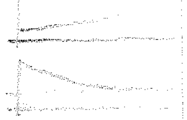
$Y(s) = 1 + \dfrac{1}{\frac{3}{8}s + \frac{3}{8}} + \dfrac{1}{3s+12}$



6-8) a) $Z(s) = \frac{2s+s^2}{3+4s+s^2} = \frac{s(s+2)}{(s+3)(s+1)} \Rightarrow RL$

$Z(\infty) > Z(0)$

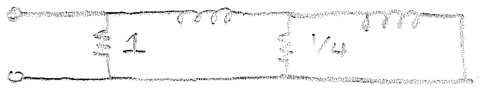
$Y(0) > Y(\infty)$



CAUER'S FIRST:

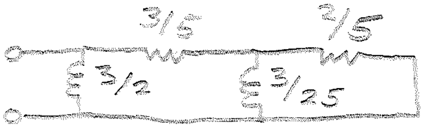
$Y(s) = \frac{3+4s+s^2}{2s+s^2}$

Z				Y
$\frac{1}{2}s$	s^2+2s	s^2+4s+3	1
	$s^2+\frac{3}{2}s$	s^2+2s
$\frac{1}{6}s$	$\frac{1}{2}s$	$2s+3$	4
	$\frac{1}{2}s$	$2s$
	0	3



CAUER'S SECOND

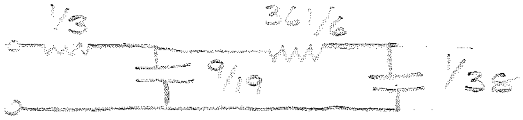
Z			Y
$\frac{3}{5}$	$2s+s^2$	$3+4s+s^2$	$\frac{2}{3s}$
	$2s+\frac{3}{5}s^2$	$3+\frac{2}{3}s$	
$\frac{2}{5}$	$\frac{2}{5}s^2$	$\frac{10}{3}s+s^2$	$\frac{5}{2}\cdot\frac{10}{3s} = \frac{50}{6s} = \frac{25}{3s}$
		$\frac{10}{3}s+s^2$	



b) $Z(s) = \dfrac{s^2+7s+4}{3s^2+2s} \Rightarrow RC$

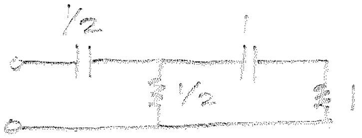
CAUER'S FIRST

Y			Z
$\frac{3\cdot3}{19}S=\frac{9}{19S}$	$3S^2+2S$	S^2+7S+4	$\frac{1}{3}$
	$3S^2+\frac{36}{19}S$	$S^2+\frac{2}{3}S$	
$\frac{1}{4}\cdot\frac{2S}{19}=\frac{1}{38S}$	$\frac{2}{19}S$	$\frac{19}{3}S+4$	$\frac{19}{3}\cdot\frac{19}{2}=\frac{361}{6}$
	$\frac{2}{19}S$	$\frac{19}{3}S$	
	0	4	

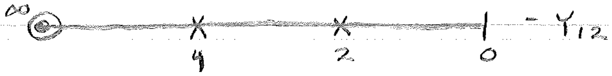


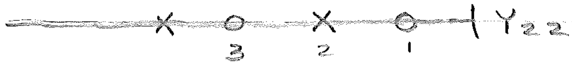
CAUER'S SECOND

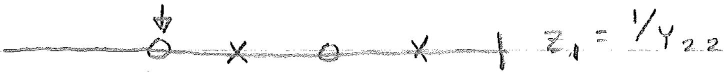
Y			Z
2	$2S+3S^2$	$4+7S+S^2$	$2/S$
	$2S+3S^2$	$4+6S$	
1	S^2	$S+S^2$	$1/S$
	S^2	S	
	0	S^2	



10.19) $Y_{22} = \frac{(s+1)(s+3)}{(s+2)(s+4)}$ $\quad -Y_{12}(s) = \frac{1}{(s+2)(s+4)}$

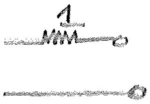






SHIFT POLE TO ∞

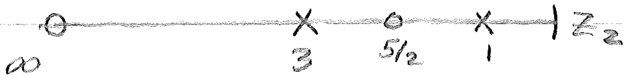
$Z_1(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)} \Rightarrow Z_1(\infty) = 1$

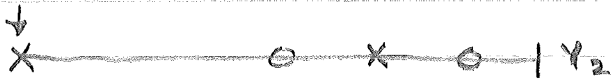


$Z_2(s) = Z_1(s) - 1$

$= \frac{(s^2+6s+8)-(s^2+4s+3)}{(s+1)(s+3)}$

$= \frac{2s+5}{(s+1)(s+3)}$

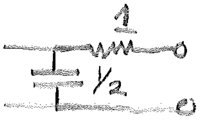




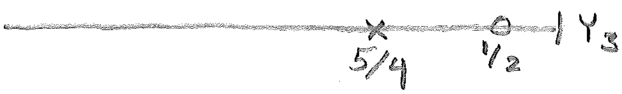
REMOVE ∞ POLE

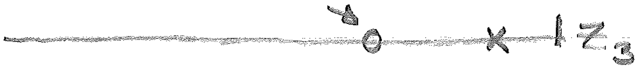
$Y_2 = \frac{s^2+4s+3}{2s+5}$

$$\begin{array}{r|l} & \frac{1}{2}s \\ \hline 2s+5 & s^2+4s+3 \\ & \underline{s^2+\frac{5}{2}s} \\ & \frac{3}{2}s+3 \end{array}$$



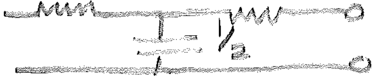
$Y_3(s) = \frac{3/2\,s+3}{2s+5} = \frac{3s+6}{4s+10}$





SHIFT 0 TO ∞

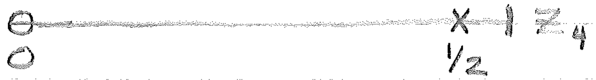
$Z_3(s) = \frac{4s+10}{3s+6} \Rightarrow Z_3(\infty) = \frac{4}{3}$



$$Z_4(s) = \frac{4s+10}{3s+6} - \frac{4}{3}$$

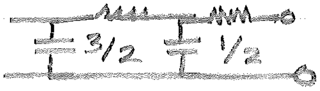
$$\begin{array}{r|l} & \frac{4}{3} \\ 3s+6 & 4s+10 \\ & \underline{4s+8} \\ & \quad\;\; 2 \end{array}$$

$$\Rightarrow Z_4(s) = \frac{2}{3s+6}$$

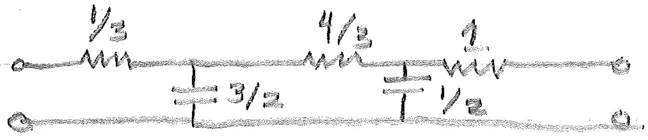


REMOVE POLE @ ∞

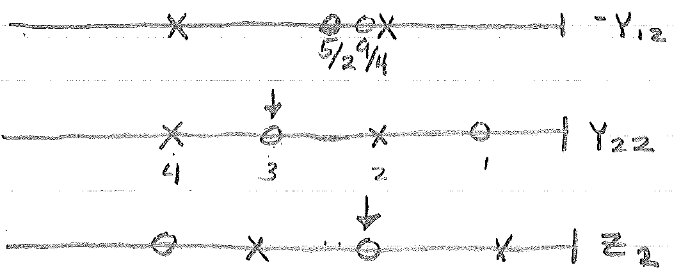
$$\begin{array}{r|l} & \frac{3}{2}s \\ 12 & 3s+6 \\ & \underline{3s} \\ & \quad 6 \end{array}$$



$Y_5 = 3$



$$10\text{-}21)\quad Y_{22}(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)} \qquad -Y_{12} = \frac{(s+5/2)(s+9/4)}{(s+2)(s+4)}$$



SHIFT ZERO TO 9/4, BY PARTIALLY REMOVING POLE @ 3

$$Z_2 = \frac{(s+2)(s+4)}{(s+1)(s+3)}$$

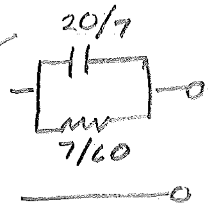
$$Z_3 = Z_2 - \frac{K}{s+3}$$

$$Z_3(-9/4) = 0$$

$$Z_2\left(-\frac{9}{4}\right) = 7/15$$

$$\left.\frac{K}{s+3}\right|_{s=-9/4} = \frac{4}{3}K$$

$$\Rightarrow 0 = \frac{7}{15} - \frac{4}{3}k \Rightarrow k = \frac{7}{20}\text{;}\quad K/s+3 \Rightarrow$$



$$\text{AND}\quad Z_3(s) = \frac{(s+\frac{45}{20})(s+\frac{68}{20})}{(s+1)(s+3)} \Rightarrow$$

$\rightsquigarrow$ FLIP & SHIFT

BOB MARKS

EE 532 NETWORK SYNTHESIS I Oct. 3, 1972

50-Minute Exam. Open Books 100

1. Determine whether or not the following functions are positive real functions of **s**.

10 a) $\frac{s + 2}{s^3 + 5s^2 + 6s + 7}$ (NO) $p(s) > q(s)$ IN DEGREE OVER 1

10 b) $\frac{s^2 + 4s \textcircled{-} 3}{s^2 + 5s + 6}$ (NO) (−) COEFF ✓

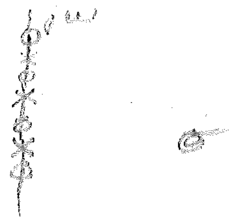
10 c) $\frac{s^4 \overset{s^3 ?}{} + 5s^2 \overset{s ?}{} + 4}{s^4 + 8s^2 + 12}$ (NO) MISSING $\overset{\text{ODD}}{\checkmark}$ PWR OF S ✓

10 d) $\frac{(s^2 + 1)^2 (s + 3)}{(s + 2)(s + 4)(S^2+5)} \Rightarrow$ Y(S) P.R. $\Rightarrow \frac{1}{Y(S)} = Z(S)$ IS P.R. : $\frac{(S+2)(S+4)(S^2+5)}{(S^2+1)(S^2+1)(S^2+5)} \Rightarrow$ DOUBLE j AXIS POLES $\Rightarrow$ NOT POS REAL

DOUBLE j AXIS ZEROS $\Rightarrow$ NOT. P.R.

10 e) $\frac{s^4 + 4s^2 + 3}{s^3 + 2s} \Rightarrow$ POSITIVE REAL

$\frac{(S^2+3)(S^2+1)}{S(S^2+2)}$



10 f) $\frac{s^3 + 2s^2 + 5s}{s^2 + 2s + 1} \Rightarrow$ POS REAL

10 g) $\frac{s^2 + s}{s^2 + 5s + 6} \Rightarrow$ NOT POS REAL ; FAILED STURM? inspection?

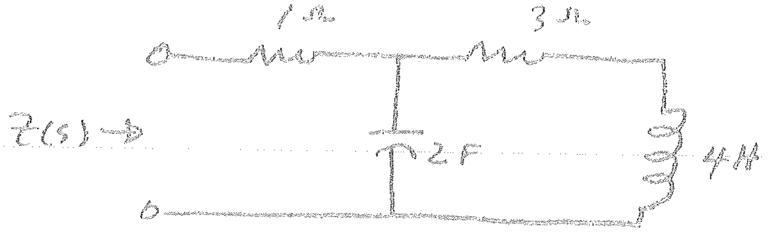
10 h) $\frac{s^3 + 2s^2 + 2s + 2}{s^4 + 4s^3 + 5s^2 + 3s + 4} \Rightarrow$ NOT POS REAL

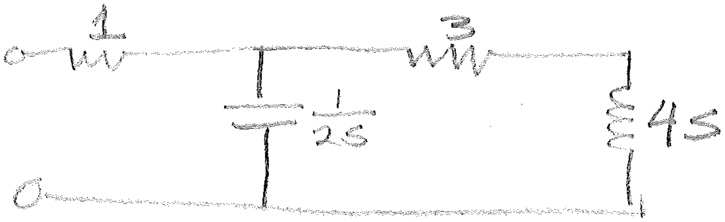
$p(s)+q(s)$ NOT STRICTLY HURWITZ common factor?

10 i) $\frac{s^3 + 5s^2 + 6s + 4}{s^3 + 2s^2 + 7s + 5} \Rightarrow$ POS REAL

EE 532 NETWORK SYNTHESIS I - Exam

2. Determine the driving-point impedance as a ratio of polynomials.





$$Z(s) = 1 + \frac{(3+4s)\frac{1}{2s}}{3+4s+\frac{1}{2s}}$$

$$= 1 + \frac{3+4s}{6s+8s^2+1}$$

$$= \frac{(6s+\cancel{8s^2}+1)+(3+\cancel{4s})}{8s^2+6s+1}$$

$$= \frac{8s^2+10s+4}{8s^2+6s+1}$$

10

e) $\dfrac{s^4+4s^2+3}{s(s^2+2)}$

$P(s)+q(s)=s^4+s^3+4s^2+2s+3$

s^4	1	4	3
s^3	1	2	0
s^2	2	3	0
s^1	$\frac{1}{2}$	0	
s^0	3		

$\Rightarrow$ [illegible]

STURM: $A(\omega^2) = 0 \overset{?}{\geq} 0 \ \forall \omega$ ✓

$A(\omega^2) = (s^4+4s^2+3)(0) = 0$

f) $p(s)+q(s) = s^3+3s^2+7s+1$

s^3	1	7
s^2	3	1
s^1	$\frac{20}{3}$	0
s^0	1	

STURM

$A(\omega) = (2s^2)(s^2+1) - (s^3+5s)2s$ ✓

$= \cancel{2s^4} + 2s^2 - \cancel{2s^4} - 10s^2$

$= -8s^2$

$\Rightarrow A(\omega^2) = -8(j\omega)^2 = 8\omega^2 \geq 0 \ \forall \omega$

$P_0(x) = x^3 - 3x^2 + 9x + 20$

$P_1(x) = 3x^2 - 6x + 9$

$$
\begin{array}{r}
\frac{1}{3}x - \frac{1}{3} \\
3x^2 - 6x + 9 \enclose{longdiv}{x^3 - 3x^2 + 9x + 20} \\
x^3 - 2x^2 + 3x \\
\hline
-x^2 + 6x + 20 \\
-x^2 + 2x - 3 \\
\hline
4x + 23
\end{array}
$$

$\Rightarrow P_2(x) = -4x - 23$

$$
\begin{array}{r}
-\frac{3}{4}x + \left(6 + \frac{69}{4}\right)/4 \\
-4x - 23 \enclose{longdiv}{3x^2 - 6x + 9} \\
3x^2 + \frac{69}{4}x \\
\hline
-\left(6 + \frac{69}{4}\right)x + 9 \\
-\left(6 + \frac{69}{4}\right)x + \frac{23\left(6 + \frac{69}{4}\right)}{4} \\
\hline
9 + \frac{23\left(6 + \frac{69}{4}\right)}{4} = -P_3
\end{array}
$$

$\frac{69}{4}$

$\Rightarrow P_3 = -\left(9 + \frac{23\left(6 + \frac{69}{4}\right)}{4}\right)$

	P_0	P_1	P_2	P_3	
0	+	+	−	−	1
∞	+	+	−	−	1

$|1 - 1| = 0$ REAL 0's

TWIXT 0 & INFINITY

✓

BOB MARKS

EE 532 Network Synthesis I Oct. 19, 1972

Hour Exam No. 2

38/40 $\boxed{95}$

1. a) Use the first Cauer's expansion (about infinity) to obtain the driving point admittance

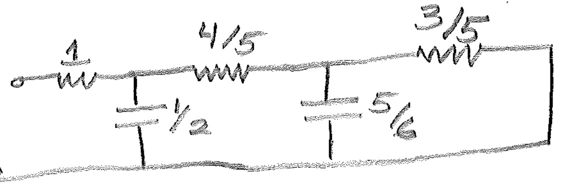
$$Y(s) = \frac{s^2+6s+5}{s^2+8s+12} \qquad Y(0)=\frac{5}{12} \;;\; Y(\infty)=1$$

$$\Rightarrow Y(0) < Y(\infty) \Rightarrow$$

$$Z(s)=\frac{s^2+8s+12}{s^2+6s+5} \qquad Z(0)=\frac{12}{5} \quad Z(\infty)=1$$

$$Z(0) > Z(\infty) \Rightarrow RC$$

$$\begin{array}{c|c|c|c}
Y & & & Z \\
\frac{1}{2}s & s^2+\overset{12}{6}s+5 & s^2+8s+12 & 1 \\
 & s^2+\frac{7}{2}s & s^2+6s+5 & \\
\hline
 & & 2s+7 & \frac{2}{5}\cdot 2=\frac{4}{5} \\
\frac{5}{6}s & \frac{5}{2}s+5 & 2s+4 & \\
 & \frac{5}{2}s & & \\
\hline
 & 5 & 3 & \frac{3}{5}
\end{array}$$



✓

b) Use the second Foster method to obtain the same driving point admittance as above.

$$Y(s)=\frac{s^2+6s+5}{s^2+8s+12}$$

$$\frac{Y(s)}{s}=\frac{s^2+6s+5}{s^3+8s^2+12s} = \frac{A}{s}+\frac{Bs}{(s+2)}+\frac{Cs}{(s+6)}$$

12
6
72
40
32

$$\begin{array}{r}
\frac{5}{12s} \\
-12s+8s^2+s^3 \overline{)\,5+6s+s^2} \\
5+\frac{40}{12}s+\frac{5}{12}s^2 \\
\hline
\frac{32}{12}s+\frac{7}{12}s^2
\end{array}$$

$$\frac{Y(s)}{s}=\frac{5}{12s}+\frac{1}{12}\left(\frac{32s+7s^2}{12s+8s^2+s^3}\right)=\frac{5}{12s}+\frac{1}{12}\,\frac{32+7s}{s^2+8s+12}$$

arith

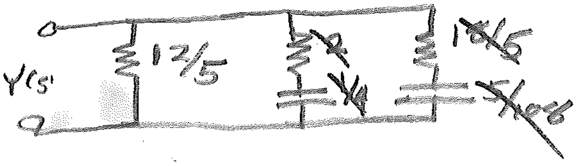
$$=\frac{5}{12s}+\frac{1}{12}\,\frac{7s+32}{(s+2)(s+6)}=\frac{5}{12s}+\frac{A}{s+2}+\frac{B}{s+6}$$

$$A=\frac{1}{12}\,\frac{18}{4}=\frac{18}{36}=\cancel{\frac{1}{2}}\;;\; B=\frac{1}{12}\,\frac{(-10)}{(-4)}=\frac{10}{36}=\cancel{\frac{5}{18}}$$

32
14
18

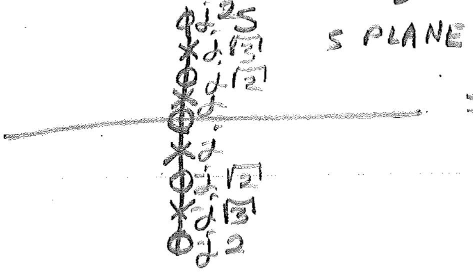
$$Y(s)/s=\frac{5}{12s}+\frac{1}{2s+4}+\frac{1}{\frac{18}{5}(s+6)} \Rightarrow Y(s)=\frac{5}{12}+\frac{1}{2+4/s}+\frac{1}{\frac{18}{5}+\frac{6\cdot 18}{5s}}$$

9



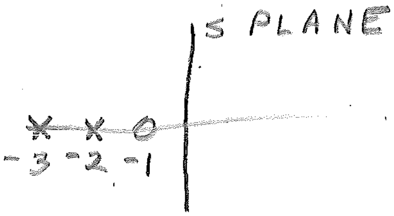
2. State whether each of the following <u>positive-real</u> functions are Z_{RC}, Z_{RL}, Z_{LC} or Z_{RLC} type functions. Give reasons.

a) $Z = \frac{s^5 + 6s^3 + 8s}{s^4 + 4s^2 + 3} = \frac{S(S^4+6S^2+8)}{(S^2+1)(S^2+3)} = \frac{S(S^2+2)(S^2+4)}{(S^2+1)(S^2+3)}$



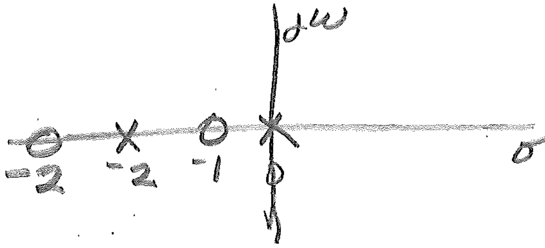
⇒ LC FUNCTION ✓

b) $Z = \frac{s + 1}{s^2 + 5s + 6} = \frac{(S+1)}{(S+2)(S+3)}$



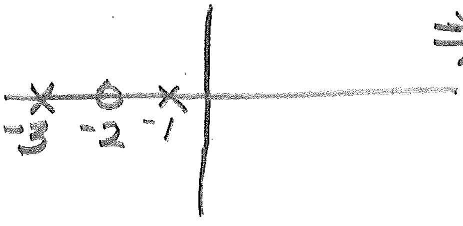
⇒ RLC ✓
(POLES & ZEROS NOT INTERLACED)

c) $Z = \frac{s^2 + 4s + 3}{s^2 + 2s} = \frac{(S+3)(S+1)}{S(S+2)}$ ✓



⇒ RC FUNCTION
(POLE NEAREST ORIGIN)
Poles & 0's INTERLACED

d) $Z = \frac{s + 2}{s^2 + 4s + 3} = \frac{(S+2)}{(S+3)(S+1)}$ ✓



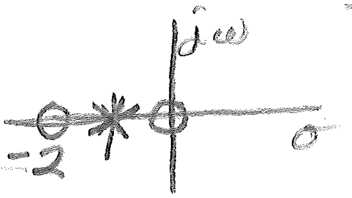
⇒ RC FUNCTION
(SAME REASONS AS ABOVE)

10

e) $Z = \frac{s^2 + 2s}{s^3 + 3s^2 + 3s + 1} = \frac{S(S+2)}{(S+1)(S^2+2S+1)}$ ✓

$= \frac{S(S+2)}{(S+1)(S+1)(S+1)}$

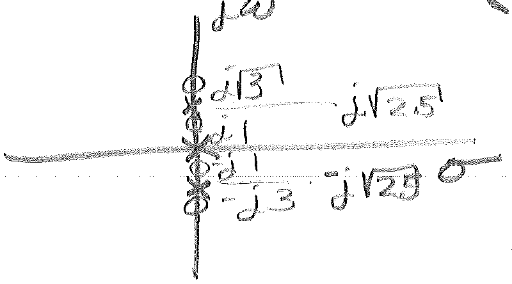
$$\begin{array}{r} S^2+2S+1 \\ S+1\overline{)S^3+3S^2+3S+1} \\ S^3+S^2 \\ \hline 2S^2+3S+1 \\ 2S^2+2S \\ \hline S+1 \end{array}$$



⇒ RLC
(Poles not interlaced)

3. Which of the following functions are positive-real? Give reasons. Note that a function which is Z_{RC}, Z_{RL} or Z_{LC} is positive real.

a) $Z = \dfrac{s^4 + 4s^2 + 3}{s^3 + 2.5s} = \dfrac{(S^2+3)(S^2+1)}{S(S^2+2.5)}$ ✓

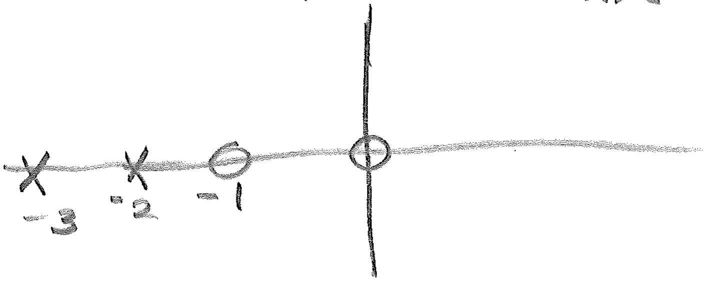


⇒ LC FUNCTION
(Poles & 0's interlaced on the jω AXIS

b) $Z = \dfrac{s^2 + s}{s^2 + 5s + 6} = \dfrac{S(S+1)}{(S+2)(S+3)}$

NOT LC, RC, LR ✓

~~ROUTH ON NUM & DEN~~)



$2S^2 + 6S + 6$

ROUTH

S^2	2	6
S^1	6	✓
S^0	6	

OK ✓

STURM

$(S^2+6)S^2 - 5S^2$

$S^4 + S^2$

$\omega^4 - \omega^2$

$X^2 - X < 0$ ✓

FOR $X = 1/2$ ⇒ NOT POS REAL

c) $Z = \dfrac{s^3 + 2s^2 + 2s + 2}{s^4 + 4s^3 + 5s^2 + 3s + 4}$

$Z(\infty) = 0$ $Z(0) = \frac{1}{2}$

$Z(\infty) < Z(0)$

(RC ?)

P.R. or Not?

$Y = \dfrac{S^4 + 4S^3 + 5S^2 + 3S + 4}{S^3 + 2S^2 + 2S + 2}$ ⇒ POLE @ ∞

CAUER'S 1ST

9

Z | Y

$\frac{1}{2}$ | $S^3 + 2S^2 + 2S + 2$ | $S^4 + 4S^3 + 5S^2 + 3S + 4$ | S

$S^3 + \frac{3}{2}S^2 + \frac{1}{2}S + 2$ | $S^4 + 2S^3 + 2S^2 + 2S$

$\frac{1}{2}$ | $\frac{1}{2}S^2 + \frac{1}{2}S$ | $2S^3 + 3S^2 + S + 4$ | $4S$

$\frac{1}{2}S^2 + \frac{1}{2}S$ | $2S^3 + 2S^2$

0 | COMMON TERM $S^2 + S + 4$

$S^2 + S$?

~~$S + 1$~~

~~$S^2+S \overline{)S^3 + 2S^2 + 2S + 2}$~~

~~$S^3 + S^2$~~

~~$S^2 + 2S + 2$~~

~~S^2~~

~~$S^2 + 3S + 2$~~

~~$S^2+S \overline{)S^4 + 4S^3 + 5S^2 + 3S + 4}$~~

~~$S^4 + S^3$~~

~~$3S^3 + 5S^2 + 3S + 4$~~

~~$3S^3 + 3S^2 + 3S$~~

~~$2S^2 + 3S + 4$~~

4. Use Foster's First method to realize the impedance

$$Z(s) = \frac{s(s^2 + 3)}{(s^2 + 2)(s^2 + 4)} \Rightarrow LC$$

$$= \frac{s^3 + 3s}{s^4 + 6s^2 + 8} \Rightarrow \text{NO POLES @ 0 OR } \infty$$

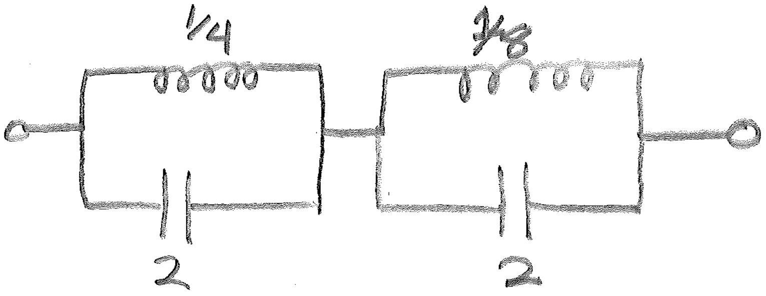
$$\frac{s(s^2+3)}{(s^2+2)(s^2+4)} = \frac{2As}{s^2+2} + \frac{2Bs}{s^2+4}$$

$$A = \left.\frac{s(s^2+3)}{(s+j\sqrt{2})(s^2+4)}\right|_{\substack{s^2=-2\\ s=j\sqrt{2}}} = \frac{\cancel{j\sqrt{2}}(1)}{j2\cancel{\sqrt{2}}(2)} = \frac{1}{4}$$

$$B = \left.\frac{s(s^2+3)}{(s^2+2)(s+j2)}\right|_{\substack{s^2=-4\\ s=j2}} = \frac{j2(-1)}{(-2)j4} = \frac{1}{4}$$

$$Z(s) = \frac{s}{2s^2+4} + \frac{s}{2s^2+8}$$

$$= \frac{1}{2s + 4/s} + \frac{1}{2s + 8/s}$$



10

BOB MARKS

EE 632 Network Systhesis I
Take-Home Exam

Nov. 2, 1972
Due Monday Noon, Nov. 13, 1972

For each of the problems:

a) Determine whether or not the function is positive real.

b) If the function is positive real, determine a network realization using the standard techniques such as:

1) the Foster preamble

2) recognition as RC, RL, or LC type of function

3) resistance (or conductance) minimization

4) Brune or Bott-Duffin procedure

c) Check your network by working backwards to determine the driving-point impedance or admittance of your network.

Your solutions must be submitted in log-book form with headings and statements or comments to enable your procedures and results to be easily understood.

100

The following statement must be signed:

I certify that I have neither given nor received aid from another person on this examination.

Robert J. Marks II
Signature

1. ✓ $Y(s) = \frac{s+2}{s^2+2s+4}$

2. ✓ $Y(s) = \frac{s^2+2}{2s^2+6s+4}$ — check this one carefully!

3. ✓ $Y(s) = \frac{s+2}{s^2+7s+12}$

4. ✓ $Y(s) = \frac{3s^2+7s}{s^4+5s^2+6}$

5. ✓ $Y(s) = \left(\frac{s+1}{s+2}\right)^2$

6. ✓ $Z(s) = \frac{s^4+4s^3+4s^2+4s+3}{s^4+6s^3+9s^2+6s+8}$

7. APG! $Z(s) = \frac{s^4+2s^3+4s^2+3s+1}{s^4+2s^3+3s^2+2s+2}$

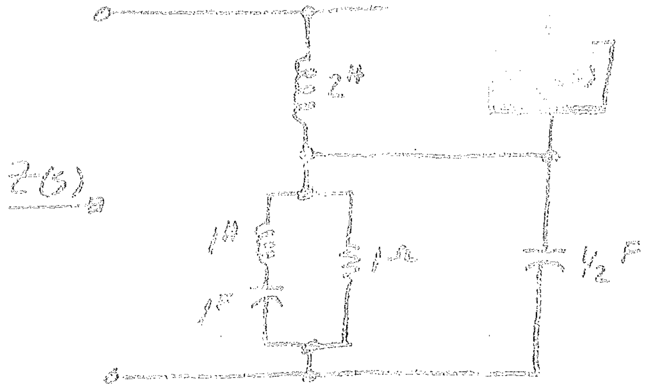
8. SHOW If positive real determine a Bott-Duffin network realiz-tion of the driving-point impedance function:

$$Z(s) = \frac{s^4+2s^3+3s^2+s+1}{s^4+s^3+3s^2+2s+1}$$

9. If positive real, realize the following driving-point impedance by the Brune procedure.

$$Z(s) = \frac{4s^4+3s^3+7s^2+4s+2}{s^4+s^3+3s^2+s+2}$$

10. ✓ A Bott-Duffin realization for a driv[illegible]t impedance is shown. Determine the realization for $Z_a(s)$



1) $Y(s) = \frac{s+2}{s^2+2s+4}$

a) POSITIVE REAL? $Y(s)$ IS REAL IF s IS REAL

ROUTH TEST ON NUM + DEN

NUM + DEN = s^2+3s+6

$$\begin{array}{lll} s^2 & 1 & 6 \\ s^1 & 3 & \\ s^0 & 6 & \end{array}$$

STURM'S TEST

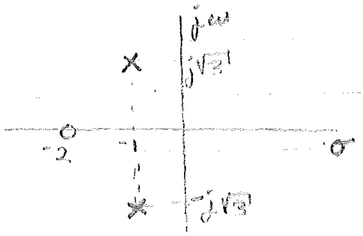
$$\begin{aligned} A(s^2) &= m_1 m_2 - n_1 n_2 \\ &= (2)(s^2+4)-(s)(2s) \\ &= 2s^2+8-2s^2 = 8 \end{aligned}$$

$8 = A(\omega^2) = A(s^2)\big|_{s=j\omega} \geq 0$ FOR ALL ω^2

$\Rightarrow Y(s)$ IS POS REAL

b) RC OR RL?

$$Y(s) = \frac{s+2}{s^2+2s+4} = \frac{s+2}{(s+1+j\sqrt{3})(s+1-j\sqrt{3})}$$



NOT RC OR RL

(DEFINITELY NOT LC)

c) FOSTER PREAMBLE (CONT.)

NO POLES @ $s=0$ OR $s=\infty$ FOR $Y(s)$

$Z(s) = \frac{s^2+2s+4}{s+2}$ $\Rightarrow$ POLE @ ∞

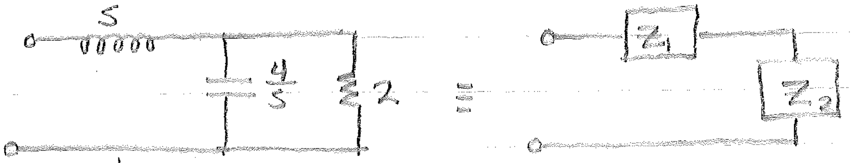
$$\begin{array}{r|l} & s \\ \hline s+2 & s^2+2s+4 \\ & \underline{s^2+2s} \\ & \qquad\qquad 4 \end{array}$$

$\Rightarrow Z(s) = s + \frac{4}{s+2}$

$Z(s) = s + \frac{4}{s+2}$; $\frac{4}{s+2}$ IS RC FUNCTION

$= s + \frac{1}{\frac{1}{4}s + \frac{1}{2}}$

c) CHECKING DRIVING POINT IMPEDANCE



$Z(s) = \frac{1}{Y(s)} = Z_1 + Z_2$

$Z_1 = s$

$Z_2 = \frac{8/s}{2 + 4/s} = \frac{8}{2s+4} = \frac{4}{s+2}$

$\Rightarrow Z(s) = s + \frac{4}{s+2}$

$= \frac{4 + s(s+2)}{s+2}$

$= \frac{4 + s^2 + 2s}{s+2}$

$\therefore Y(s) = \frac{1}{Z(s)} = \frac{s+2}{s^2+2s+4}$

2) $Y(s) = \frac{s^2+2}{2s^2+6s+4}$

a) POS. REAL?

1) NUM + DEN $= 3s^2+6s+6$

$$\begin{array}{lll} s^2 & 3 & 6 \\ s' & 6 & \\ s^0 & 6 & \end{array}$$

$\Rightarrow$ STRICTLY HURWITZ

2) $A(s^2) = m_1 m_2 - n_1 n_2$

$= (s^2+2)(2s^2+4) - (0)$

$= 2s^4+8s^2+8$

$A(\omega^2) = A(s^2)\big|_{s=j\omega} = 2\omega^4 - 8\omega^2 + 8$

$P_0(x) = A(\omega^2)\big|_{x=\omega^2} = 2x^2 - 8x + 8$

$= 2(x^2-4x+4)$

$= 2(x-2)^2 \geq 0 \;\forall\; x$

3) $Y(s)$ IS REAL WHEN s IS REAL

$\Rightarrow$ $Y(s)$ IS POSITIVE REAL

b) FOSTER PREAMBLE (CONT.)

$$Y(s) = \frac{s^2+2}{2s^2+6s+4}$$

$$Z(s) = \frac{1}{Y(s)} = \frac{2s^2+6s+4}{s^2+2}$$

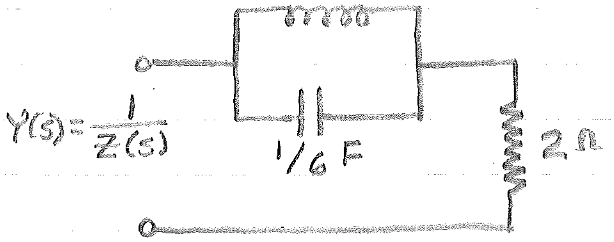
j-AXIS POLES!

LET $Z(s) = \frac{2s^2+6s+4}{s^2+2} = \frac{2As}{s^2+2} + Z_1(s)$

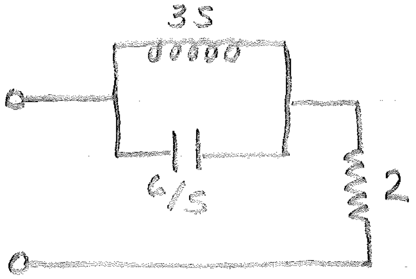
AND NOTE: $Z(s) = \frac{6s+2(s^2+2)}{s^2+2} = \frac{6s}{s^2+2} + \frac{2(s^2+2)}{s^2+2}$

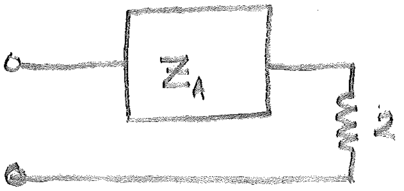
$$= \frac{6s}{s^2+2} + 2$$

$$= \frac{1}{s/6 + 1/3s} + 2$$



c) CHECKING DRIVING POINT IMPEDANCE





$$Z_a = \frac{1}{\frac{1}{3s} + s/6} = \frac{3s}{1+\frac{1}{2}s^2}$$

$$\Rightarrow Z = 2 + Z_a$$

$$= 2 + \frac{6s}{2+s^2}$$

$$= \frac{2s^2+6s+4}{s^2+2}$$

3) $Y(s) = \dfrac{s+2}{s^2+7s+12}$

a) POS REAL?

1) NUM + DEN $= s^2+8s+14$

s^2	$\underline{1}$	14
s^1	8	
s^0	14	

$\Rightarrow Y(s)$ IS STRICTLY HURWITZ

2) STURM

$$A(s^2) = m_1 m_2 - n_1 n_2$$
$$= 2(s^2+12) - s(7s)$$
$$= -5s^2 + 24$$

$$A(\omega^2) = 5\omega^2 + 24 > 0 \ \forall \ \omega^2$$

3) $Y(s)$ IS REAL FOR s REAL

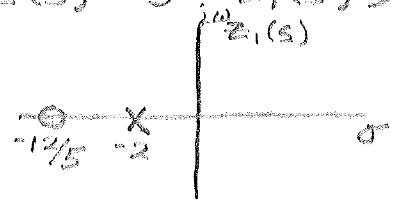
$\Rightarrow Y(s)$ IS POS REAL

b) FOSTER PREAMBLE (CONT.)

$Z(s) = \dfrac{1}{Y(s)} = \dfrac{s^2+7s+12}{s+2}$; POLE @ ∞

$$s+2 \,\big|\, s^2+7s+12 \,\big|\, s$$
$$\underline{s^2 + 2s}$$
$$5s+12$$

$Z(s) = s + Z_1(s) \Rightarrow Z_1(s) = \dfrac{5s+12}{s+2}$



$Z_1(s)$ IS AN RC. FUNCTION

1) POLES & ZEROS "INTERLACED" AND REAL

2) SINGULARITY NEARED $j\omega$ AXIS IS A POLE

USING FOSTER'S FIRST

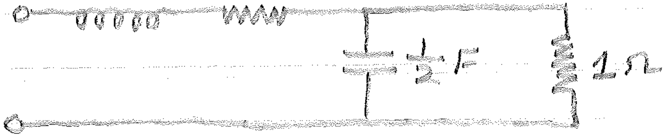
$Z_1(s) = \dfrac{5s+12}{s+2}$ $\Rightarrow$ NO POLES @ 0 OR ∞

$$s+2 \,\big|\, 5s+12 \,\big|\, 5$$
$$\underline{5s+10}$$
$$2$$

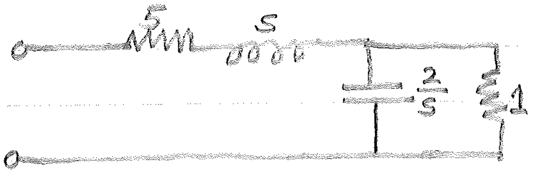
$$Z_1(s) = 5 + \frac{2}{s+2} = 5 + \frac{1}{s/2 + 1}$$

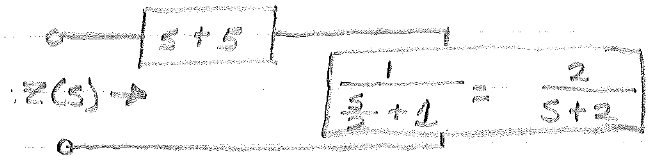
$$Z(S) = Z_1(S) + S$$

$$= S + 5 + \frac{1}{S/2 + 1}$$



c) CHECKING DRIVING POINT IMPEDENCE





$$Z(S) = S + 5 + \frac{2}{S+2}$$

$$= \frac{(S+5)(S+2)+2}{S+2}$$

$$= \frac{S^2 + 7S + 12}{S+2}$$

$$\Rightarrow Y(S) = \frac{S+2}{S^2 + 7S + 12}$$

4) $Y(s) = \dfrac{3s^2+7s}{s^4+5s^2+6}$ ⇒ DEN HAS MISSING ODD TERM AND NUM & DEN DIFFER BY POWER OF 2

⇒ NOT POS. REAL. SHOWN ANALYTICALLY:

1) STURM TEST:

$$A(s^2) = m_1 m_2 - n_1 n_2$$
$$= (3s^2)(s^4+5s^2+6) - (0)$$
$$= 3[s^6+5s^4+6s^2]$$

$$A(\omega^2) = A(s^2)\big|_{s=j\omega} = 3[-\omega^6+5\omega^4-6s^2]$$

$$P_0(x) = A(\omega^2)\big|_{\omega^2=x} = 3[-x^3+5x^2-6x]$$
$$= -3x[x^2-5x+6]$$
$$= -3x(x-2)(x-3)$$

$$P_0(4) = -12(2)(1) = -24 < 0$$

✓

⇒ Y(s) IS NOT POS. REAL

5) $Y(s) = \left(\frac{s+1}{s+2}\right)^2 = \frac{s^2+2s+1}{s^2+4s+4}$

a) POS REAL?

1) NUM + DEN = $2s^2+6s+5$

$$\begin{array}{lll} s^2 & 2 & 5 \\ s^1 & 6 & 0 \\ s^0 & 5 & \end{array}$$

$\Rightarrow$ STRICTLY HURWITZ

2) $A(s^2) = m_1 m_2 - n_1 n_2$

$= (s^2+1)(s^2+4) - 8s^2$

$= s^4 + 5s^2 + 4 - 8s^2$

$= s^4 - 3s^2 + 4$

$A(\omega^2) = A(s^2)\big|_{s=j\omega} = \omega^4 + 3\omega^2 + 4 > 0 \ \forall \ \omega^2$

3) $Y(s)$ IS REAL FOR s REAL

b) FOSTER PREAMBLE (CONT.)

NO $j\omega$ POLES; NO POLES @ 0 OR ∞

SO LETS TAKE OUT A RESISTOR; $Z = \frac{s^2+4s+4}{s^2+2s+1}$

$Re[Z(j\omega)] = Ev[Z(s)]_{s=j\omega}$

$$Ev[Z(s)] = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$= \frac{A(s^2)}{m_2^2 - n_2^2}$$

$$= \frac{A(s^2)}{(s^2+1)^2 - 4s^2}$$

$$= \frac{A(s^2)}{s^4 - 2s^2 + 1}$$

$$\Rightarrow Re[Z(j\omega)] = \frac{\omega^4+3\omega^2+4}{\omega^4+2\omega^2+1} = \frac{\omega^4+3\omega^2+4}{(\omega^2+1)^2}$$

FIND MINIMUM:

$$Re\left[Z(j\omega)\right]_{\omega^2=x} = \frac{x^2+3x+4}{(x+1)^2} = \frac{p}{q}$$

$$\frac{d\,Re\left[Z(j\omega)\right]}{dx} = \frac{qp'-q'p}{q^2} = \frac{(x+1)^2(2x+3)-2(x+1)(x^2+3x+4)}{(x+1)^2} = 0$$

$$\Rightarrow (x+1)\quad(2x+3)- \qquad 2(x^2+3x+4)=0 \qquad ; \; x \neq -1$$

$$(2x^2+5x+3)-(2x^2+6x+8)=0$$

$$+5x+5=0$$

MIN? @ $x=-1 \Rightarrow \omega^2=-1 \Rightarrow \omega=j$

$$Re\left[Z(j\omega)\right]_{\omega^2=-1} = \infty$$

$\Rightarrow \omega^2=-1$ GIVES $\underline{\underline{\text{MAX}}}$ $Re\left[Z(j\omega)\right]$

CHECK $Re\left[Z(j\omega)\right]$ @ 0 & ∞

$$Re\left[Z(j\omega)\right]_{\omega^2=x=0} = 4$$

$Re\left[Z(j\omega)\right]_{\omega^2=x=\infty} = 1 \Leftarrow$ MIN $Re\left[Z(j\omega)\right]$

SO LET'S TAKE OUT AN OHM

$$s^2+2s+1 \,\overline{\big)\, s^2+4s+4}^{\;1}$$

$$\underline{s^2+2s+1}$$

$$2s+3$$

LEAVING:

$$Z(s) = 1 + Z_1(s)$$

$$\ni Z_1(s) = \frac{2s+3}{s^2+2s+1}$$

$\frac{1}{Z_1(s)} = Y_1(s) = \frac{s^2+2s+1}{2s+3}$ HAS A POLE @ ∞

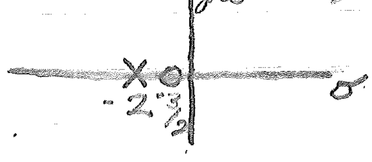
$$2s+3 \,\overline{\big)\, s^2+2s+1}^{\;\frac{1}{2}s}$$

$$\underline{s^2+\tfrac{3}{2}s}$$

$$\tfrac{1}{2}s+1$$

$$\Rightarrow Y_1(s) = \tfrac{1}{2}s + Y_2(s) \ni Y_2(s) = \frac{s+2}{4s+6}$$

$$Z_2(s) = \frac{1}{Y_2(s)} = \frac{4s+6}{s+2}$$



Z₂(s) IS AN RL FUNCTION

1) POLES & O'S INTERLACED AND REAL

2) NEAREST jω AXIS SINGULARITY, IS A POLE

USING FOSTER'S FIRST:

$$Z_2(s) = \frac{4s+6}{s+2} \quad ; \text{ NO } 0 \text{ OR } \infty \text{ POLES}$$

$$\begin{array}{r} 3 \\ 2+s\,\overline{)\,6+4s} \\ \underline{6+3s} \\ s \end{array}$$

$$\Rightarrow Z_2(s) = 3 + \frac{s}{2+s}$$

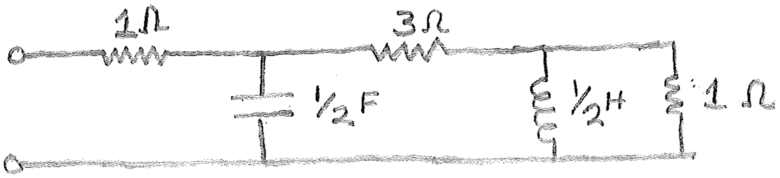
$$= 3 + \frac{1}{\frac{2}{s}+1}$$

PULLING EVERYTHING TOGETHER:

$$Z(s) = 1 + Z_1(s)$$

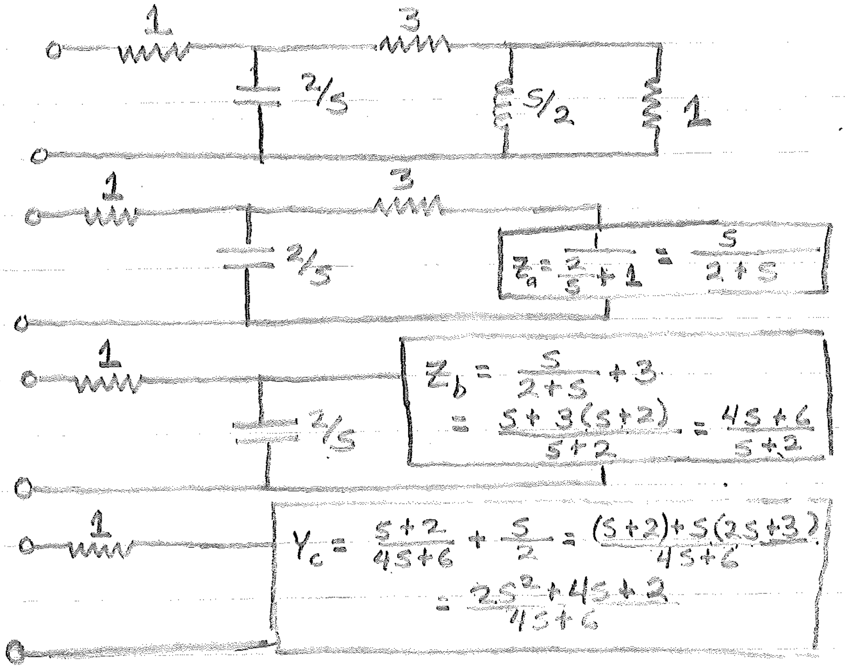
$$= 1 + \frac{1}{\frac{1}{2}s + Y_2(s)}$$

$$= \underset{z}{1} + \cfrac{1}{\underset{Y}{\tfrac{1}{2}s} + \cfrac{1}{\underset{z}{3} + \cfrac{1}{\underset{Y}{\tfrac{2}{s}+1}}}}$$



✓

c) CHECKING DRIVING POINT IMPEDENCE:



$$\boxed{Z_a = \frac{1}{\frac{2}{s}+1} = \frac{s}{2+s}}$$

$$\boxed{\begin{aligned} Z_b &= \frac{s}{2+s}+3 \\ &= \frac{s+3(s+2)}{s+2} = \frac{4s+6}{s+2} \end{aligned}}$$

$$\boxed{\begin{aligned} Y_c &= \frac{s+2}{4s+6}+\frac{s}{2} = \frac{(s+2)+s(2s+3)}{4s+6} \\ &= \frac{2s^2+4s+2}{4s+6} \end{aligned}}$$

$$Z(s) = 1 + \frac{4s+6}{2s^2+4s+2}$$

$$= \frac{4s+6+(2s^2+4s+2)}{2s^2+4s+2}$$

$$= \frac{2s^2+8s+8}{2s^2+4s+2}$$

$$= \frac{s^2+4s+4}{s^2+2s+1}$$

$$= \frac{(s+2)^2}{(s+1)^2}$$

$$\therefore Y(s) = \frac{1}{Z(s)} = \frac{(s+1)^2}{(s+2)^2}$$

6) $Z(s) = \dfrac{s^4 + 4s^3 + 4s^2 + 4s + 3}{s^4 + 6s^3 + 9s^2 + 6s + 8}$ (ARG!)

a) POS. REAL?

NUM + DEN = $2s^4 + 10s^3 + 13s^2 + 10s + 11$

s^4	2	13	11	
s^3	10	10		
s^2	11	11	$\Rightarrow s^2 + 1$	
s^1	0			
s^0				

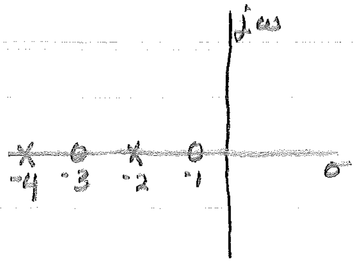
COMMON FACTOR OF $(s^2 + 1)$?

$$
\begin{array}{r l}
 & s^2 + 4s + 3 \\
s^2 + 1 \big) & s^4 + 4s^3 + 4s^2 + 4s + 3 \\
 & \underline{s^4 \qquad\quad + s^2} \\
 & 4s^3 + 3s^2 + 4s + 3 \\
 & \underline{4s^3 \qquad\quad + 4s} \\
 & 3s^2 \quad + 3 \\
 & \underline{3s^2 \quad + 3} \\
 & 0
\end{array}
$$

$$
\begin{array}{r l}
 & s^2 + 6s + 8 \\
s^2 + 1 \big) & s^4 + 6s^3 + 9s^2 + 6s + 8 \\
 & \underline{s^4 \qquad\quad + s^2} \\
 & 6s^3 + 8s^2 + 6s + 8 \\
 & \underline{6s^3 \qquad\quad + 6s} \\
 & 8s^2 \quad + 8 \\
 & \underline{8s^2 \quad + 8} \\
 & 0
\end{array}
$$

YEP!

$$\Rightarrow Z(s) = \frac{(s^2+1)(s^2+4s+3)}{(s^2+1)(s^2+6s+8)}$$
$$= \frac{s^2+4s+3}{s^2+6s+8}$$
$$= \frac{(s+3)(s+1)}{(s+2)(s+4)}$$



Z(s) IS AN RL FUNCTION

1) INTERLACED POLES & ZEROS ON NEGATIVE REAL AXIS

2) NEAREST SINGULARITY TO $j\omega$ AXIS IS A ZERO

b) NETWORK DETERMINATION

$$Z(s) = \frac{s^2+4s+3}{s^2+6s+8} \quad ; \text{ NO } 0 \text{ OR } \infty \text{ POLES}$$

FOSTER'S FIRST

$$\frac{Z(s)}{s} = \frac{s^2+4s+3}{s^3+6s^2+8s}$$

$$\begin{array}{r|l} & \frac{3}{8s} \\ \hline 8s+6s^2+s^3 & 3+4s+s^2 \\ & \underline{3+\frac{9}{4}s+\frac{3}{8}s^2} \\ & \frac{7}{4}s+\frac{5}{8}s^2 \end{array}$$

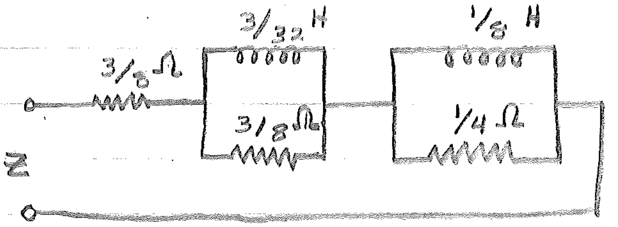
$$\Rightarrow \frac{Z(s)}{s} = \frac{3}{8s} + \frac{5s^2+14s}{8(s^3+6s^2+8s)}$$
$$= \frac{3}{8s} + \frac{5s+14}{8(s^2+6s+8)}$$
$$= \frac{3}{8s} + \frac{5s+14}{8(s+4)(s+2)}$$
$$= \frac{3}{8s} + \frac{A}{s+4} + \frac{B}{s+2}$$

$$A = \left.\frac{5s+14}{8(s+2)}\right|_{s=-4} = \frac{(-20)+14}{8(-2)} = \frac{+6}{16} = \frac{3}{8}$$
$$B = \left.\frac{5s+14}{8(s+4)}\right|_{s=-2} = \frac{-10+14}{8(2)} = \frac{4}{16} = \frac{1}{4}$$

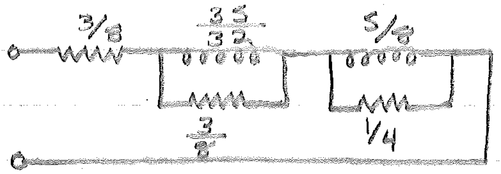
$$\Rightarrow \frac{Z(s)}{s} = \frac{3}{8s} + \frac{3}{8(s+4)} + \frac{1}{4(s+2)}$$

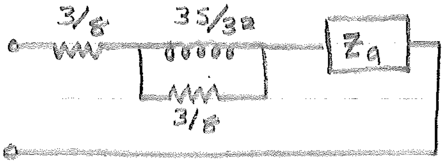
$$\therefore Z(s) = \frac{3}{8} + \frac{3}{8(1+4/s)} + \frac{1}{4(1+2/s)}$$

$$= \frac{3}{8} + \frac{1}{\frac{8}{3}+\frac{32}{3s}} + \frac{1}{4+8/s}$$

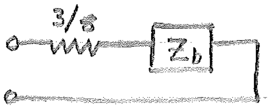


c) CHECKING DRIVING POINT IMPEDANCE





$$Z_a = \frac{1}{4 + 8/s} = \frac{s}{4s + 8}$$



$$Z_b = Z_a + \frac{1}{\frac{8}{3} + \frac{32}{3s}} = \frac{3s}{8s + 32} + \frac{s}{4s + 8}$$

$$= \frac{3}{8}\left(\frac{s}{s + 4}\right) + \frac{2}{8}\,\frac{s}{(s + 2)}$$

$$= \frac{2s(s + 4) + 3s(s + 2)}{8(s + 4)(s + 2)}$$

$$= \frac{(2s^2 + 8s) + (3s^2 + 6s)}{8(s + 4)(s + 2)}$$

$$= \frac{5s^2 + 14s}{8(s + 4)(s + 2)}$$

$$Z(s) = \frac{3}{8} + \frac{5s^2 + 14s}{8(s + 4)(s + 2)}$$

$$= \frac{3(s + 4)(s + 2) + (5s^2 + 14s)}{8(s + 4)(s + 2)}$$

$$= \frac{(3s^2 + 18s + 24) + (5s^2 + 14s)}{8(s + 4)(s + 2)}$$

$$= \frac{8s^2 + 32s + 24}{8(s + 4)(s + 2)}$$

$$= \frac{s^2 + 4s + 3}{(s + 4)(s + 2)}$$

$$\therefore\ Z(s) = \frac{(s + 3)(s + 1)}{(s + 4)(s + 2)}$$

THE ORIGINAL REDUCED EXPRESSION

7) $Z(s) = \dfrac{s^4+2s^3+4s^2+3s+1}{s^4+2s^3+3s^2+2s+2}$

a) POS REAL?

1) NUM + DEN $= 2s^4+4s^3+7s^2+5s+3$

$$\begin{array}{llll} s^4 & 2 & 7 & 3 \\ s^3 & 4 & 5 & \\ s^2 & 9/2 & 3 & \\ s^1 & 7/3 & & \\ s^0 & 3 & & \end{array}$$

$\Rightarrow$ STRICTLY HURWITZ

2) $A(s^2) = m_1 m_2 - n_1 n_2$

$= (s^4+4s^2+1)(s^4+3s^2+2) - (2s^3+3s)(2s^3+2s)$

$= s^8+3s^6+5s^4+5s^2+2$

$A(\omega^2) = A(s^2)\big|_{s=j\omega} = \omega^8 - 3\omega^6 + 5\omega^4 - 5\omega^2 + 2$

$P_0(x) = A(\omega^2)\big|_{x=\omega^2} = x^4 - 3x^3 + 5x^2 - 5x + 2$

$P_0(1) = 0$

$$\begin{array}{r|l} & x^3 - 2x^2 + 3x - 2 \\ \hline x-1 & x^4 - 3x^3 + 5x^2 - 5x + 2 \\ & \underline{-x^3 + 2x^2 - 3x + 2} \\ & -2x^3 + 3x^2 - 2x + 0 \end{array}$$

$P_0(x) = (x-1)(x^3 - 2x^2 + 3x - 2)$

ANOTHER ROOT @ ONE!

$$\begin{array}{r|l} & x^2 - x + 2 \\ \hline x-1 & x^3 - 2x^2 + 3x - 2 \\ & \underline{-x^2 + x - 2} \\ & -x^2 + 2x + 0 \end{array}$$

$\Rightarrow P_0(x) = (x-1)^2 (x^2 - x + 2)$

$(x-1)^2 \geq 0 \;\forall\; x \geq 0$

SO IT REMAINS ONLY TO SHOW $x^2 - x + 2 \geq 0 \,\forall x \geq 0$

$$P_0'(x) = x^2 - x + 2$$

$$P_1'(0) = \frac{d\,P_0'(x)}{dx} = 2x - 1$$

$$\begin{array}{r|l} & \frac{1}{2}x - \frac{1}{4} \\ \hline 2x-1 & x^2 - x + 2 \\ & \underline{x^2 - \frac{1}{2}x} \\ & \quad -\frac{1}{2}x + 2 \\ & \quad \underline{-\frac{1}{2}x + \frac{1}{4}} \\ & \qquad\quad 7/4 \end{array}$$

$$P_2(x) = -7/4$$

	P_0'	P_1	P_2	
0	+	−	−	1
∞	+	+	−	1

$\Rightarrow$ 0 REAL ZERO'S 'TWIXT 0 & ∞

3) $Z(S)$ IS REAL FOR REAL S

$\therefore$ $Z(S)$ IS POS. REAL

b) FOSTER PREAMBLE (CONT.)

$$Z(s) = \frac{s^4+2s^3+4s^2+3s+1}{s^4+2s^3+3s^2+2s+2}$$

NO POLES @ 0 OR ∞

j-AXIS POLES?

ROUTH ON DEN.

$$\begin{array}{llll} s^4 & 1 & 3 & 2 \\ s^3 & 2 & 2 & \\ s^2 & 2 & 2 & \leftarrow (s^2+1) \\ s^1 & \epsilon & & \\ s^0 & 2 & & \end{array}$$

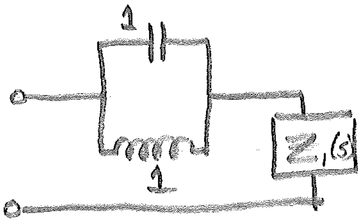
FACTORING DEN

$$\begin{array}{r} s^2+2s+2 \\ s^2+1 \overline{) \, s^4+2s^3+3s^2+2s+2} \\ \underline{s^4 \qquad +s^2 \qquad\qquad} \\ 2s^3+2s^2+2s+2 \\ \underline{2s^3 \qquad +2s \qquad} \\ 2s^2 + \qquad 2 \\ \underline{2s^2 + \qquad 2} \end{array}$$

$$Z(s) = \frac{s^4+2s^3+4s^2+3s+1}{(s^2+1)(s^2+2s+2)} = \frac{2As}{s^2+1} + Z_1(s)$$

$$A = \left.\frac{s^4+2s^3+4s^2+3s+1}{2s(s^2+2s+2)}\right|_{s=j} = \frac{1}{2}$$ (REALLY!)

$$\Rightarrow Z(s) = \frac{s}{s^2+1} + Z_1(s)$$



$$
\begin{aligned}
Z_1(s) &= Z(s) - \frac{s}{s^2+1} \\
&= \frac{s^4+2s^3+4s^2+2s+1}{(s^2+1)(s^2+2s+2)} - \frac{s}{s^2+1} \\
&= \frac{s^4+2s^3+4s^2+2s+1) - (s^3+2s^2+2s)}{(s^2+1)(s^2+2s+2)} \\
&= \frac{s^4+s^3+2s^2+s+1}{(s^2+1)(s^2+2s+2)}
\end{aligned}
$$

$$
\begin{array}{r l}
 & s^2+s+1 \\
\cline{2-2}
s^2+1 \, \big) & s^4+s^3+2s^2+s+1 \\
 & \underline{s^4 \qquad + s^2} \\
 & \qquad s^3+s^2+s \\
 & \qquad \underline{s^3 \qquad +s} \\
 & \qquad\qquad s^2 \quad +1 \\
 & \qquad\qquad \underline{s^2+1}
\end{array}
$$

$$\Rightarrow Z_1(s) = \frac{s^2+s+1}{s^2+2s+2}$$

NO 0, ∞, OR j AXIS POLES, SO TAKE OUT A RESISTOR,
IF $Z_1(s)$ ISN'T MINIMUM

SINCE MINIMIZING $Re[Z(j\omega)]$ BECAME HAIRY, LET'S LOOK @ $Re[Y(j\omega)]$

$$Y(s) = \frac{1}{Z(s)} = \frac{s^2+2s+2}{s^2+s+1}$$

$$Ev(Y(s)) = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$= \frac{(s^2+2)(s^2+1) - 2s^2}{(s^2+1)^2 - s^2}$$

$$= \frac{(s^4+3s^2+2) - 2s^2}{(s^4+2s^2+1) - s^2}$$

$$= \frac{s^4+s^2+2}{s^4+s^2+1}$$

$$\Rightarrow Re[Y(j\omega)] = Ev[Y(s)]\Big|_{s=j\omega} = \frac{\omega^4-\omega^2+2}{\omega^4-\omega^2+1}$$

$$Y(x) = Re[Y(j\omega)]\Big|_{\omega^2=x} = \frac{x^2-x+2}{x^2-x+1}$$

$$\frac{d}{dx}Y(x) = \frac{(x^2-x+1)(2x-1) - (x^2-x+2)(2x-1)}{[x^2-x+1]^2}$$

$$= \frac{(2x-1)[(x^2-x+1) - (x^2-x+2)]}{[x^2-x+1]^2}$$

$$= \frac{(2x-1)(-1)}{[x^2-x+1]^2} = 0 \Rightarrow x = \frac{1}{2}$$

$$Y(x)\big|_{x=1/2} = 7/3$$

$$Y(x)\big|_{x=0} = 2$$

$$Y(x)\big|_{x=\infty} = 1$$

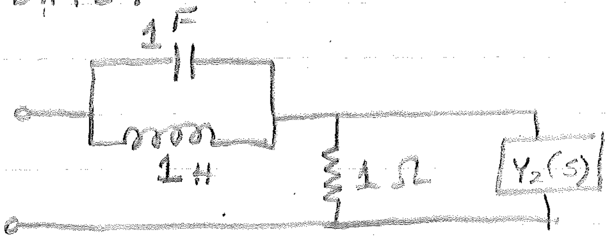
SO TAKE OUT A MHO

$$Y_1(s) = \frac{s^2+2s+2}{s^2+s+1}$$

$$\begin{array}{r l} & 1 \\ s^2+s+1 \,\big) & s^2+2s+2 \\ & \underline{s^2+s+1} \\ & \quad s+1 \end{array}$$

$$\Rightarrow Y_1(s) = 1 + Y_2(s) \quad \Rightarrow Y_2(s) = \frac{s+1}{s^2+s+1}$$

SO TO DATE:

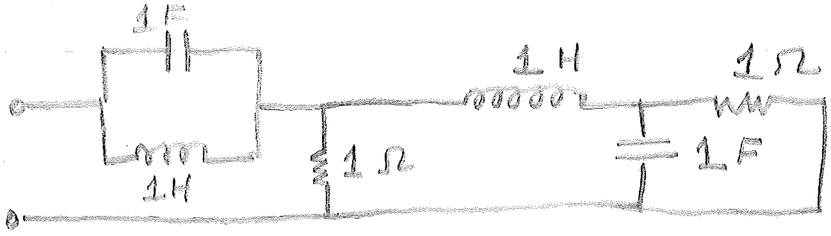


$$Z_2(s) = \frac{1}{Y_2(s)} = \frac{s^2+s+1}{s+1}$$ HAS A POLE @ ∞

$$\begin{array}{r l} & s \\ s+1 \,\big) & s^2+s+1 \\ & \underline{s^2+s} \\ & \qquad 1 \end{array}$$

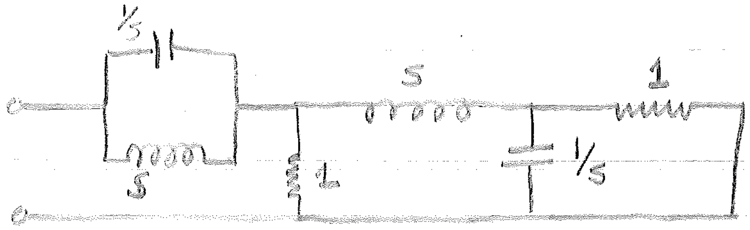
$$\Rightarrow Z_2(s) = s + \frac{1}{s+1} \quad \Rightarrow \frac{1}{s+1}$$ IS AN RC NETWORK

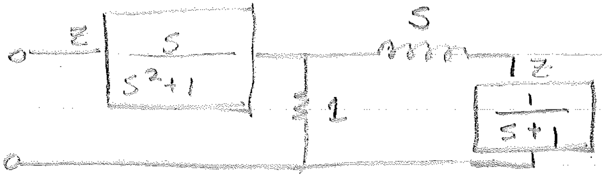
SO THE SYNTHESIZED NETWORK IS:

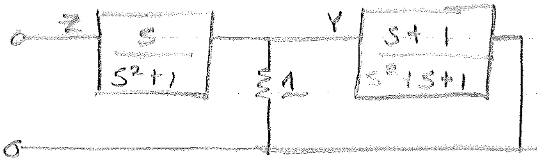


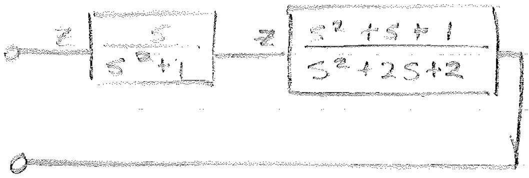
✓

c) CHECKING DRIVING POINT IMPEDANCE:









$$\Rightarrow Z(s) = \frac{s}{s^2+1} + \frac{s^2+s+1}{s^2+2s+2}$$

$$= \frac{(s^3+2s^2+2s)+(s^2+1)(s^2+s+1)}{(s^2+1)(s^2+2s+2)}$$

$$= \frac{(s^3+2s^2+2s)+(s^4+s^3+2s^2+s+1)}{s^4+2s^3+3s^2+2s+2}$$

$$= \frac{s^4+2s^3+4s^2+3s+1}{s^4+2s^3+3s^2+2s+2} \quad \checkmark$$

THE ORIGINAL EQUATION

8) $Z(s) = \dfrac{s^4 + 2s^3 + 3s^2 + s + 1}{s^4 + s^3 + 3s^2 + 2s + 1}$

a) POS. REAL?

1) NUM + DEN $= 2s^4 + 3s^3 + 6s^2 + 3s + 2$

$$\begin{array}{lccc} s^4 & 2 & 6 & 2 \\ s^3 & 3 & 3 & \\ s^2 & 4 & 2 & \\ s^1 & \frac{3}{2} & & \\ s^0 & 2 & & \end{array}$$

2) $A(s^2) = m_1 m_2 - n_1 n_2$

$$= (s^4 + 3s^2 + 1)(s^4 + 3s^2 + 1) - (2s^3 + s)(s^3 + 2s)$$

$$= (s^8 + 3s^6 + s^4) + (3s^6 + 9s^4 + 3s^2) + (s^4 + 3s^2 + 1)$$

$$- (2s^6 + 5s^4 + 2s^2)$$

$$= s^8 + 4s^6 + 6s^4 + 4s^2 + 1$$

$$A(\omega^2) = A(s^2)\Big|_{s=j\omega} = \omega^8 - 4\omega^6 + 6\omega^4 - 4\omega^2 + 1$$

$$P_0(x) = A(\omega^2)\Big|_{\omega^2 = x} = x^4 - 4x^3 + 6x^2 - 4x + 1$$

$$= (x-1)^4 \geq 0 \;\forall\; x$$

3) $Z(s)$ IS REAL FOR s REAL

$\therefore$ $Z(s)$ IS POS REAL

b) FOSTER PREAMBLE (CONT.)

1) j AXIS POLES ?

ROUTH ON NUM

s^4	1	3	1
s^3	2	1	
s^2	5/2	1	
s^1	1/5		
s^0	1		

ROUTH ON DEN

s^4	1	3	1
s^3	1	2	
s^2	1	1	
s^1	1		
s^0	1		

$\Rightarrow$ NO j AXIS POLES

2) NO 0 OR ∞ POLES EITHER

3) MINIMUM R ?

$$Re[Z(j\omega)] = Ev[Z(s)]_{s=j\omega} = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = A(\omega^2) \text{ FROM STURM TEST}$$

$$A(\omega^2) = (\omega^2 - 1)^4 = 0 \text{ FOR } \omega = 1$$

$$\Rightarrow Re[Z(j1)] = 0$$

$\therefore$ $Z(s)$ IS A MINIMUM FUNCTION AS GIVEN

c) BOTT-DUFFIN: $R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)}$

1) $Z(s) = \frac{R(s)\,kZ(k) + sZ(k)}{k + sR(s)}$

$$= \frac{1}{\frac{1}{R(s)Z(k)} + \frac{s}{kZ(k)}} + \frac{1}{\frac{k}{sZ(k)} + \frac{R(s)}{Z(k)}}$$

$$= \frac{1}{\frac{1}{Z_1(s)} + \frac{1}{Z_c}} + \frac{1}{\frac{1}{Z_L} + \frac{1}{Z_2(s)}}$$

$Z_1(s) = Z(k)\,R(s)$; $Z_L = \frac{sZ(k)}{k} \Rightarrow L = \frac{Z(k)}{k}$

$Z_c = \frac{k}{s}Z(k) \Rightarrow C = \frac{1}{kZ(k)}$; $Z_2(s) = \frac{Z(k)}{R(s)}$

2) $Z(s=j) = \frac{1 - j2 - 3 + j + 1}{1 - j - 3 + j2 + 1} = \frac{1+j}{1-j}\left(\frac{1+j}{1+j}\right) = \frac{j2}{2} = j = jX(s=j)$

$\Rightarrow X(s=j) = 1$

LET $R(s=j) = 0$

a) SOLVING FOR L & C

$\Rightarrow Z(s=j) = 1 = \frac{1}{\frac{1}{0} + \frac{s}{kZ(k)}\Big|_{j1}} + \frac{1}{\frac{k}{sZ(k)}\Big|_{j} + 0} = \frac{sZ(k)}{k}$

THUS: $1 = sL\,\Big|_{s=j1}$

$1 = jL \Rightarrow L = 1\ H$

$L = \frac{Z(k)}{k} = 1$

$\Rightarrow Z(k) - k = 0 = \frac{k^4 + 2k^3 + 3k^2 + k + 1}{k^4 + k^3 + 3k^2 + 2k + 1} - k$

$$0 = \frac{k^4 + 2k^3 + 3k^2 + k + 1 - (k^5 + k^4 + 3k^3 + 2k^3 + k)}{k^4 + k^2 + 3k^2 + 2k + 1}$$

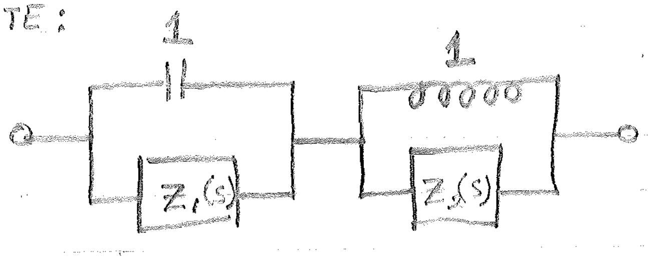
$$= -k^5 - k^3 + k^2 + 1$$

$\Rightarrow k = 1$

$Z(1) = \frac{1+2+3+1+1}{1+1+3+2+1} = \frac{8}{8} = 1$

$\therefore C = \frac{1}{kZ(k)} = 1\ F$

TO DATE:



$$R(s) = \frac{KZ(s) - sZ(K)}{KZ(K) - sZ(s)}$$

$$= \frac{Z(s) - s}{1 - sZ(s)} \qquad Z(s) = p/q$$

$$= \frac{p/q - s}{1 - sp/q}$$

$$= \frac{p - qs}{q - ps}$$

$$= \frac{(s^4 + 2s^3 + 3s^2 + s + 1) - (s^5 + s^4 + 3s^3 + 2s^2 + s)}{(s^4 + s^3 + 3s^2 + 2s + 1) - (s^5 + 2s^4 + 3s^3 + s^2 + s)}$$

$$= \frac{-s^5 - s^3 + s^2 + 1}{-s^5 - s^4 - 2s^3 + 2s^2 + s + 1}$$

FACTORING OUT AN $s-1$

$$\begin{array}{r}
-s^4 - s^3 - 2s^2 - s - 1 \\
s-1\,\overline{)\,-s^5 \qquad - s^3 + s^2 \qquad + 1} \\
\underline{s^4 + s^3 + 2s^2 + s + 1} \\
-s^4 - 2s^3 - s^2 - s
\end{array}$$

$$\begin{array}{r}
-s^4 - 2s^3 - 4s^2 - 2s - 1 \\
s-1\,\overline{)\,-s^5 - s^4 - 2s^3 + 2s^2 + s + 1} \\
\underline{s^4 + 2s^3 + 4s^2 + 2s + 1} \\
-2s^4 - 4s^3 - 2s^2 - s
\end{array}$$

$$\therefore R(s) = \frac{s^4 + s^3 + 2s^2 + s + 1}{s^4 + 2s^3 + 4s^2 + 2s + 1}$$

$$= \frac{s^4 + s^3 + 2s^2 + s + 1}{(s+1)^4}$$

(s^2+1) TERM MUST BE IN DENOM. DO TO THE EARLIER STIPULATION $R(s=j) = 0$

$$\begin{array}{r}
s^2 + s + 1 \\
s^2+1\,\overline{)\,s^4 + s^3 + 2s^2 + s + 1} \\
\underline{s^4 \qquad + s^2 \qquad\qquad} \\
s^3 + s^2 + s + 1 \\
\underline{s^3 \qquad + s \qquad} \\
s^2 \qquad + 1
\end{array}$$

THEN: $R(s) = \frac{(s^2+1)(s^2+s+1)}{(s+1)^4}$

LET'S GET OUT THE LC TANKS:

1) $Z_2(s) = \frac{Z(k)}{R(s)} = \frac{1}{R(s)} = \frac{(s+1)^4}{(s^2+1)(s^2+s+1)}$

$$\frac{(s+1)^4}{(s^2+1)(s^2+s+1)} = \frac{2As}{s^2+1} + Z_4(s)$$

$$A = \frac{(s+1)^4}{2s(s^2+s+1)}\Big|_{s=j} = \frac{(j+1)^4}{j2(j)} = \left(\frac{-2}{-2}\right) = 1$$

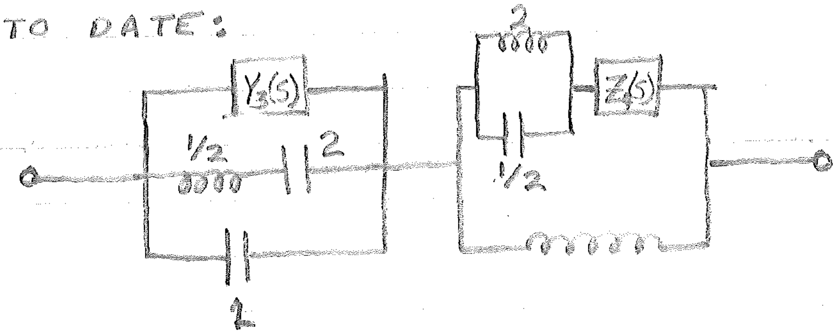
2) $Y_1(s) = \frac{1}{Z_1(s)} = \frac{1}{Z(k)R(s)} = \frac{1}{R(s)}$

$\therefore Y_1(s) = Z_2(s) = \frac{2s}{s^2+1} + Y_3(s)$

AND $Y_3(s) = Z_4(s)$

$$Y_1(s) = \frac{1}{s/2 + \frac{1}{2s}} \quad ; \quad Z_2(s) = \frac{1}{s/2 + \frac{1}{2s}}$$

SO, TO DATE:



AGAIN $Z_2(S) = \frac{2S}{S^2+1} + Z_4(S)$

$$\Rightarrow Z_4(S) = Z_2(S) - \frac{2S}{S^2+1}$$

$$= \frac{(S+1)^4}{(S^2+1)(S^2+S+1)} - \frac{2S}{S^2+1}$$

$$= \frac{(S^4+2S^3+4S^2+2S+1) - 2S(S^2+S+1)}{(S^2+1)(S^2+S+1)}$$

$$= \frac{(S^4+2S^3+4S^2+2S+1) - (2S^3+2S^2+2S)}{(S^2+1)(S^2+S+1)}$$

$$= \frac{S^4+2S^2+1}{(S^2+1)(S^2+S+1)}$$

$$= \frac{(S^2+1)^2}{(S^2+1)(S^2+S+1)}$$

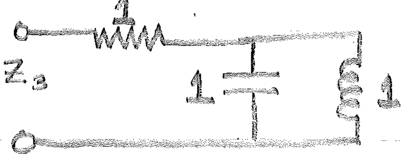
$$= \frac{(S^2+1)}{S^2+S+1}$$; (SIMILAR TO PROB. 2)

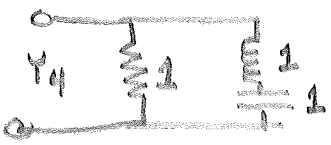
$$Y_4(S) = \frac{S^2+S+1}{S^2+1}$$

$$= \frac{S^2+1}{S^2+1} + \frac{S}{S^2+1}$$

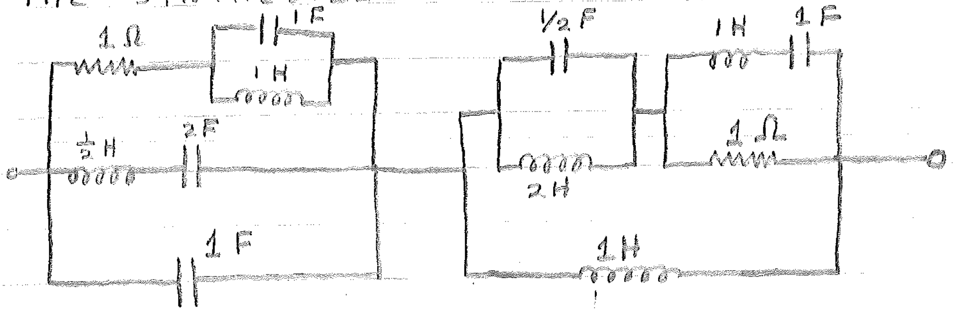
$$= 1 + \frac{1}{S + 1/S}$$

AND SINCE $Y_3 = Z_4$; $Z_3 = Y_4 = 1 + \frac{1}{S + 1/S}$



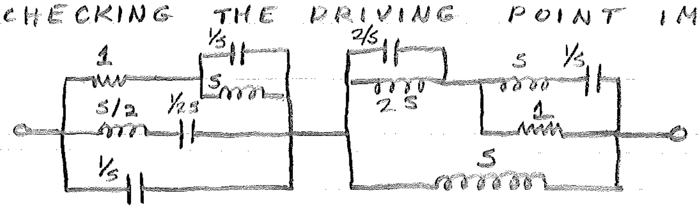


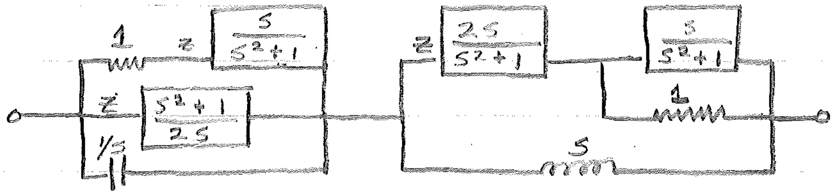
SO THE SYNTHESIZED NETWORK IS:

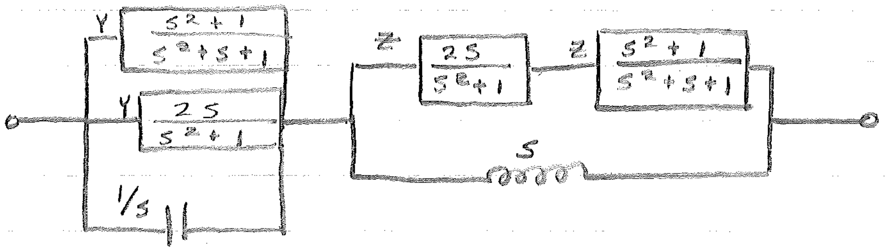


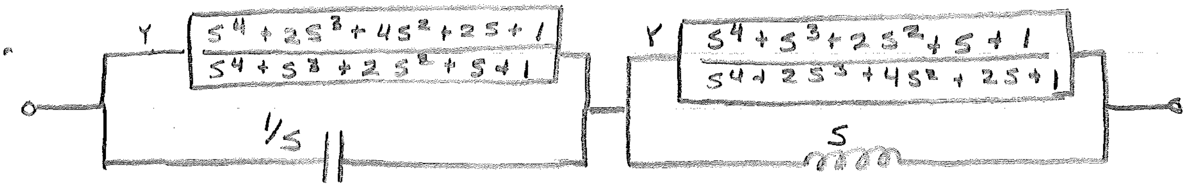
✓

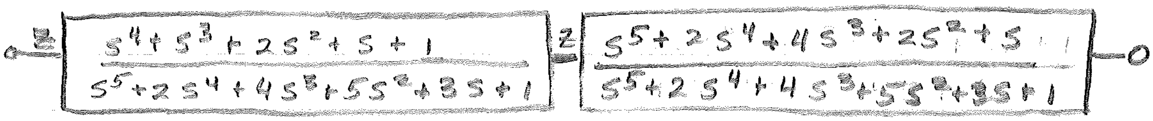
d) CHECKING THE DRIVING POINT IMPEDANCE

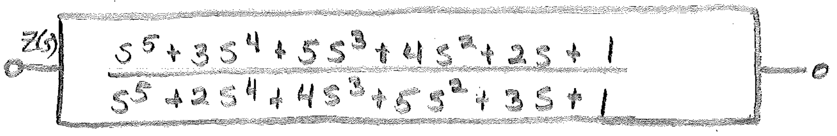












HOPEFULLY, THERE'S A COMMON FACTOR:

$$\begin{array}{rl} & s+1 \\ s^4+s^3+3s^2+2s+1 \big) & \overline{s^5+2s^4+4s^3+5s^2+3s+1} \\ & \underline{s^5+s^4+3s^3+2s^2+s} \\ & \quad s^4+s^3+3s^2+2s+1 \\ & \quad \underline{s^4+s^3+3s^2+2s+1} \end{array}$$

$$\begin{array}{rl} & s+1 \\ s^4+2s^3+3s^2+s+1 \big) & \overline{s^5+3s^4+5s^3+4s^2+2s+1} \\ & \underline{s^5+2s^4+3s^3+s^2+s} \\ & \quad s^4+2s^3+3s^2+s+1 \\ & \quad \underline{s^4+2s^3+3s^2+s+1} \end{array}$$

SO, FACTORING OUT AN $(s+1)$ FROM THE NUMERATOR & THE DENOMINATOR LEAVES:

$$Z(s) = \frac{s^4+2s^3+3s^2+s+1}{s^4+s^3+3s^2+s+1}$$, THE ORIGINAL EXPRESSION

9) $Z(s) = \dfrac{4s^4+3s^3+7s^2+4s+2}{s^4+s^3+3s^2+s+2}$

a) POS REAL?

1) NUM + DEN $= 5s^4+4s^3+10s^2+5s+4$

$$\begin{array}{llll} s^4 & 5 & 10 & 4 \\ s^3 & 4 & 5 & \\ s^2 & \frac{15}{4} & 4 & \\ s^1 & \frac{11}{5} & & \\ s^0 & 4 & & \end{array}$$

2) $A(s^2) = m_1 m_2 - n_1 n_2$

$= (4s^4+7s^2+2)(s^4+3s^2+2) - (3s^3+4s)(s^3+s)$

$= (4s^8+19s^6+31s^4+20s^2+4) - (3s^6+7s^4+4s^2)$

$= 4s^8+16s^6+24s^4+16s^2+4$

$A(\omega^2) = A(s^2)\big|_{s=j\omega} = 4\omega^8-16\omega^6+24\omega^4-16\omega^2+4$

$= 4(\omega^8-4\omega^6+6\omega^4-4\omega^2+1)$

$P_0(x) = A(\omega^2)\big|_{x=\omega^2} = 4(x^4-4x^3+6x^2-4x+1)$

$= 4(x-1)^4 \geq 0 \ \forall x$

3) $Z(s)$ IS REAL WHEN s IS REAL

$\Rightarrow Z(s)$ IS POSITIVE REAL

b) FOSTER PREAMBLE (CONT.)

1) NO POLES @ 0 OR ∞

2) j AXIS POLES

ROUTH ON DEN:

$$\begin{array}{llll} s^4 & 1 & 3 & 2 \\ s^3 & 1 & 1 & \\ s^2 & 2 & 2 & \\ s^1 & \epsilon & & \\ s^0 & 2 & & \end{array} \quad \Rightarrow (s^2+1) \text{ FACTOR!}$$

$$\begin{array}{r l}
 & s^2+s+2 \\
s^2+1 \,\big) & s^4+s^3+3s^2+s+2 \\
 & \underline{s^4 \qquad +s^2} \\
 & \quad s^3+2s^2+s+2 \\
 & \quad \underline{s^3 \qquad +s} \\
 & \qquad 2s^2 \;+2 \\
 & \qquad \underline{2s^2+2}
\end{array}$$

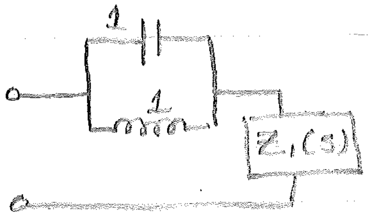
$$\Rightarrow Z(s) = \frac{4s^4+3s^3+7s^2+4s+2}{(s^2+1)(s^2+s+2)} = \frac{2As}{s^2+1} + Z_1(s)$$

$$A = \left.\frac{4s^4+3s^3+7s^2+4s+2}{2s(s^2+s+2)}\right|_{s=j}$$

$$= \frac{4-j3-7+j4+2}{j2(1+j)} = \frac{-(1-j)\;(1-j)}{j2(1+j)(1-j)} = \frac{j2}{j2\;(2)} = \frac{1}{2}$$

$$\Rightarrow Z(s) = \frac{s}{s^2+1} + Z_1(s)$$
$$= \frac{1}{s+1/s}$$

TO DATE:



$$Z_1(s) = Z(s) - \frac{s}{s^2+1}$$

$$= \frac{4s^4+3s^3+7s^2+4s+2}{(s^2+1)(s^2+s+2)} - \frac{s}{s^2+1}$$

$$= \frac{(4s^4+3s^3+7s^2+4s+2)-(s^3+s^2+2s)}{(s^2+1)(s^2+s+2)}$$

$$= \frac{4s^4+2s^3+6s^2+2s+2}{(s^2+1)(s^2+s+2)}$$

$$\begin{array}{r|l} & 4s^2+2s+2 \\ \hline s^2+1 & 4s^4+2s^3+6s^2+2s+2 \\ & \underline{4s^2 \qquad\quad +4s^2} \\ & 2s^3+2s^2+2s+2 \\ & \underline{2s^3 \qquad\quad +2s} \\ & 2s^2 \qquad +2 \end{array}$$

$$\Rightarrow Z_1(s) = \frac{2(2s^2+s+1)}{s^2+s+2} = \frac{4s^2+2s+2}{s^2+s+2}$$

FINDING MINIMUM RESISTANCE

FIRST, LOOK @ $m_1 m_2 - n_1 n_2$

$$= A(s^2) = (4s^2+2)(s^2+2)-(2s^2)$$

$$= (4s^4+10s^2+4)-2s^2$$

$$= 4s^4+8s^2+4$$

$$= 4(s^4+2s^2+1)$$

$$A(\omega^2) = 4(\omega^4-2\omega^2+1)$$

$$= 4(\omega^2-1)^2 = 0 \quad \text{FOR } \omega=1$$

$$\Rightarrow \min[\operatorname{Re} Z_1(j\omega)] = 0 \quad \text{FOR } \omega = 1$$

$Z_1(s)$ HAS NO 0 OR ∞ POLES,
AND NO j-AXIS POLES

$\therefore$ $Z_1(s)$ IS A MINIMUM FUNCTION,
READY TO BE BRUNIED

c) BRUNE'S METHOD

$$Z_1(s) = \frac{4s^2+2s+2}{s^2+s+2}$$

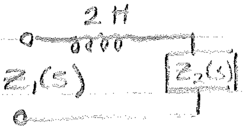
$$Re\left[Z_1(s=j)\right]=0$$

$$\Rightarrow Z_1(s=j) = jX = \frac{-4+j2+2}{-1+j+2} = \frac{-2(1-j)}{1+j}\cdot\frac{(1-j)}{(1-j)} = \frac{+2(+j2)}{2} = j2$$

$$\Rightarrow X = 2$$

$$L_1 = \frac{X}{\omega} = \frac{2}{1} = 2$$

$$\Rightarrow Z_1(s) = 2s + Z_2(s) \Rightarrow$$



$$Z_2(s) = Z_1(s) - 2s$$

$$= \frac{4s^2+2s+2}{s^2+s+2} - 2s$$

$$= \frac{(4s^2+2s+2)-(2s^3+2s^2+4s)}{s^2+s+2}$$

$$= \frac{-2s^3+2s^2-2s+2}{s^2+s+2}$$

$$\frac{1}{Z_2(s)} = Y_2(s) = \frac{s^2+s+2}{-2s^3+2s^2-2s+2}$$

$Y_2(s)$ SHOULD HAVE A POLE @ $s^2+\omega_1^2 = s^2+1$

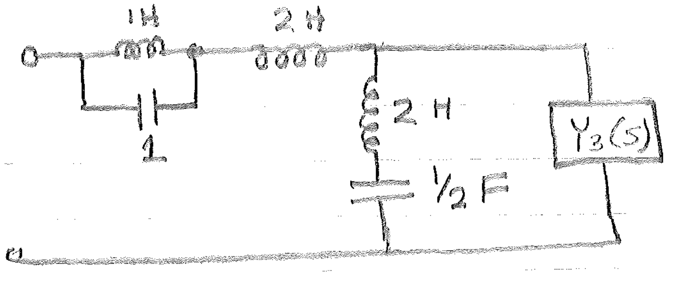
$$\begin{array}{r}
-2s+2 \\
s^2+1\,\big)\overline{-2s^3+2s^2-2s+2} \\
\underline{-2s^3\qquad\quad -2s\qquad} \\
2s^2\qquad\quad +2 \\
\underline{2s^2\qquad\quad +2}
\end{array}$$

$$\therefore Y_2(s) = \frac{s^2+s+2}{(-2s+2)(s^2+1)} = \frac{2As}{s^2+1} + Y_3(s)$$

$$A = \left.\frac{s^2+s+2}{2s(-2s+2)}\right|_{s=j} = \frac{-1+j+2}{j2(-j2+2)} = \frac{j+1}{j4(-j+1)} = \frac{j+1}{4(j+1)} = \frac{1}{4}$$

$$\Rightarrow \frac{2As}{s^2+1} = \frac{s}{2s^2+2} = \frac{1}{2s+2/s}$$

TO DATE:



$$Y_2(s) = \frac{s}{2s^2+2} + Y_3(s)$$

$$\Rightarrow Y_3(s) = Y_2(s) - \frac{s}{2s^2+2}$$

$$= \frac{s^2+s+2}{(-2s+2)(s^2+1)} - \frac{s}{2(s^2+1)}$$

$$= \frac{s^2+s+2}{2(-s+1)(s^2+1)} - \frac{s}{2(s^2+1)}$$

$$= \frac{(s^2+s+2) - s(-s+1)}{2(-s+1)(s^2+1)}$$

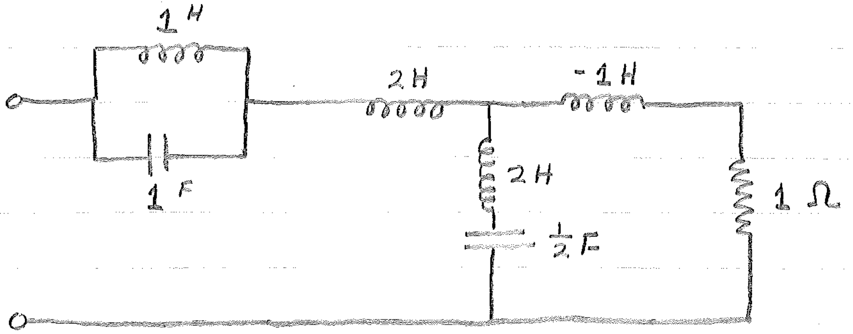
$$= \frac{s^2+s+2+s^2-s}{2(-s+1)(s^2+1)}$$

$$= \frac{2s^2+2}{2(-s+1)(s^2+1)}$$

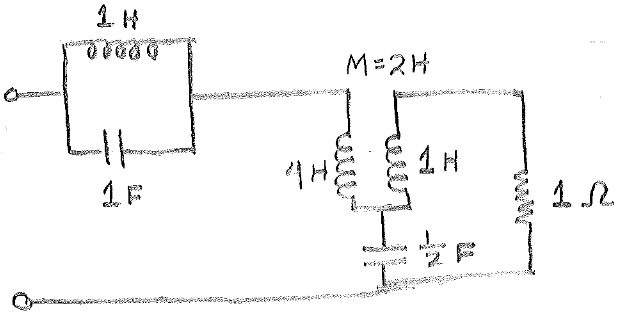
$$= \frac{2(s^2+1)}{2(-s+1)(s^2+1)}$$

$$= \frac{1}{-s+1}$$

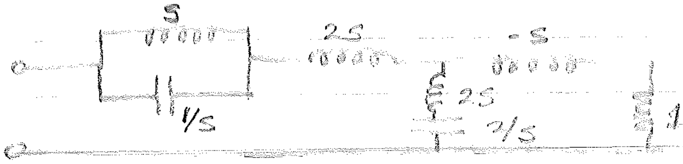
GIVING THE FINAL NETWORK:

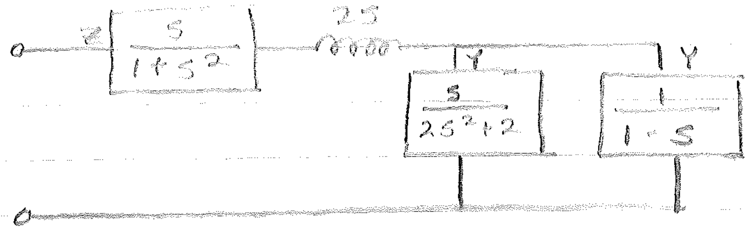


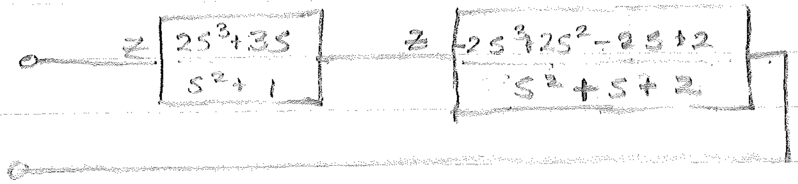
OR, EQUIVALENTLY



d) CHECKING RESULT:







$$Z(S) = \frac{2S^3+3S}{S^2+1} + \frac{-2S^3+2S^2-2S+2}{S^2+S+2}$$

$$= \frac{P_1}{q_1} + \frac{P_2}{q_2}$$

$$= \frac{P_1q_2 + P_2q_1}{q_1q_2}$$

1) P_1q_2

$$\begin{array}{rrrrr}
 & & S^2 & +S & +2 \\
 & 2S^3 & & +3S & \\
\hline
 & 3S^3 & 3S^2 & 6S & \\
2S^5 \quad 2S^4 & 4S^3 & & & \\
\hline
2S^5+2S^4 & +7S^3 & +3S^2 & +6S &
\end{array}$$

2) $p_2 q_1$

$$\begin{array}{r} -2s^3+2s^2-2s+2 \\ \underline{s^2 \qquad\quad +1} \\ -2s^3+2s^2-2s+2 \\ \underline{-2s^5\ 2s^4-2s^3+2s^2 \qquad\qquad} \\ -2s^5+2s^4-4s^3+4s^2-2s+2 \end{array}$$

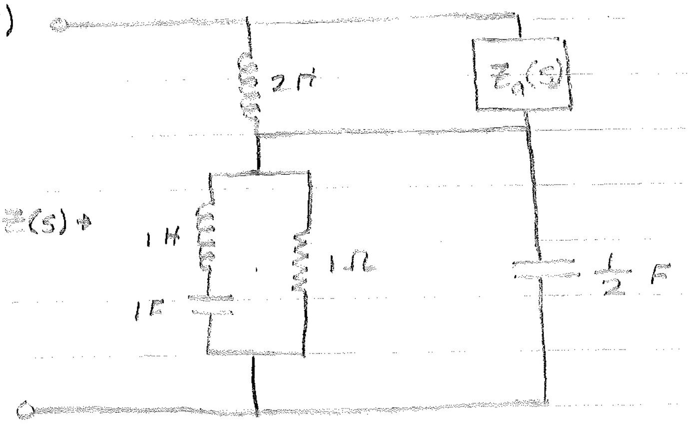
$$\Rightarrow p_1 q_2 + q_2 p_1 = 4s^4+3s^3+7s^2+4s+2$$

3) $q_1 q_2$

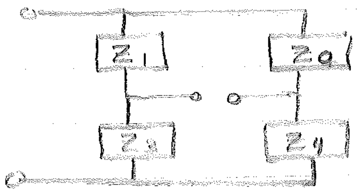
$$\begin{array}{r} s^2+s+2 \\ \underline{s^2 \quad +1} \\ s^2+s+2 \\ \underline{s^4 \quad s^3 \quad 2s^2 \qquad\qquad} \\ s^4+s^3+3s^2+s+2 \end{array}$$

$$\Rightarrow Z(s) = \frac{4s^4+3s^3+7s^2+4s+2}{s^4+s^3+3s^2+s+2}$$, THE ORIGINAL EXPRESSION

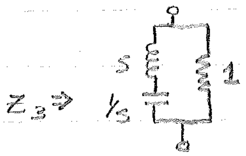
10) a)



THE BOTT DUFFIN NETWORK IS A BALANCED BRIDGE (SHOWN ON PG 180 OF TEXT)



THUS $Z_3 Z_a = Z_1 Z_4$



$$Z_3 = \frac{1}{1 + \frac{1}{s + 1/s}}$$

$$= \frac{1}{1 + \frac{s}{s^2+1}}$$

$$= \frac{s^2+1}{s^2+s+1}$$

$Z_4 = \frac{2}{s}$

$Z_1 = 2s$

$$\Rightarrow Z_a(s) = \frac{Z_1 Z_4}{Z_3}$$

$$= \frac{4s^2+4s+4}{s^2+1}$$

b) $Z_a(s) = 4\left[\frac{s^2+s+1}{s^2+1}\right]$

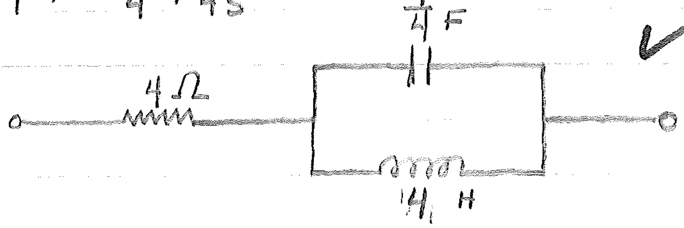
SIMILAR TO PROBLEM 2:

$\therefore Z_a(s) = 4\left[\frac{s^2+1}{s^2+1} + \frac{s}{s^2+1}\right]$

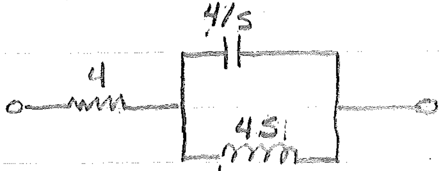
$= 4 + \frac{4s}{s^2+1}$

$= 4 + \frac{1}{\frac{s}{4} + \frac{1}{4s}}$

$Z_a(s)$:



c) GOING BACKWARDS:



$Z_a(s) = 4 + \frac{1}{\frac{s}{4} + \frac{1}{4s}}$

$= 4 + \frac{4s}{s^2+1}$

$= \frac{4s^2+4s+4}{s^2+1}$

$= 4\left[\frac{s^2+s+1}{s^2+1}\right]$

$Z_a(s)\,Z_3(s) = 4\left[\frac{s^2+s+1}{s^2+1}\right]\left[\frac{s^2+1}{s^2+s+1}\right] = 4 = Z_1 Z_4$

BOB MARKS

(56)

EE 532 Network Synthesis I
Final Examination

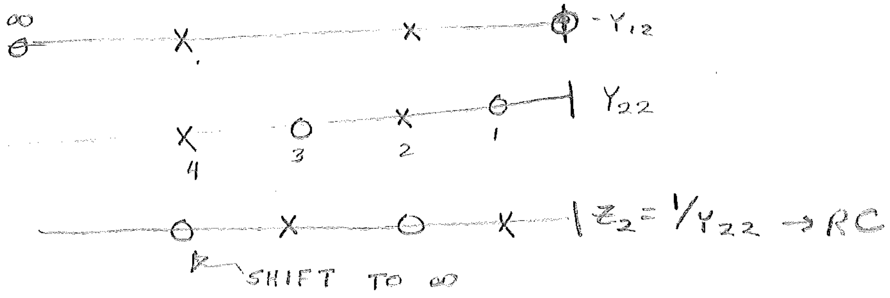
Four Hours

Nov. 21, 1972
Open Books & Notes

39/70

1. Determine an RC ladder network to realize the short-circuit admittance function specifications:

$$y_{22} = \frac{(s+1)(s+3)}{(s+2)(s+4)} \qquad -y_{12} = k\frac{s}{(s+2)(s+4)}$$

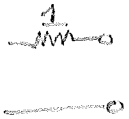


$$Z_2 = \frac{(s+2)(s+4)}{(s+1)(s+3)} \Rightarrow Z_2(\infty) = 1$$

TAKE OUT AN OHM

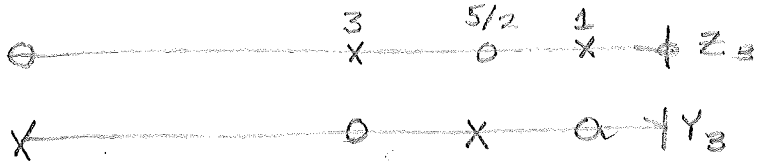
$$= \frac{s^2+6s+8}{s^2+4s+3}$$

$$\begin{array}{r|l} & 1 \\ s^2+4s+3 & s^2+6s+8 \\ & s^2+4s+3 \\ \hline & 2s+5 \end{array}$$



$$Z_3(s) = \frac{2s+5)}{s^2+4s+3} = \frac{(2\ s+5)}{(s+3)(s+1)}$$

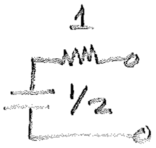
7



TAKE OUT POLE @ ∞

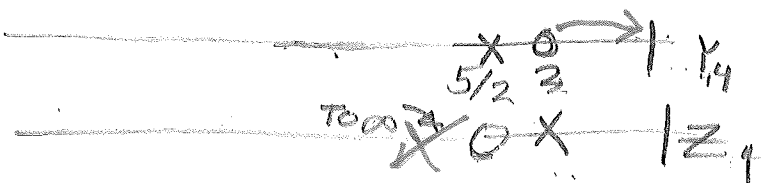
$$Y_3(s) = \frac{s^2+4s+3}{2s+5}$$

$$\begin{array}{r|l} & \frac{1}{2}s \\ 2s+5 & s^2+4s+3 \\ & s^2+\frac{5}{2}s \\ \hline & 3/2\,s+3 \end{array}$$



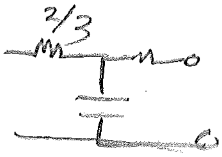
$$Y_4(s) = \frac{3/2\,s+3}{2s+5} = \frac{1}{2}\frac{3s+6}{2s+5}$$

3
5/2



$Z_4(s) = \frac{2s+5}{3s+6} \Rightarrow Z_4(\infty) = \frac{2}{3}$

$$\begin{array}{r|l} & \frac{2}{3} \\ 3s+6 & 2s+5 \\ & \underline{2s+4} \\ & \quad 4 \end{array}$$



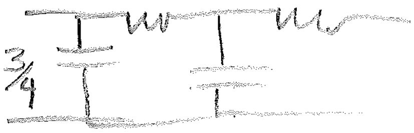
$Z_5 = \frac{4}{3s+6}$

$\overset{0}{\infty}$ ——— $\underset{2}{x}$ | Z_4

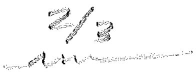
X ← TAKE OUT ——— 0 | Y_4

$Y_4 = \frac{3s+6}{4}$

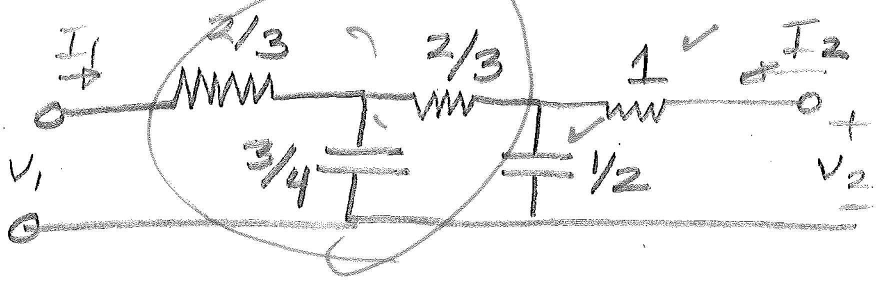
$$\begin{array}{r|l} & \frac{3}{4}s \\ 4 & 3s+6 \\ & \underline{3s -} \\ & \quad 6 \end{array}$$



$Y_5 = \frac{6}{4} \Rightarrow Z_5 = \frac{4}{6} = \frac{2}{3}$



GIVING



2. Determine an unloaded, symmetrical, RC lattice network to realize the short-circuit admittance function specifications:

$$y_{22} = \frac{(s+1)\,(s+3)}{(s+2)\,(s+4)} \qquad -y_{12} = k\frac{s}{(s+2)\,(s+4)}$$

where k is selected to be as large as possible.

$$Y_A = Y_{22} - Y_{12}$$

$$= \frac{(s+1)(s+3) + ks}{(s+2)(s+4)}$$

$$= \frac{s^2 + 4s + 3 + ks}{(s+2)(s+4)}$$

$$= \frac{s^2 + (4+k)s + 3}{(s+2)(s+4)}$$

$$= \frac{s^2 + (4+k)s + 3}{s^2 + 6s + 8}$$

Text method is easier. $\frac{y_{22}}{s}$ must have pos. residues!

$$RT = \frac{-(4-k) \pm \sqrt{(4-k)^2 - 12}}{2}$$

ROUTH ON NUM:

$$\begin{array}{lll} s^2 & 1 & 3 \\ s^1 & k+4 & \\ s^0 & 3 & \end{array} \Rightarrow k > {}^{-}4$$

FOR Y_A TO BE RC
k MUST YIELD ~~RTS.~~
ZEROS INTERLACING
THE POLES. SAME
FOR Y_b

$$Y_B = Y_{11} + Y_{12}$$

$$= \frac{(s+1)(s+3) - ks}{(s+2)(s+4)}$$

$$= \frac{s^2 + (4-k)s + 3}{s^2 + 6s + 8} \Rightarrow k \geq 4$$

$$-\sqrt{12} < 4 - k < \sqrt{12}$$

$$-\sqrt{12} > k - 4 > \sqrt{12}$$

$$4 - \sqrt{12} < k$$

zero must be nearest the origin also!

$$Z_B = \frac{s^2 + 6s + 8}{s^2 + (4-k)s + 3}$$

TO AUX SHEET:

Y Z

$$\frac{5}{(k+2)} \quad s^2 + (4-k)s + 3 \,\big)\, s^2 + 6s + 8 \quad 1$$

$$\frac{5s}{k+2} \qquad s^2 + (4-k)s + 3$$

$$\left[(4-k) - \frac{5}{k+2}\right]s + 3 \qquad (k+2)s + 5$$

$$\frac{k+2}{\left[(4-k) - \frac{5}{k+2}\right]}$$

$$\frac{3(k+2)}{(4-k) - \frac{5}{k+2}}$$

$k > -2$

3

2) (CONT)

$$Y_{22} = \frac{s^2+4s+3}{s^2+6s+8} \qquad -Y_{12} = \frac{ks}{s^2+6s+8}$$

~~TO~~ ~~$Y_4 + Y_{12} = \frac{s^2+(4+k)s+3}{s^2+6s+8}$~~

K MAX = 4 TO SATISFY COEFF CONDITIONS

SO $Y_a = \frac{s^2+8s+3}{(s+2)(s+4)} = \frac{s^2+8s+3}{s^2+6s+8}$

$$\frac{-8 \pm \sqrt{64 - 12}}{2} \qquad \frac{-8\pm\sqrt{52}}{2}$$

$$Z_a = \frac{s^2+6s+8}{s^2+8s+3}$$

$$\frac{-6\pm\sqrt{4}}{2} \qquad -3 \pm 1$$

~~STURM~~

$(s^2+8)(s^2+3) - 48s^2$

$s^4 + 11s^2 + 24 - 48s^2$

$s^4 - 37s^2 + 24$

$\omega^4 + 37\omega^2 + 24$

$(s^2+3)^2 - 64s^2$

$s^4 + 6s^2 - 64s^2 + 9$

$\omega^4 + 58\omega^2 + 9$

$$Re[Z_a] = \frac{\omega^4+37\omega^2+24}{\omega^4+58\omega^2+9}$$

$$= \frac{X^2+37X+24}{X^2+58X+9}$$

$$0 = \frac{dRe[Z_a]}{dX} = (X^2+58X+9)(2X+37) - (X^2+37X+24)(2X+58)$$

$$\Rightarrow 2X^3 + 37X^2 + 116X^2 + (58)(37)X + (9)(37)$$

$$Z_a = \frac{(s+8-\sqrt{52})(s+8+\sqrt{52})}{(s+2)\quad(s+4)}$$

↑

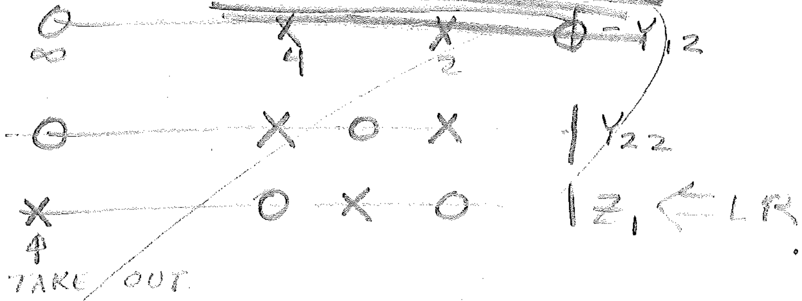
NEXT STEP IS TO
SYNTHESIZE

(OVER)

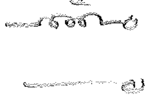
3. ✓ Determine an unloaded symmetrical RC lattice network to realize the short-circuit admittance function specifications with as few elements as possible: TO AUX SHEET

$$-y_{12} = k \frac{s}{(s+2)(s+4)}$$ ~~; $Y_{22} = \frac{s(s+3)}{(s+2)(s+4)}$ ← RC~~

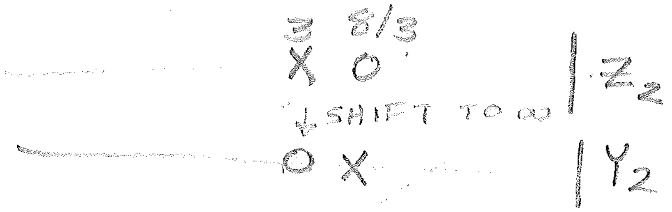
~~TO AUX SHEET~~



$$\begin{array}{r} s \\ s+3 \overline{) s^2+6s+8} \\ s^2+3s \\ \hline 3s+8 \end{array}$$

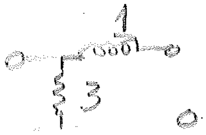


$$Z_2 = \frac{3s+8}{s+3}$$



5

$$Y_2(s) = \frac{s+3}{3s+8} \Rightarrow Y_2(\infty) = \frac{1}{3}$$

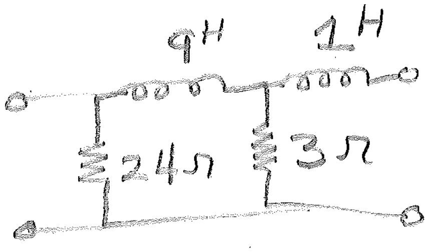


$$\begin{array}{r} 1/3 \\ 3s+8 \overline{) s+3} \\ s+8/3 \\ \hline 1/3 \end{array}$$

$$Y_3(s) = \frac{1}{3(3s+8)}$$

$$Z_3(s) = 9s+24$$

$$Z_4(s) = 24$$



③ $Y_{12} = \frac{1}{2}(Y_B - Y_A)$

$$Y_B - Y_A = 2Y_{12} = \frac{2ks}{(s+2)(s+4)} = \frac{R_1}{s+2} + \frac{R_2}{s+4}$$

$$R_1 = \frac{2ks}{s+4}\Big|_{s=-2} = \frac{-4k}{2} = -2k$$

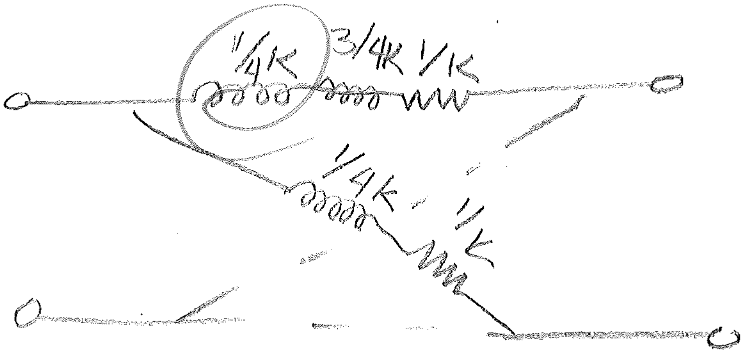
$$R_2 = \frac{2ks}{s+2}\Big|_{s=-4} = \frac{-8k}{-2} = 4k$$

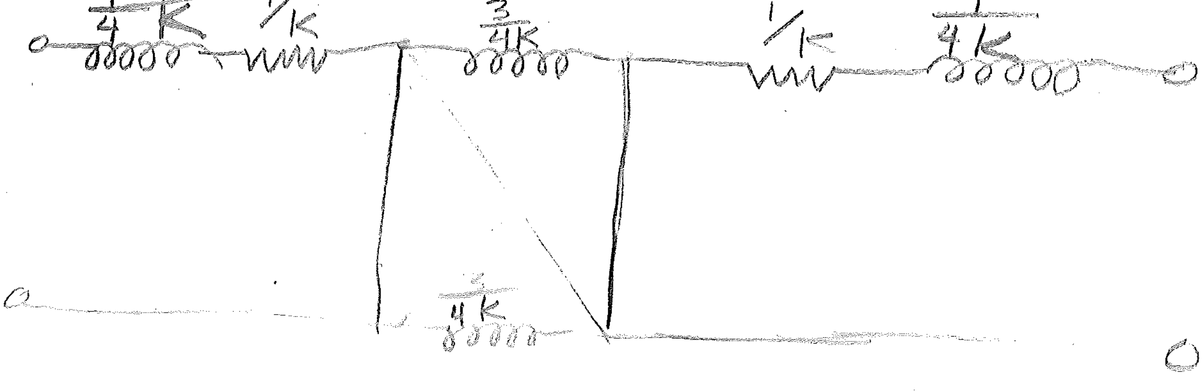
SO LET:

$$Y_A = \frac{+2k}{s+2} = \frac{1}{\frac{s}{k} + \frac{1}{k}} \Rightarrow$$

$$Y_B = \frac{4k}{s+4} = \frac{1}{\frac{s}{4k} + \frac{1}{k}}$$

$$Z_a = \frac{s}{k} + \frac{1}{k} \; ; \; Z_b = \frac{s}{4k} + \frac{1}{k}$$





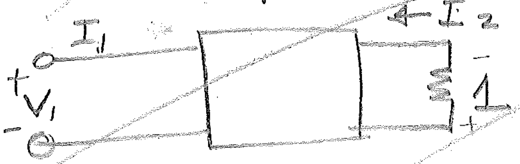
ETC

✓ 4. Given the transfer admittance function $-Y_{12}(s)$ for $Y_L = 1$ mho, determine the equations for the Z_a and Z_b arms of the symmetrical constant-resistance lattice which can be used to realize this transfer admittance. (TO AUX SHEET)

~~$Z_a = \dfrac{R - Z_{12}}{1 + Z_{12}/R} \ni R_L = \dfrac{1}{Y_L} = 1$~~

~~$\Rightarrow Z_a = \dfrac{1 - Z_{12}}{1 + Z_{12}}$~~

$-Y_{12} = \dfrac{I_2}{V_1}$



$Z_{12} = \dfrac{V_2}{I_1}$ $\Rightarrow$

$2Z_a + 1 = \dfrac{1}{Y_{12}}$

$2Z_a = \left(-1 + \dfrac{1}{Y_{12}}\right)$

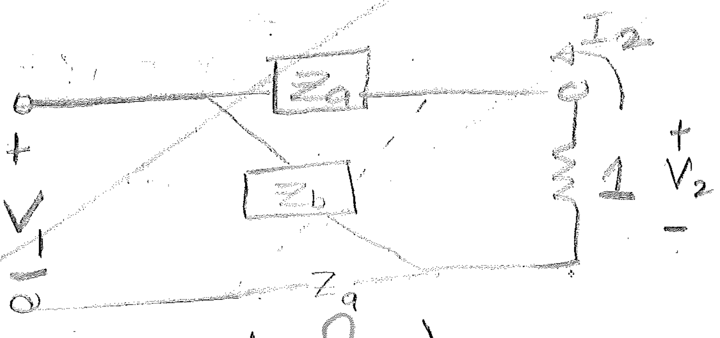
$Z_a = +\dfrac{1}{2}\left(-1 + \dfrac{1}{Y_{12}}\right)$

$Z_b = 1/Z_a$

$-Y_{12} = \dfrac{I_2}{V_1} = \dfrac{-V_2}{V_1} = -G_{12}; \; I_2 = -V_2$

$+Y_{12} = +G_{12}$ ✓ FOR $Z_L = 1$

$Y_{12} = G_{12} = \dfrac{-Y_{12}}{Y_{22} + 1}$



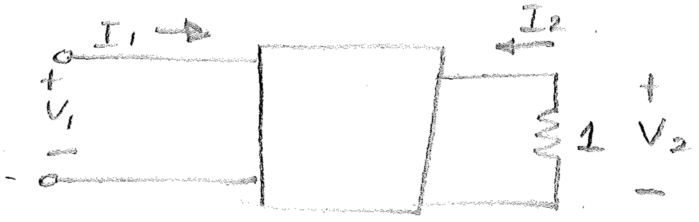
2

$V_2 = \left(\dfrac{1}{2Z_a + 1}\right) V_1$?

$\Rightarrow \dfrac{V_2}{V_1} = \dfrac{1}{2Z_a + 1} = \dfrac{-I_2}{V_1} = Y_{12}$

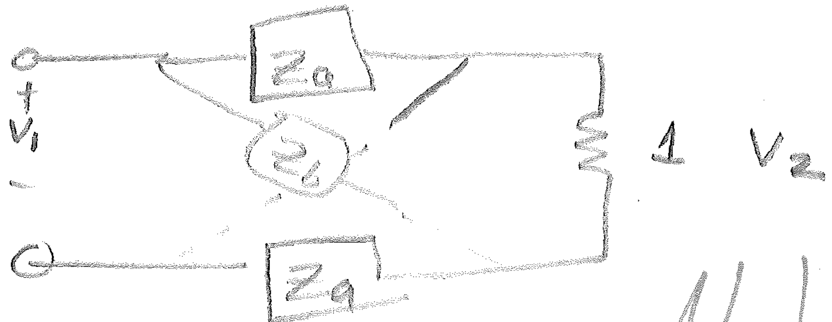
$\Rightarrow +Y_{12} = \dfrac{1}{2Z_a + 1}$

④



$$\mp Y_{12} = \frac{I_2}{V_1}$$

$$= \frac{-V_2/R_L}{V_1} = \frac{-V_2}{V_1}$$



No! where from?

$$\boxed{V_2 = \frac{1}{1+2Z_a} V_1}$$

$$\mp Y_{12} = -\frac{V_2}{V_1} = \frac{1}{1+2Z_a} \Rightarrow 1+2Z_a = \frac{\mp 1}{Y_{12}}$$

$$\mp 2Z_a = \frac{1}{Y_{12}} - 1$$

$$\cancel{Z_a = \mp\frac{1}{2}\left(\frac{1}{Y_{12}} - 1\right)}$$

$$\cancel{Z_b = 1/Z_a}$$

✓ 5. Use the results of problem 4 to obtain a symmetrical constant-resistance lattice realization of the transfer admittance function

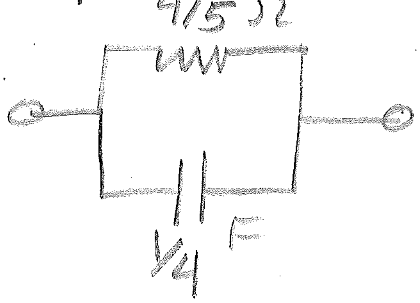
$$-Y_{12} = \frac{2}{s+3} \qquad \text{for } Z_L = R = 1 \text{ Ohm.}$$

$$Z_a = \frac{1}{2}\left(\left[\frac{s+3}{-2} - 1\right]\right)$$

$$= -\frac{1}{2}\left[\frac{s+5}{-2}\right]$$

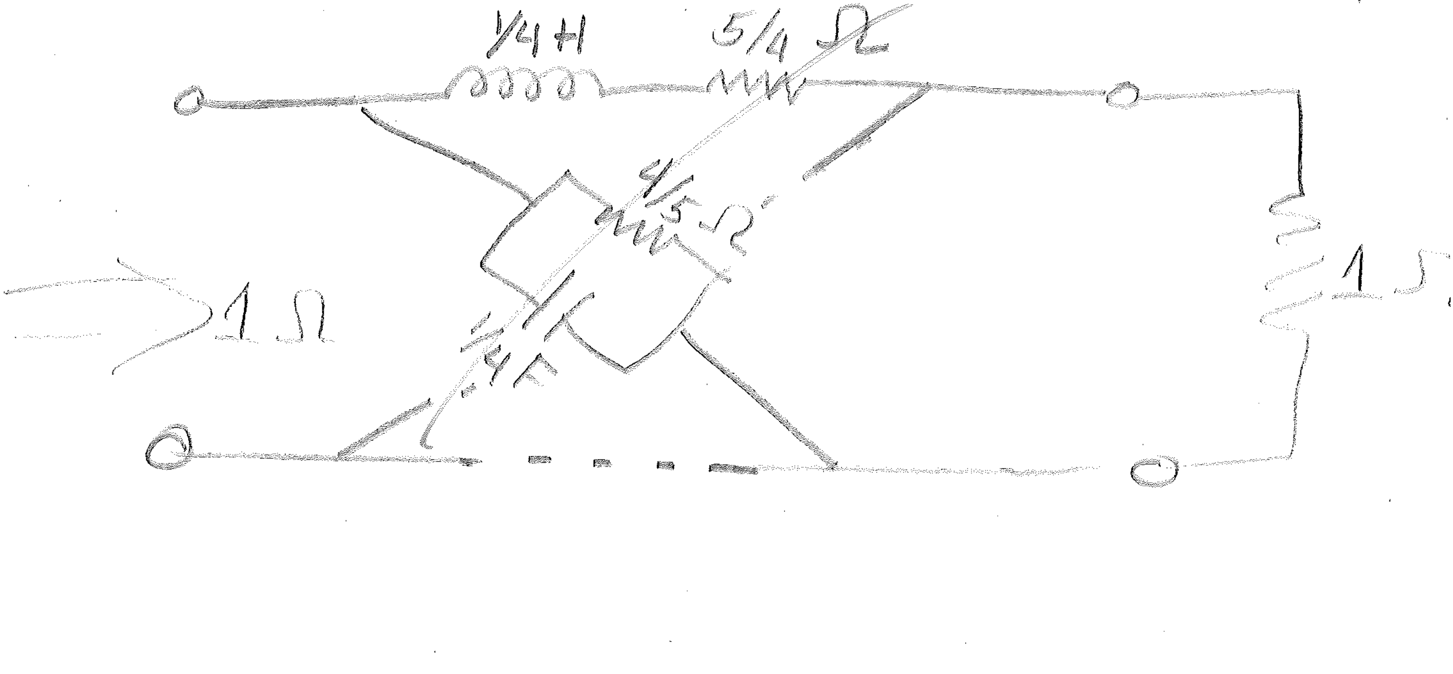
$$= \frac{s+5}{4}$$

$$Z_b = \frac{4}{s+5} = \frac{1}{\frac{s}{4} + \frac{5}{4}}$$



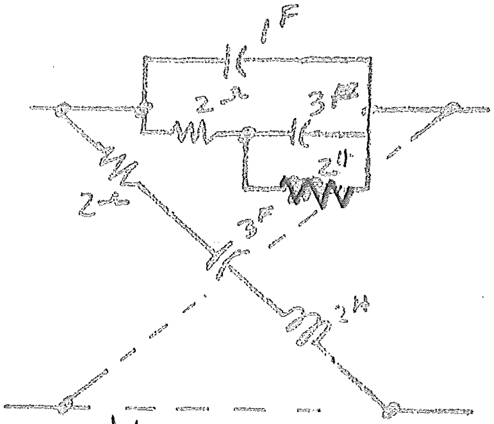
error in 4

(10



6. Decompose this symmetrical lattice network into an unbalanced network.

I FORGOT THE THIRD WAY FOR DEC. Sooooo

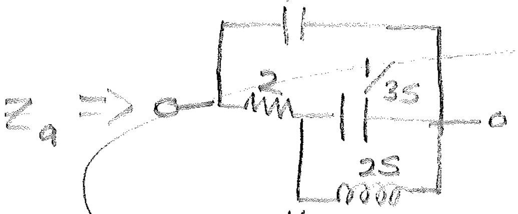


$$Z_b = 2 + \frac{1}{3s} + 2s$$

$$= \frac{6s + 1 + 6s^2}{3s}$$

$$= \frac{6s^2 + 6s + 1}{3s}$$

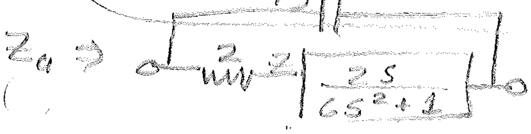
TO AUX SHEET



$$Y = 3s + \frac{1}{2s}$$

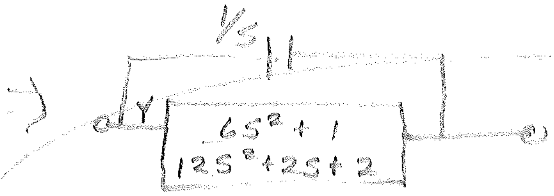
$$= \frac{6s^2 + 1}{2s}$$

$$Z = \frac{2s}{6s^2 + 1}$$



$$\frac{2s}{6s^2+1} + 2$$

$$= \frac{2s + 12s^2 + 2}{6s^2 + 1}$$



3

$$Y = \frac{6s^2 + 1}{12s^2 + 2s + 2} + s$$

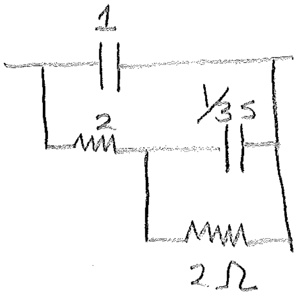
$$= \frac{(6s^2 + 1) + (12s^3 + 2s^2 + 2s)}{12s^2 + 2s + 2}$$

$$= \frac{12s^3 + 8s^2 + 2s + 1}{12s^2 + 2s + 2}$$

$$\Rightarrow Z_a = \frac{12s^2 + 2s + 2}{12s^3 + 8s^2 + 2s + 1}$$

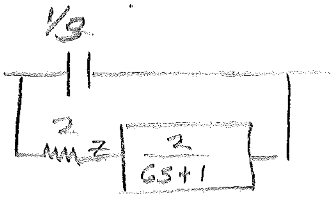
$$= \frac{6(s^2 + s + 1)}{12s^3 + 8s^2 + 2s + 1}$$

6)



$Y = 3s + \frac{1}{2}$

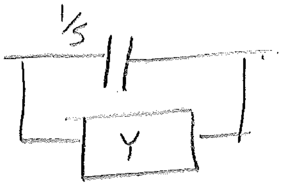
$= \frac{6s+1}{2}$



$Z = \frac{2}{6s+1} + 2$

$= \frac{2+12s+2}{6s+1}$

$= \frac{12s+4}{6s+1}$



$Y_A = \frac{6s+1}{12s+4} + s$

$= \frac{6s+1+12s^2+4s}{12s+4}$

$= \frac{12s^2+10s+1}{12s+4}$

$\Rightarrow Z_A = \frac{12s+4}{12s^2+10s+1}$

$Z_b = \frac{6s^2+6s+1}{3s}$

~~$Z_b = \frac{6s^3+6s+1}{3s}$~~

~~$Z_a = \frac{12s+1}{12s^2+10s+1}$~~

~~$Z_{11} = \frac{1}{2}(Z_a + Z_b)$~~

~~$= \frac{1}{2}\left[\frac{12s+1}{12s^2+10s+1} + \frac{6s^2+6s+1}{3s}\right]$~~

~~$= \frac{1}{2}\left[(12s+1)3s+6\right.$~~

(6) CONT

$$Z_{11} = \frac{1}{2}(Z_a + Z_b)$$

$$= \frac{1}{2}\left[\frac{12S+4}{12S^2+10S+1} + \frac{6S^2+6S+1}{3S}\right]$$

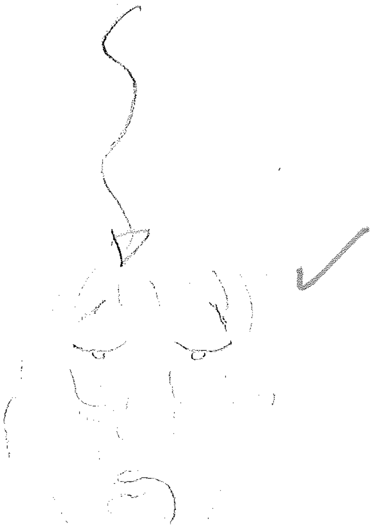
$$= \frac{1}{2}\left[\frac{(3S)(12S+4)+(10S^2+10S+1)(6S^2+6S+1)}{(12S^2+10S+1)(3S}\right]$$

$$= \frac{1}{2}\left[\frac{36S^2+12S+60S^4+60S^3+10S^2+60S^3+60S^3+10S+6S^2+6S+1}{(12S^2+10S+1)(3S)}\right]$$

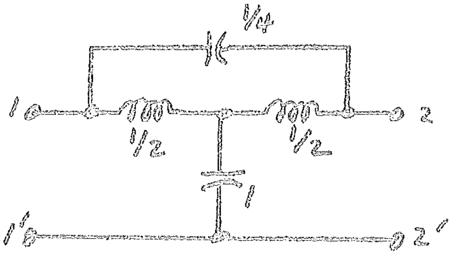
$$= \frac{1}{2}\left[\frac{60S^4+180S^3+52S^2+28S+1}{\cancel{13}\,36S^3+10S^2+3S}\right]$$

$$Z_{22} = \frac{1}{2}(Z_b - Z_a)$$

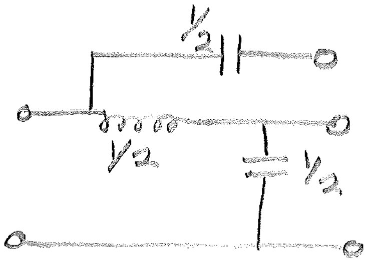
$$= \frac{1}{2}\left[\frac{6S^2+6S+1}{3S} - \frac{12S+4}{12S^2+10S+1}\right]$$



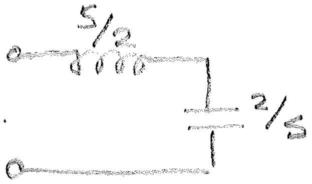
7. Determine the z_{12} and $z_{11} = z_{22}$ functions for this symmetrical Bridged-T network by first obtaining the impedance functions (Z_a and Z_b) of the equivalent symmetrical lattice network.



BARTLETT'S BISECTION THEM:

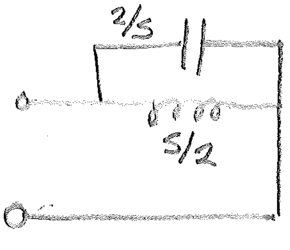


a) $Z_b = Z_{\frac{1}{2}\,o.c.}$



$\Rightarrow Z_b = \frac{S}{2} + \frac{2}{S} = \frac{S^2}{2S} + \frac{4}{2S} = \frac{S^2+4}{2S}$

b) $Z_a = Z_{\frac{1}{2}\,s.c.}$



$\Rightarrow Y_A = \frac{S}{2} + \frac{2}{S} = \frac{S^2+4}{2S} \Rightarrow Z_A = \frac{2S}{S^2+4}$

$$Z_{22} = Z_{11} = \frac{1}{2}(Z_A + Z_b) = \frac{1}{2}\left[\frac{S^2+4}{2S} + \frac{2S}{S^2+4}\right] = \frac{1}{2}\,\frac{(S^2+4)^2+4S^2}{(2S)(S^2+4)}$$
$$= \frac{S^4+12S^2+16}{4(S^3+4S)}$$

9

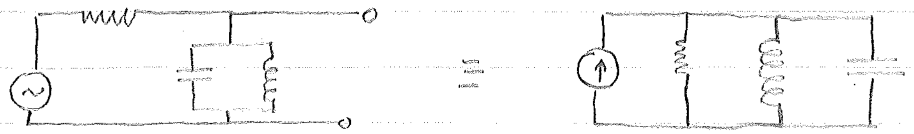
$$Z_{12} = Z_{21} = \frac{1}{2}(Z_b - Z_a) = \frac{1}{2}\left\{\frac{-4S^2+(S^2+4)^2}{(2S)(S^2+4)}\right\} = \frac{+S^4+16}{-4(S^3+4S)}$$

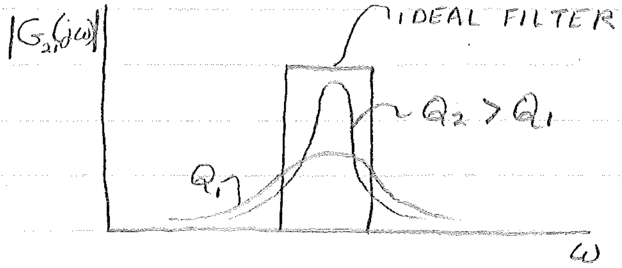
NETWORK
SYNTHESIS II

12-5-71

CHAPTER 13:

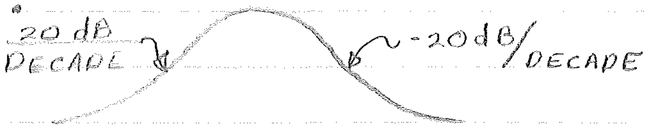
BUTTERWORTH FILTERS }
CHEBYCHEV FILTERS } BEFORE CHRISTMAS

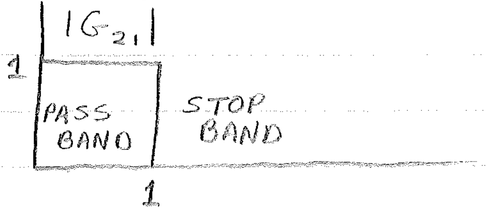




$$G_{21} = \frac{V_2}{V_1}\bigg|_{I_2=0}$$

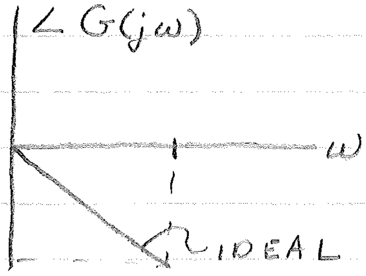
BODE:



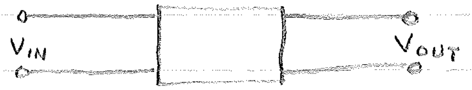


FILTER TYPES:
- HIGH PASS
- BAND PASS
- BAND ELIMINATION

PHASE PLOTS:



CONSIDER BLACK BOX:

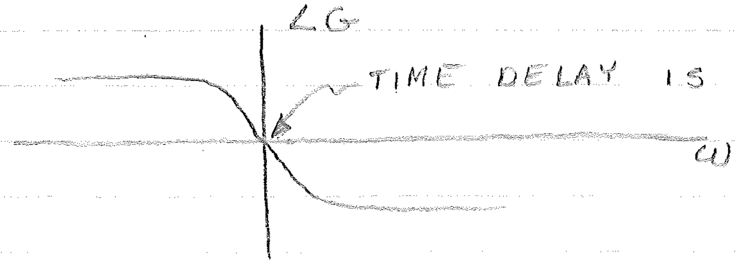


$V_{IN} = A_1 \cos \omega t = A_1 \cos \theta_1$

$V_{OUT} = A_2 \cos(\omega t + \phi) = A_2 \cos \theta_2$

$\frac{d\theta_1}{d\omega} = \frac{d\theta_2}{d\omega} = t$ = ENVELOPE OR GROUP DELAY

$\frac{\phi}{\omega}$ = PHASE DELAY

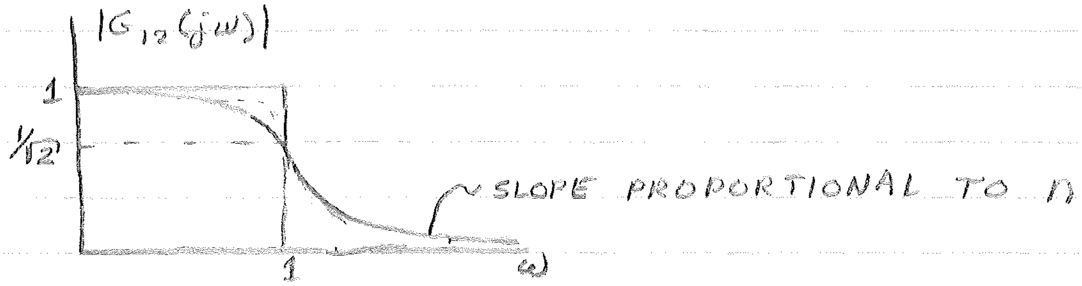


BUTTERWORTH FILTER (CHAPTER 13.1)

$|G_{12}(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}} \Rightarrow n^{TH}$ ORDER BUTTERWORTH FILTER

$|G(j1)| = \frac{\sqrt{2}}{2}$

FOR $\omega >> 1$; $G_{12}(j\omega) = \omega^{-n}$



FILTER IS MAXIMALLY FLAT @ $\omega = 0$

12-7-72

13-3, 13-4, 13-5, 13-10, 13-14 (pp 374-385)

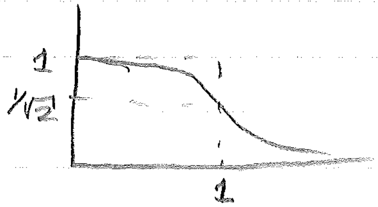
BUTTERWORTH

$|G_{12}(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}}$

1) MAXIMALLY FLAT @ $\omega = 0$

2) $|G_{12}(j\,1)] = 1/\sqrt{2}$

3) $|G_{12}(j\omega)|_{\omega>>} = \frac{1}{\omega^n}$ (20n $\frac{db}{DECADE}$)



<u>POLE LOCATIONS</u>

$-s^2 = \omega^2$

$|G_{12}(j\omega)|^2 = G_{12}(j\omega)\,\overline{G_{12}(j\omega)}$ CONJUGATE

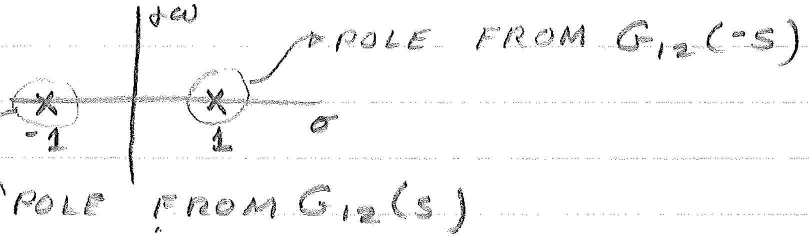
$= G_{12}(j\omega)\,G_{12}(-j\omega)$

$= [G_{12}(s)\,G_{12}(-s)]_{s=j\omega}$

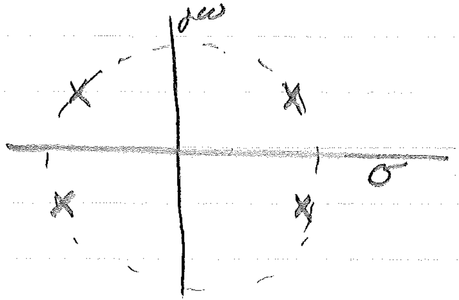
THEN: $|G_{12}(s)|^2 = \frac{1}{1+(-s^2)^n}$

ROOTS FROM: $(-s^2)^n = -1$

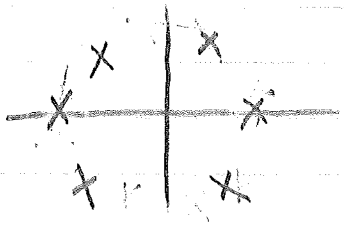
a) $n=1 \Rightarrow$



b) n=2



c) n=3



ROOTS ALWAYS LIE ON UNIT CIRCLE

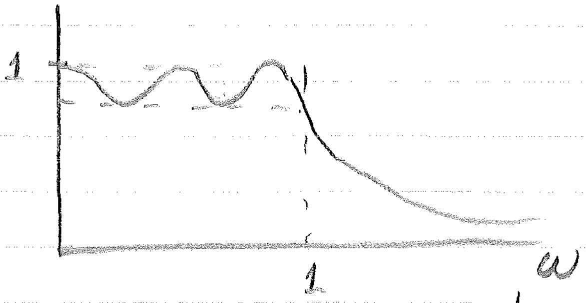
$$s_k = e^{j\left(\frac{2k-1}{n}\right)\frac{\pi}{2}} \quad ; \; n = \text{EVEN}$$

$$s_k = e^{j\left(\frac{2k}{n}\right)\frac{\pi}{2}} \quad ; \; n = \text{ODD}$$

$$G_{12}(s) = \frac{1}{1 + a_1 s + a_2 s^2 + \dots + a_n s^n} \quad \leftarrow \text{COEFFICIENTS}$$

FROM TABLE 13-2, Pg 37

CHEBYSHEV



$$|G_{12}(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)} \quad \ni C_n(\omega) \text{ IS CHEBYSHEV FUNCTION}$$

$$C_n(\omega) \triangleq \begin{cases} \cos[n \cos^{-1}\omega] & ; |\omega| \le 1 \\ \cosh[n \cosh^{-1}\omega] & ; |\omega| \ge 1 \end{cases}$$

CONSIDER $\alpha = \cos^{-1}(\omega)$

$$C_n(\omega) = \cos n\alpha = C_n(\alpha)$$

$$C_{n+1}(\alpha) = \cos(n+1)\alpha$$

$$= \cos n\alpha \cos\alpha - \sin n\alpha \sin\alpha$$

$$C_{n-1}(\alpha) = \cos(n-1)\alpha$$

$$= \cos n\alpha \cos\alpha + \sin n\alpha \sin\alpha$$

$$\Rightarrow C_{n+1}(\alpha) + C_{n-1}(\alpha) = n \cos n\alpha \cos\alpha$$

$$\cos\alpha = \omega \; ; \quad \cos n\alpha = C_n(\alpha)$$

THEN:

$$C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega)$$

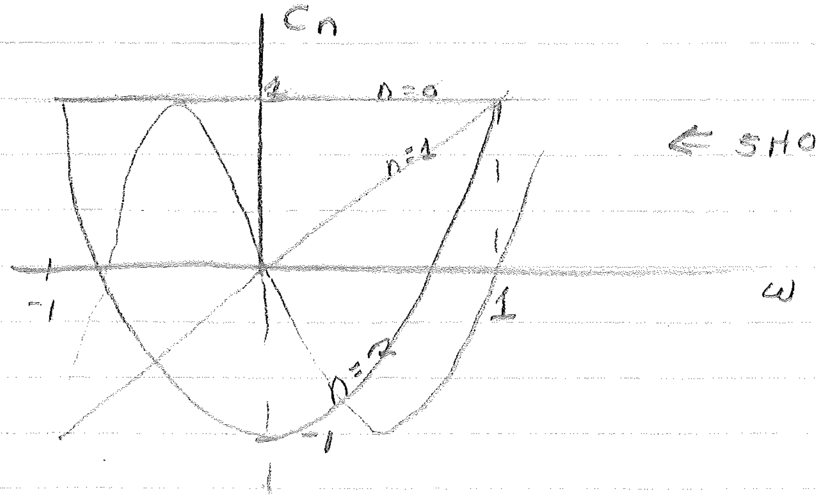
THUS: $C_0 = 1$; $C_1(\omega) = \omega$

GIVES: $C_2(\omega) = 2\omega^2 - 1$, ETC.

LISTED ON Pg. 377 OF TEXT

FOR $|\omega| < 1$, THEN $C_n(\omega) = \cos[n \cos^{-1}\omega]$

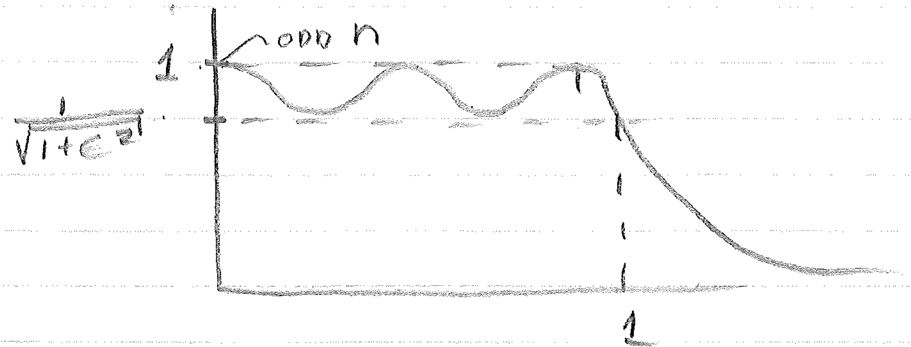
$$|C_n(\omega)| \leq 1$$

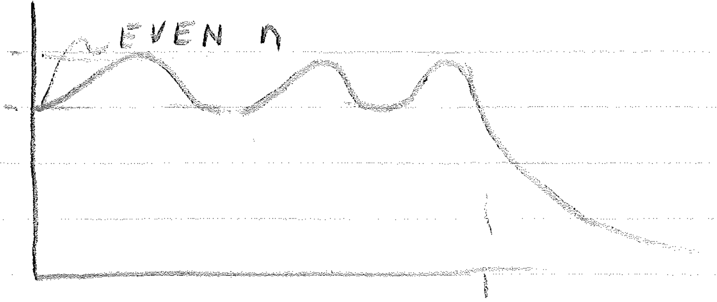


⇐ SHOWN ON Pg. 378

AGAIN:

$$|G_{12}(j\omega)| = \frac{1}{1 + \epsilon^2 C_n^2(\omega)}$$





$$\text{RIPPLE} = 1 - \frac{1}{\sqrt{1 - \epsilon^2}} \overset{\text{(EXPANSION)}}{=} 1 - \left[1 - \frac{\epsilon^2}{2} + \ldots\right]$$

FOR $\epsilon \ll 1$:

$$\text{RIPPLE} = \frac{\epsilon^2}{2}$$

FOR $|\omega| > 1$

LET $B = \cosh^{-1}\omega$

$$\cosh nB = \frac{e^{nB} + e^{-nB}}{2}$$

$$e^{B} = \cosh B + \sinh B$$

$$\Rightarrow \cosh nB = \frac{(\cosh B + \sinh B)^n + (\cosh B + \sinh B)^{-n}}{2}$$

$$\cosh B = \omega$$

$$\sinh B = \pm\sqrt{\omega^2 - 1}$$

$$\Rightarrow \cosh nB = \frac{1}{2}\left[(\omega \pm \sqrt{\omega^2-1})^n + (\omega \pm \sqrt{\omega^2-1})^{-n}\right]$$

FOR $n \geq 2$ & $\omega >> 1$

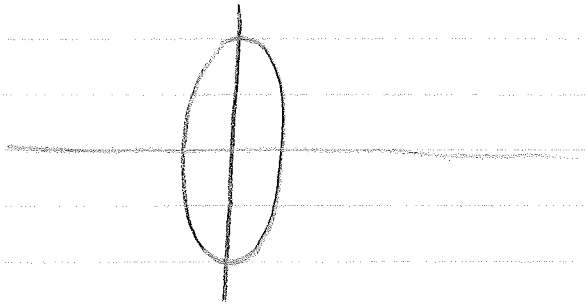
$$\cosh[n\cosh^{-1}\omega] \cong \frac{(\omega + \sqrt{\omega^2-1})^n}{2} = C_n(\omega)$$

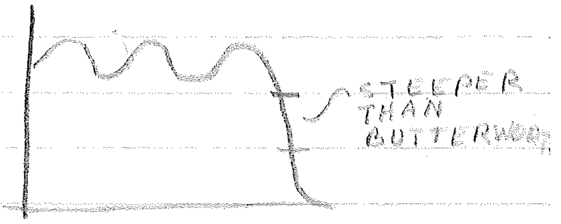
EMPLOYING (+) SIGN

$$\sim 2^{n-1}\omega^n$$

THEN $|G_{12}(j\omega)|_{\omega >> 1} = \dfrac{1}{\epsilon C_n(\omega)} = \dfrac{1}{2^{n-1}\epsilon\omega^n}$

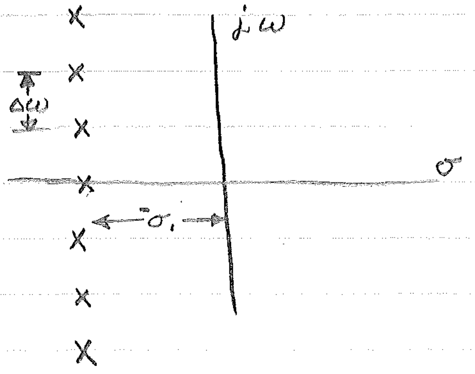
POLES ON ELLIPSE:





12-12-72

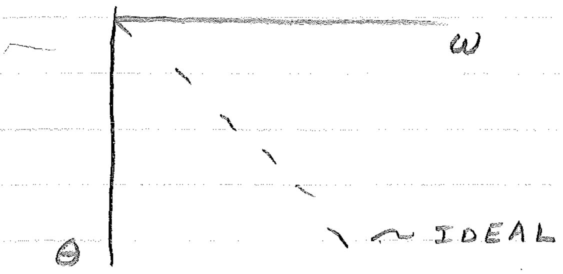
13-10)



$$G_{12}(s) = \frac{1}{(s+\sigma_1)(s+\sigma_1+j\Delta\omega)(s+\sigma_1-j\Delta\omega)(\quad)\cdots}$$

$$= \frac{1}{(s+\sigma_1)[(s+\sigma_1)^2+\Delta\omega^2][(s+\sigma_1)^2+2\Delta\omega^2][\quad]\cdots}$$

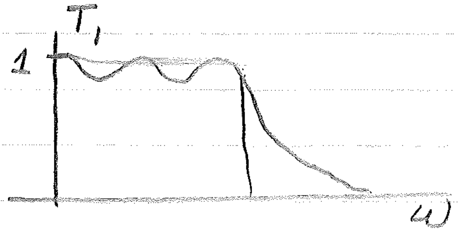
$$\angle G_{12}(j\omega) = -\mathrm{atan}\frac{\omega}{\sigma_1} - \mathrm{atan}\frac{2\omega\sigma_1}{\sigma_1^2+\omega^2+\Delta\omega^2} + \ldots = \theta$$



$$T = \frac{d\theta}{d\omega}$$

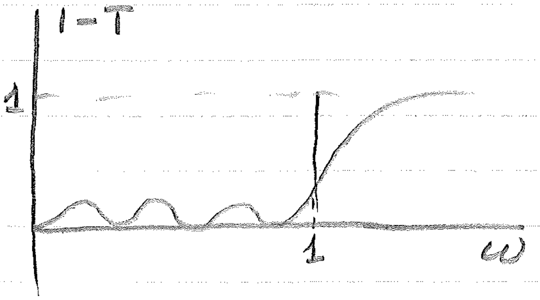
13-14)

$$T_1 \quad |G_{12}(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)}$$



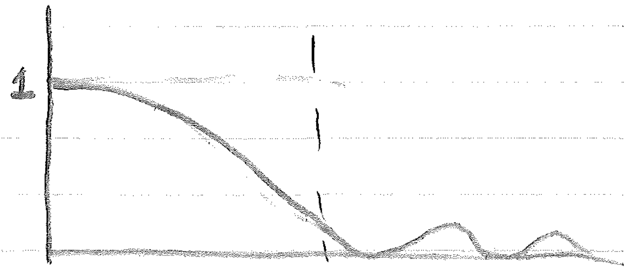
$$1 - T_1 = \frac{1+\epsilon^2 C_n^2(\omega) - 1}{1+\epsilon^2 C_n^2(\omega)}$$

$$= \frac{\epsilon^2 C_n^2(\omega)}{1+\epsilon^2 C_n^2(\omega)} \Rightarrow$$

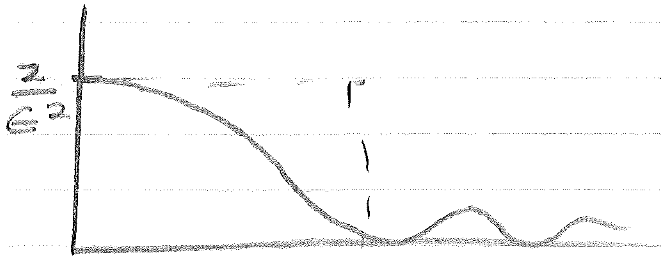


$$\frac{1}{\omega} \Rightarrow \omega$$

$$\Rightarrow \frac{\epsilon^2 C_n^2(1/\omega)}{1+\epsilon^2 C_n^2(1/\omega)} \Longrightarrow$$



$$\frac{2C_n^2(1/\omega)}{1+\epsilon^2 C_n^2(1/\omega)} \Rightarrow$$



$$\cosh^{-1} x = \ln \frac{x \pm \sqrt{x^2 - 1}}{2}$$

12-14-72

NEPERS

$$\frac{V_2}{V_1} = H(s)$$

$$= e^{\alpha + j\beta}$$

$$= e^{\alpha} e^{j\beta}$$

$$= e^{\alpha} \angle \beta$$

$\beta \Rightarrow$ PHASE

$e^{\alpha} \Rightarrow$ AMPLITUDE

α IS MEASURED IN NEPERS

$$\left|\frac{V_2}{V_1}\right| = e^{\alpha}$$

$$20 \log_{10}\left|\frac{V_2}{V_1}\right| = 20 \log_{10} e^{\alpha}$$

$$\Rightarrow \quad db \propto \alpha$$

12-19-72

11-21-72

14: 1, 13, 14, 16, 17, 4, 5

RECALL THAT FOR THE SYMMETRICAL LATTICE: $Z_a = Z_{11} - Z_{12}$

$Z_b = Z_{11} + Z_{12}$

FOR CONSTANT R:

$$Z_a = 1 - Z_{12} / 1 + Z_{12} = \frac{1}{Z_b}$$

1-8-72

QUADRADRANTAL SYMMETRY - CONSTELLATION MUST HAVE REFLECTION ON BOTH AXES.

(14.13e)

$$|Z_{12}(j\omega)|^2 = \frac{(1+\omega^2)^2}{1-\omega^2+\omega^6}$$

PROB; 14.1,13, 14, 16, 4, 5

PROB; 15.4,5,6

1-9-73

1-12-73

TAKE HOME TEST

SEC. 11-2$^{3}_{1}$, Pg 305

SEC 11-4, Pg 313

OMIT REST OF CHAPT. 11

SEC 14.2, Pg 402

START

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$V_2 = -I_2$

$$\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22}+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$Z_{11} = \frac{V_1}{I_1}\Big|_{V_2 = I_2} = \frac{Z_{11}Z_{22} + Z_1 - Z_{12}^2}{Z_{22}+1}$$

$$= \frac{\Delta_Z + Z_{11}}{Z_{22}+1}$$

$$\text{BUT } Y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0} = Z_{11}/\Delta_Z$$

$$\Rightarrow Z_{11} = \frac{Z_{11}(1/Y_{22}+1)}{Z_{22}+1} \longrightarrow \text{Eq } 14.22,\ \text{Pg } 403$$

PROB. 14-17, 15-7, 8

1-13-73

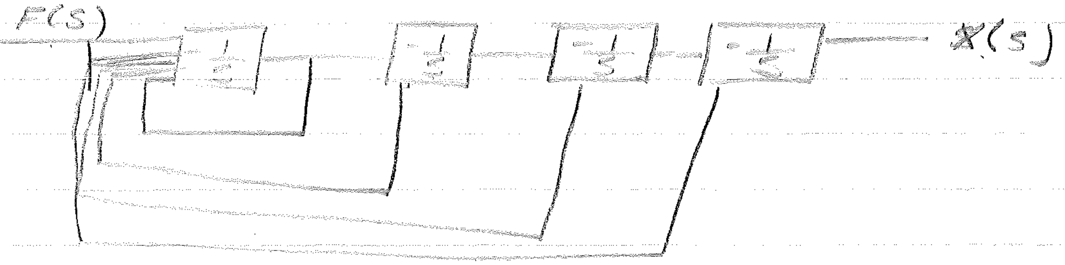
SEC. 14-2

SUMERAZATION ON Pg. 419

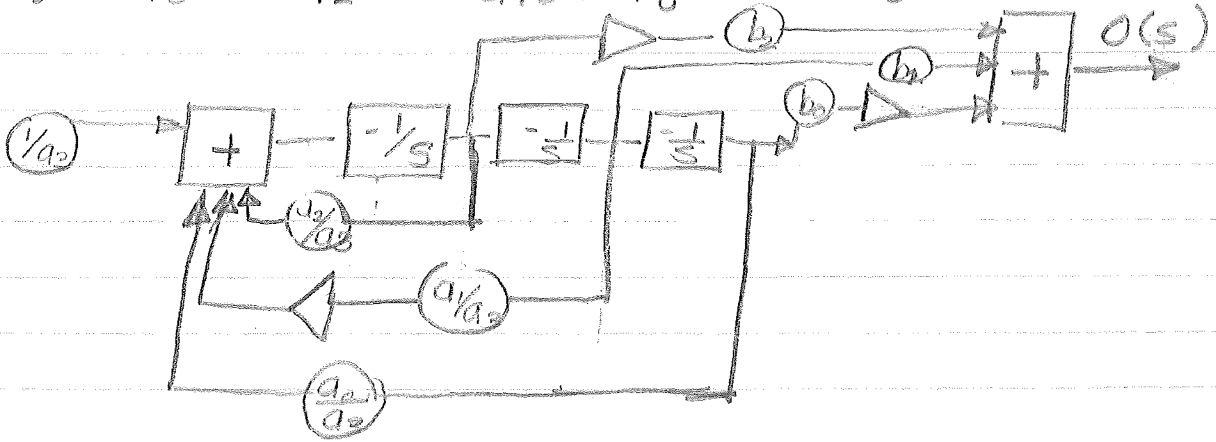
1-19-72

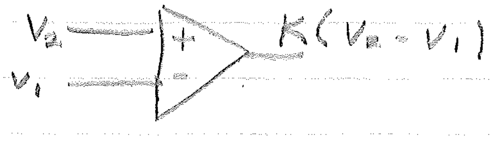
Pg. 97

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

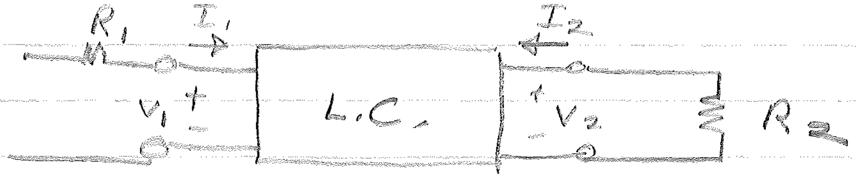


$$O(s) = \frac{b_2^2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} I(s)$$





SCATTERING PARAMETERS



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y'_{11} & Y'_{12} \\ Y'_{21} & Y'_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

DEFINE:

$$a_1 \triangleq \frac{V_1 + R_1 I_1}{2\sqrt{R_1}} \qquad b_1 \triangleq \frac{V_1 - R_1 I_1}{2\sqrt{R_1}}$$

$$a_2 \triangleq \frac{V_2 + R_2 I_2}{2\sqrt{R_2}} \qquad b_2 \triangleq \frac{V_2 - R_2 I_2}{2\sqrt{R_2}}$$

THEN

$$a_1 + b_1 = V_1/\sqrt{R_1} \qquad a_2 + b_2 = V_2/\sqrt{R_2}$$

$$a_1 - b_1 = \sqrt{R_1}\, I_1 \qquad a_2 - b_2 = \sqrt{R_2}\, I_2$$

CONSIDER $R_1 = R_2 = R_0$

$$\begin{bmatrix} \dfrac{a_1 - b_1}{\sqrt{R_0}} \\ \dfrac{a_2 - b_2}{\sqrt{R_0}} \end{bmatrix} = \begin{bmatrix} Y'_{11}\sqrt{R_0}(a_1 + b_1) + Y'_{12}\sqrt{R_0}(a_2 + b_2) \\ Y'_{21}\sqrt{R_0}(a_1 + b_1) + Y'_{22}\sqrt{R_0}(a_2 + b_2) \end{bmatrix}$$

1-30-72

13.3) $n=5;\ \epsilon^2=0.2$

a) $|G(j\omega)|_{MAX} = 1$

$$|G(j\omega)|_{MIN} = \frac{1}{\sqrt{1+\epsilon^2}}$$

$$= \frac{1}{\sqrt{1.2}}$$

$$= 0.914$$

b) WIDTH $= 1.0 - 0.914 = 0.086$

OR $\quad dB = 20 \log_{10} \frac{|G(j\omega)|_{MIN}}{|G(j\omega)|_{MAX}}$

$$= 20 \log_{10} 0.914$$

$$= -20 \log_{10} \frac{1}{0.914}$$

$$= -20 \log_{10} 1.096$$

$$= -20\,(0.398)$$

$$= -0.796\ dB$$

c) $|G_{12}(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)} = \frac{1}{2}$

$$\Rightarrow \epsilon^2 C_n^2(\omega) = 1$$

$$C_n(\omega) = \frac{1}{\epsilon} = \cosh(n \cosh^{-1}\omega)$$

$$\cosh^{-1}\frac{1}{\epsilon} = n \cosh^{-1}\omega$$

$$\frac{1}{n}\cosh^{-1}\frac{1}{\epsilon} = \cosh^{-1}\omega$$

$$\omega = \cosh \frac{1}{n} \cosh^{-1}\frac{1}{\epsilon}$$

$$= \cosh[0.2 \cosh^{-1} 2.237]$$

$$= \cosh[(0.2)(1.445)]$$

$$= \cosh[0.2890]$$

$$= 1.042\ \text{HZ}$$

13.4) RIPPLE WIDTH = 0.5 db

$|G(j\omega_{1/2})|^2 = 1/2 \ni \omega_{1/2} = 1.1$

a) $20 \log_{10} \frac{1}{\sqrt{1+\epsilon^2}} = 0.5$

$\log_{10} \frac{1}{\sqrt{1+\epsilon^2}} = {}^-0.025$

$\frac{1}{\sqrt{1+\epsilon^2}} = 0.944$

$\epsilon^2 = \left(\frac{1}{0.944}\right)^2 - 1$

$= 0.124$

$\epsilon = 0.1114$

b) $|G(j\omega_{1/2})|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega_{1/2})} = \frac{1}{2}$

$\epsilon^2 C_n^2(\omega_{1/2}) = 1$

$C_n(\omega_{1/2}) = \frac{1}{\epsilon} = \cosh[n \cosh^{-1}\omega]$

$\cosh^{-1}\frac{1}{\epsilon} = n \cosh^{-1}\omega$

$n = \frac{\cosh^{-1} 1/\epsilon}{\cosh^{-1}\omega}$

$= \frac{\cosh^{-1}(1/0.1114)}{\cosh^{-1}(1.1)}$

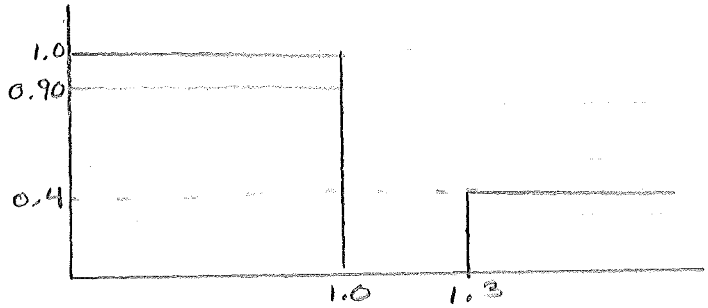
$= \frac{\cosh^{-1}(8.97)}{\cosh^{-1}(1.1)}$

$= \frac{2.884}{0.445}$

$= 6.4$

⇒ 7 STAGES ARE NEEDED

13.5)



$\frac{1}{\sqrt{1+\epsilon^2}} = 0.9$

$1+\epsilon^2 = \frac{1}{0.81}$

$\epsilon^2 = 1.235 - 1 = 0.235$

$\Rightarrow \epsilon = 0.4849$

$|G_{12}(j1.3)| = 0.4$

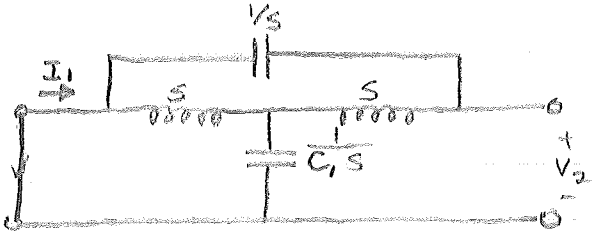
$|G_{12}(j1.3)|^2 = 0.16 = \frac{1}{1+\epsilon^2 C_n^2(\omega)}$

$\frac{1}{0.16} = 1+\epsilon^2[\cosh n[\cosh^{-1} 1.3)]^2$

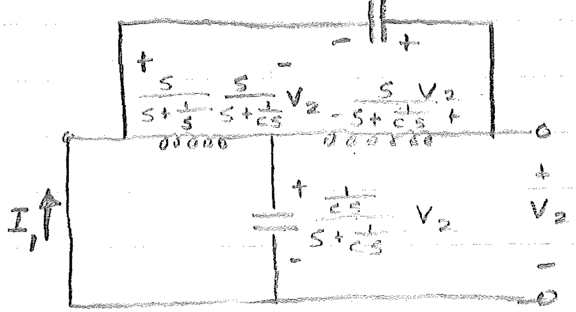
$\Rightarrow n = 2.937$

$\Rightarrow$ 3 STAGES ARE NEEDED

14·14)



$$Z_{12} = \left.\frac{V_2}{I_1}\right|_{V_1=0} \qquad \frac{\frac{1}{s}}{s+\frac{1}{s}}\;\frac{s}{s+\frac{1}{cs}}\,V_2$$



$$I_1 = \frac{\left[\frac{s}{s+\frac{1}{s}}\;\frac{s}{s+\frac{1}{cs}} + \frac{\frac{1}{cs}}{s+\frac{1}{cs}}\right]V_2}{s+\frac{1}{cs}}$$

$$\frac{V_2}{I_1} = \frac{s+\frac{1}{cs}}{\frac{s^2}{(s+\frac{1}{s})(s+\frac{1}{cs})} + \frac{\frac{1}{cs}}{s+\frac{1}{cs}}}$$

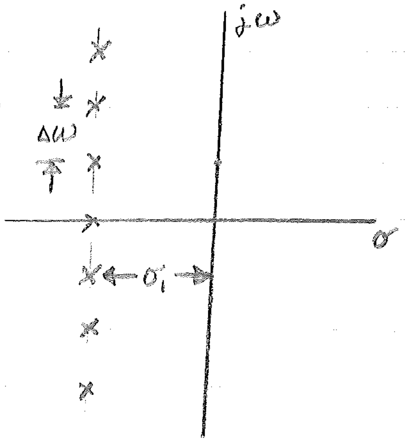
$$= \frac{cs^2+1}{\frac{cs^3}{(s+\frac{1}{s})(s+\frac{1}{cs})} + \frac{1}{s+\frac{1}{cs}}}$$

$$= \frac{cs^2+1}{\frac{cs^5}{(s^2+1)(s^2+\frac{1}{c})} + \frac{s}{s^2+\frac{1}{c}}}$$

$$= \frac{cs^2+1}{\frac{cs^5+s(s^2+1)}{(s^2+1)(s^2+\frac{1}{c})}}$$

$$NUM = (cs^2+1)(s^2+1)(cs^2+1)$$
$$= (cs^2+1)^2(s^2+1)$$

13-10)



$$G(j\omega) = \frac{1}{(s+\sigma_1)\prod_{n=1}^{3}(s+\sigma_1+jn\Delta\omega)(s+\sigma_1-jn\Delta\omega)}$$

$$\angle G_{12}(j\omega) = -\operatorname{atan}\frac{\omega}{\sigma_1} - \sum_{n=1}^{3}\operatorname{atan}\frac{2\omega\sigma_1}{\sigma_1^2+\omega^2+n^2\Delta\omega^2}$$

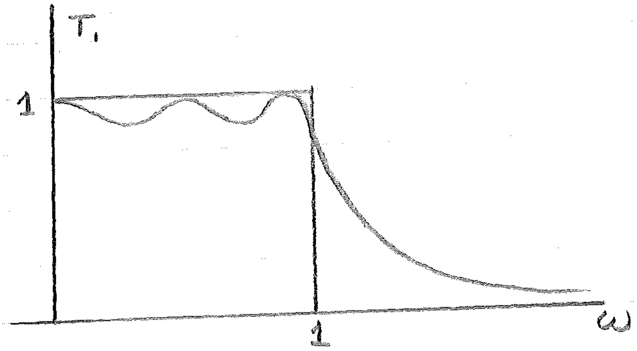
$$\frac{d}{d\omega}\angle G_{12}(j\omega) = \frac{-1/\sigma}{1+\left(\frac{\omega}{\sigma}\right)^2} - \sum_{n=1}^{3}\frac{\frac{d}{d\omega}\left[\frac{2\omega\sigma_1}{\sigma_1^2+\omega^2+n^2\Delta\omega^2}\right]}{1+\left(\frac{2\omega\sigma_1}{\sigma_1^2+\omega^2+n^2\Delta\omega^2}\right)^2}$$

$$\frac{d}{d\omega}\left[\frac{2\omega\sigma_1}{\sigma_1^2+\omega^2+n^2\Delta\omega^2}\right] = \frac{2\sigma_1(\sigma_1^2+\omega^2+n^2\Delta\omega^2)-4\omega\sigma_1}{[\sigma_1^2+\omega^2+n^2\Delta\omega^2]^2}$$

$$T = -\frac{1}{\sigma+\omega^2/\sigma} - \sum_{n=1}^{3}\frac{2\sigma_1[(\sigma_1^2+\omega^2+n^2\Delta\omega^2)-2\omega]}{[\sigma_1^2+\omega^2+n\Delta\omega^2]^2+(2\omega\sigma_1)^2}$$

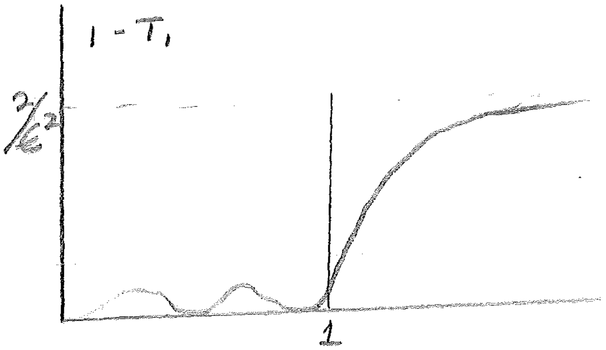
13-14) $|G_{12}|^2 = \dfrac{2C_n^2(1/\omega)}{1+\epsilon^2 C_{n_1}^2(1/\omega)}$

a) LET $T_1^{\,2} = \overline{1+\epsilon^2 C_n^2(\omega)}$

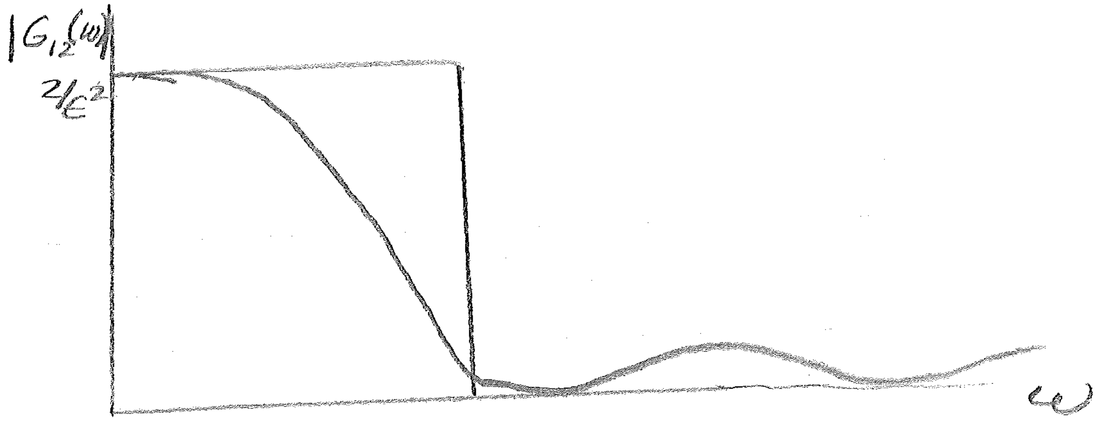


$|G_{12}|^2 \alpha 1 - T_1^{\,2} = 1 - \dfrac{1}{1+\epsilon^2 C_n^2(\omega)}$

$= \dfrac{\epsilon^2 C_n^2(\omega)}{1+\epsilon^2 C_n^2(\omega)} \xrightarrow{\text{SCALING}} |G_{12}(\tfrac{1}{\omega})|^2 = \dfrac{2}{\epsilon^2}\left[1 - T_1^{\,2}\right]$

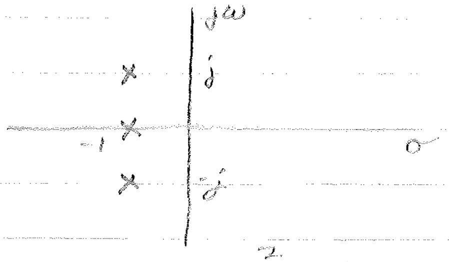


MAPPINING 0-1 ONTO 1-∞
AND 1-∞ ONTO 0-1



14.1) a) $Z_{12}(s) = \frac{2}{(s+1)(s^2+2s+2)}$

$= \frac{2}{(s+1)(s+1+j)(s+1-j)}$



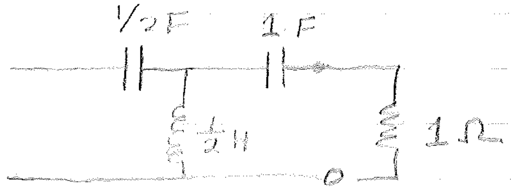
$$Z_{12}(s) = \frac{2}{(s^3+2s^2+2s)+(s^2+2s+2)}$$

$$= \frac{2}{s^3+3s^2+4s+4}$$

$$= \frac{\frac{2}{s^3+4s}}{1+\frac{3s^2+4}{s^3+4s}}$$

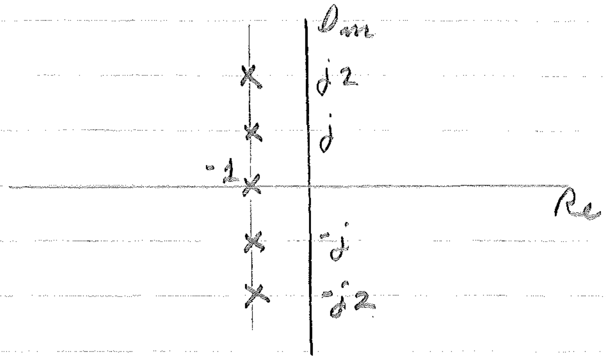
$$Z_{22} = \frac{3s^2+4}{s^3+4s} \quad ; \quad Z_{12}(s) = \frac{2}{s^3+4s}$$

$$\begin{array}{c|c|c|c}
Y & & & Z \\
2/s & 4s+s^3 & 4+3s^2 & \frac{1}{s} \\
 & 4s & 4+s^2 & \\
\hline
 & s^3 & 2s^2 & \frac{2}{s} \\
 & & 2s^2 &
\end{array}$$



b) $Z_{12}(s) = \dfrac{10}{(s+1)(s^2+2s+2)(s^2+2s+5)}$

$= \dfrac{10}{(s+1)(s+1+j)(s+1-j)(s+1+j2)(s+1-j2)}$



$Z_{12}(s) = \dfrac{10}{(s^3+3s^2+4s+2)(s^2+2s+5)}$

$$\begin{array}{rrrrrr}
 & & s^3 & +3s^2 & +4s & +2 \\
 & & & s^2 & +2s & +5 \\
\hline
 & & 5s^3 & 15s^2 & 20s & 10 \\
 & 2s^4 & 6s^3 & 8s^2 & 4s & \\
s^5 & 3s^4 & 4s^3 & 2s^2 & & \\
\hline
s^5 & +5s^4 & +15s^3 & +25s^2 & +24s & +10
\end{array}$$

$\Rightarrow Z_{12}(s) = \dfrac{10}{s^5+5s^4+15s^3+25s^2+24s+10}$

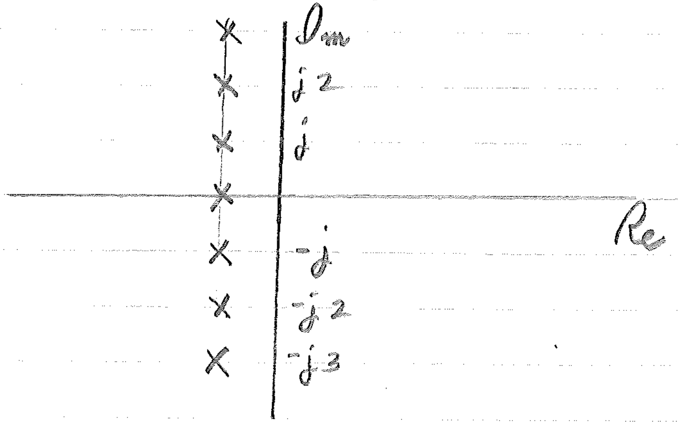
$\therefore p(s) = 10; q_1(s) = 5s^4+25s^2+10\ ;\ q_2(s) = s^5+15s^3+24s$

$Z_{12}(s) = \dfrac{p}{q_2} = \dfrac{10}{s^5+15s^3+24s}$

$Z_{22}(s) = \dfrac{q_1}{q_2} = \dfrac{5s^4+25s^2+10}{s^5+15s^3+24s}$

c) $$Z_{12} = \frac{100}{(s+1)(s^2+2s+2)(s^2+2s+5)(s^2+2s+10)}$$

$$= \frac{100}{(s+1)(s+1+j)(s+1-j)(s+1+j2)(s+1-j2)(s+1+j3)(s+1-j3)}$$



$$Z_{12}(s) = 100\left[(s^5+5s^4+15s^3+25s^2+24s+10)(s^2+2s+10)\right]^{-1}$$

$$\begin{array}{rrrrrrrr}
 & & & s^5+5s^4+15s^3+25s^2+24s+10 \\
 & & & & & s^2+2s+10 \\
\hline
 & & 10s^5 & 50s^4 & 150s^3 & 250s^2 & 240s & 100 \\
 & 2s^6 & 10s^5 & 30s^4 & 50s^3 & 48s^2 & 20s & \\
s^7 & 5s^6 & 15s^5 & 25s^4 & 24s^3 & 10s^2 & & \\
\hline
s^7+ & 7s^6 + & 35s^6+ & 105s^4+ & 124s^3+ & 308s^2+ & 260s + & 100
\end{array}$$

$$\Rightarrow Z_{12}(s) = \frac{100}{s^7+7s^6+35s^5+105s^4+124s^3+308s^2+260s+100}$$

$$p(s) = 100\ ;\ q_1(s) = 7s^6+105s^4+308s^2+100$$
$$q_2(s) = s^7+35s^5+124s^3+260s$$

$$Z_{12}(s) = \frac{p}{q_2(s)} = \frac{100}{s^7+35s^5+124s^3+260s}$$

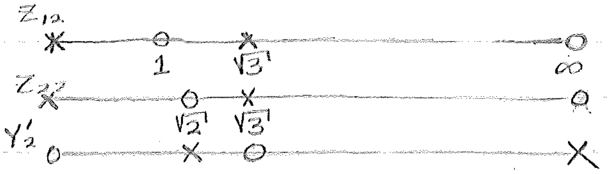
$$Z_{22}(s) = \frac{q_1(s)}{q_2(s)} = \frac{7s^6+105s^4+308s^2+100}{s^7+35s^5+124s^3+260s}$$

14.11) $Z_{12}(S) = \frac{S^2+1}{S^3+S^2+3S+2}$

$$= \frac{\frac{S^2+1}{S^3+3S}}{1+\frac{S^2+2}{S^3+3S}}$$

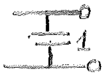
$$\therefore\ Z_{22} = \frac{S^2+2}{S^3+3S} = \frac{(S+j\sqrt{2})(S-j\sqrt{2})}{S(S+j\sqrt{3})(S-j\sqrt{3})}$$

$$Z_{12} = \frac{S^2+1}{S^3+3S} = \frac{(S+j)(S-j)}{S(S+j\sqrt{3})(S-j\sqrt{3})}$$



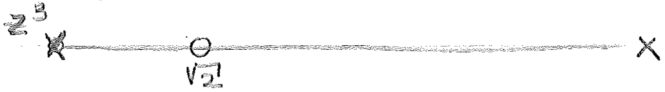
TAKE OUT POLE OF Y_2 @ ∞

$$\begin{array}{r} S \\ S^2+2\ \overline{)\ S^3+3S} \\ \underline{S^3+2S} \\ S \end{array}$$



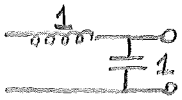
$\Rightarrow Y_2 = \frac{S}{S^2+2}$

$Z_3 = \frac{S^2+2}{S} = \frac{(S+j\sqrt{2})(S-j\sqrt{2})}{S}$

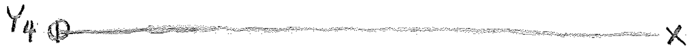


TAKE OUT POLE OF Z_3 @ ∞

$$\begin{array}{r} S \\ S\ \overline{)\ S^2+2} \\ \underline{S^2} \\ 2 \end{array}$$

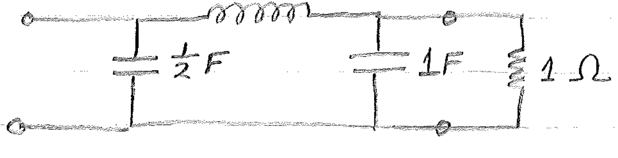


$Z_4 = \frac{2}{S} \Rightarrow Y_4 = \frac{S}{2}$



TAKE OUT POLE @ ∞ OF Y_4

YIELDING NETWORK:



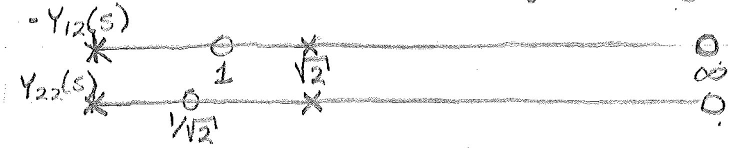
14.5) $Y_{12}(S) = \dfrac{S^2+1}{S^3+2S^2+2S+1}$

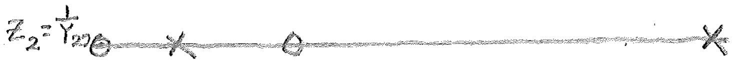
$$= \frac{\frac{S^2+1}{S^3+2S}}{1+\frac{2S^2+1}{S^3+2S}}$$

$$-Y_{12}(S) = \frac{S^2+1}{S^3+2S} = \frac{(S+j)(S-j)}{S(S+j\sqrt{2})(S-j\sqrt{2})}$$

$$Y_{22}(S) = \frac{2S^2+1}{S^3+2S} = \frac{(\sqrt{2}S+j)(\sqrt{2}S-j)}{S(S+j\sqrt{2})(S-j\sqrt{2})}$$

$$= \frac{2(S+j/\sqrt{2})(S-j/\sqrt{2})}{S(S+j\sqrt{2})(S-j\sqrt{2})}$$

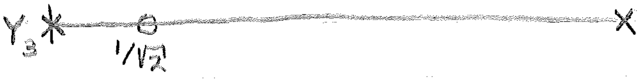




TAKE OUT POLE AT ∞ FROM Z_2

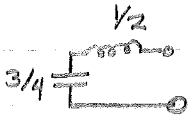
$$\begin{array}{r} \frac{1}{2}S \\ 2S^2+1 \overline{\big) S^3 + 2S} \\ \underline{S^3 + \frac{1}{2}S} \\ \frac{3}{2}S \end{array}$$

$$Z_3 = \frac{3}{2}\,\frac{S}{2S^2+1} \Rightarrow Y_3 = \frac{2}{3}\;\frac{2S^2+1}{S} = \frac{4}{3}\;\frac{(S+j/\sqrt{2})(S-j/\sqrt{2})}{S}$$

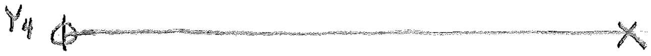


TAKE OUT Y_3 POLE @ ∞

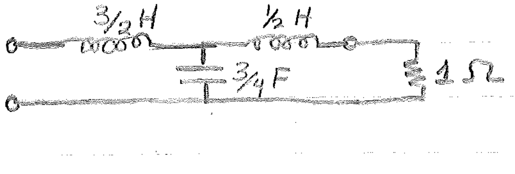
$$\begin{array}{r} \frac{4}{3}S \\ \frac{3}{2}S \overline{\big) 2S^2 + 1} \\ \underline{2S^2} \\ 1 \end{array}$$



$$\Rightarrow Y_4 = \frac{1}{\frac{3}{2}S} = \frac{2}{3S} \Rightarrow Z_4 = \frac{3S}{2}$$



YIELDING THE NETWORK:



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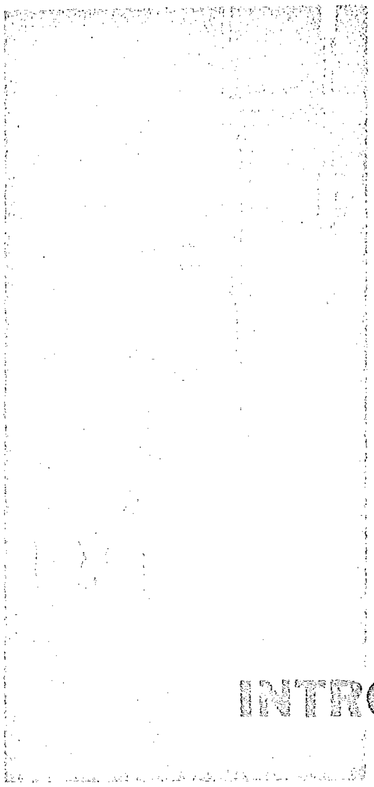

INTRODUCTION TO FILTERS

A. I. ZVEREV, Westinghouse Electric Corporation

FILTER THEORY and design is one of the most highly developed and most widely applied areas of electrical engineering.

Filters can be found in all kinds of commercial and military electronics equipment. They help to channel energy, divide the frequency spectrum, extract Doppler information, integrate coherent signals and maintain accuracy of separation between frequency sources.

The special properties of filters allow their application in many other systems, including mechanical and acoustical systems. These properties are the ability to coordinate the action of several resonant elements to produce uniform output over a prescribed frequency range, and the ability to deliver energy over the widest possible frequency range. Filter theory shows how to combine resonant mechanical or electromechanical elements to produce a uniform conversion of electrical to mechanical energy (or vice versa) over a frequency range, and to determine the greatest bandwidth that can be obtained without loss of efficiency.

All recent progress in electronics has been closely accompanied by progress in filter technology and network theory. The performance of passive and active networks, in the form of transmission filters, can be predicted and analytically described under actual operating conditions in an exact mathematical form. This fact explains why, in new electronic systems, much emphasis has been placed on such networks.

The wide diversity of filter configurations, response characteristics and applications has produced a wealth of terminology and design concepts which often proves confusing to the engineer who is not a filter expert. This article classifies filters in terms of the types of filter elements used, the functions that the filters perform, the response characteristics of filters and methods used to design them.

A number of specific design tools (polynomials, design equations, transformations and curves) are given, but the purpose of the article is to indicate the extent, rather than to exhaust the content, of the field of filter theory and design.

Anatol Ivan Zverev was born and educated in Russia, where he earned a diploma of engineering (M.S.) from the Leningrad Electro-Technique Institute, and a degree in advanced technical science from the Academy of Transport in Moscow, where he was Assistant Professor of transmission-line theory.

In 1953 he joined the Electronics Division of Westinghouse Electric Corporation as a Senior Engineer. Over the past ten years he has been involved in the design of communication, radar, data transmission, and satellite navigation systems. He has been section manager of network synthesis since 1959 and is director of the Network Synthesis Seminar.

His most recent publications include "Application of Network Synthesis to Broadbanding Transmitting System," TPE 4795, 1963, and "Helical Filters," NS 6312, 1963.

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Associate Editor

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INTRODUCTION TO FILTERS

THE ELECTRIC-WAVE FILTER was independently discovered by George Campbell in America and Karl Willy Wagner in Germany in 1917. Their results evolved from earlier work on loaded transmission lines and the classical theory of vibrating systems. The first filters were ladder structures, although lattice sections were discussed by Campbell in 1920–1922. Next came Zobel's invention of *m*-derived and image-parameter theory based on transmission-line analogy. Zobel began with simple iterative structures and extended them to complex or composite (*m*-derived) sections, each with its useful impedance and attenuation properties. The catalog of elementary sections is sufficiently extended to cover most practical requirements. A filter may thus be designed as a cascade connection of sections with matched image impedance. Zobel's filters permitted control of a wide range of filter characteristics, but the basic assumption of the theory—that such a filter behaves as a transmission line terminated at its characteristic impedance—is only approximate. The image-parameter approach suffers from three drawbacks.

1. The network obtained by this approach may be a lattice which cannot be reduced to a ladder network and is consequently difficult to realize as a microwave structure.
2. The network may be unnecessarily complex.
3. The network is frequently poorly matched at the extreme ends of the passband due to the fixed load and source impedance.

As far as we know, the first case of an exact filter design fitting a prescribed characteristic was given in 1929 by E. L. Norton and W. R. Bennett in U. S. patents. This work covered what is now known as a maximally flat type of response. In the late 1940's and early 1950's this method was extended to cover the Chebyshev type of response by Milton Dishal and others. In Norton's paper, the method of design starting from a prescribed insertion loss is established.

In 1939, a major advance in filter theory was made by S. Darlington in the United States, W. Cauer in Germany, and G. Cocci in Italy, when they solved the general synthesis problem for four-terminal reactor networks. They demonstrated the necessary and sufficient condition for a response to be realized by a physical network and developed a synthesis procedure for the realization of filter networks which could exhibit a prescribed response. Darlington also devised a method for precompensating the dissipative distortions.

Recently this theory has been widely used by filter specialists and scientists for the design of individual filters, and tables of design data covering a wide variety of cases have been compiled. Insertion-loss designs based on this synthesis procedure provide exact filter performance characteristics, and such designs lead to networks which exhibit desired responses with a minimum number of circuit elements.

Types of Filters

From the frequency-domain point of view, an ideal filter would be one that passes, without attenuation, all frequencies inside certain frequency limits while providing infinite attenuation for all other frequencies. The transfer function $|H(j\Omega)|$, the ratio of output to input quantities in the frequency domain for ideal filter, is shown graphically in Fig. 1.

From the time-domain point of view, an ideal filter is one whose output is identical to its input except for delay τ_o, or

$$e_{\text{out}}(t) = e_{\text{in}}(t - \tau_o)$$

Taking the Laplace transform of the above equation and looking at the transfer function in the frequency domain we obtain the ideal transfer function,

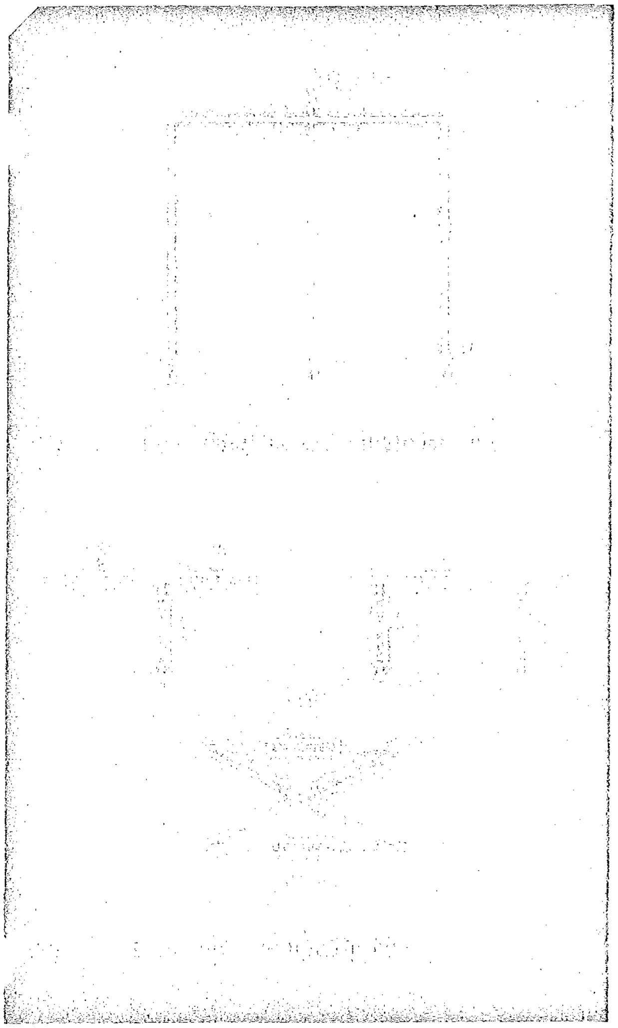
$$H(s) = e^{-\tau_o s}$$

Letting $s = j\Omega$,

$$H(j\Omega) = e^{-j\Omega\tau_o}$$

This function is not frequency-selective since it has unity amplitude; its phase decreases linearly with frequency. These conditions may be realized in practice when the delay approximates a constant for the range of frequencies over which the attenuation is small.

Electrical filters can be classified in many different ways. In terms of the frequency spectrum they are classified as audio-frequency filters, radio-frequency filters, and microwave filters. In terms of the circuit configuration of the basic elements, they may be ladder filters (in the form of T or pi) or lattice filters



(Fig. 2), the latter being the most general type of network. In terms of the character of the elements, filters may be classified as LC filters, containing lumped inductors and capacitors; RC filters, containing resistors and capacitors; transmission-line filters, containing distributed components such as stripline filters or coaxial filters; electromechanical filters such as piezoelectric filters, magnetostrictive filters, etc. If a filter circuit has an internal source of energy, it is classified as an active filter. An i-f amplifier is an example of an active filter circuit. Filters with no source of energy within the network are termed passive. A classification of passive frequency filters is shown in Fig. 3.

The following five basic types of selective networks are used for frequency discrimination in electronic equipment.

1. The low-pass filter (Fig. 4) passes the package of wave energy from zero frequency up to a certain cutoff frequency and rejects all energy beyond this limit. For example, the effective transmission of the human voice requires a frequency band ranging from near zero to 4000 cps.

2. The high-pass filter (Fig. 5) prevents the transmission of frequencies below a certain point, and appears to be electrically transparent to frequencies beyond this point. The waveguide, used at microwave frequencies, behaves as a typical high-pass filter, and usually does not pass signals below several hundred megacycles.

3. The bandpass filter (Fig. 6) passes a package of waves between certain lower and upper frequency limits, and stops all energy outside these two limits. This filter is, by far, the most important and most commonly used in electronic equipment.

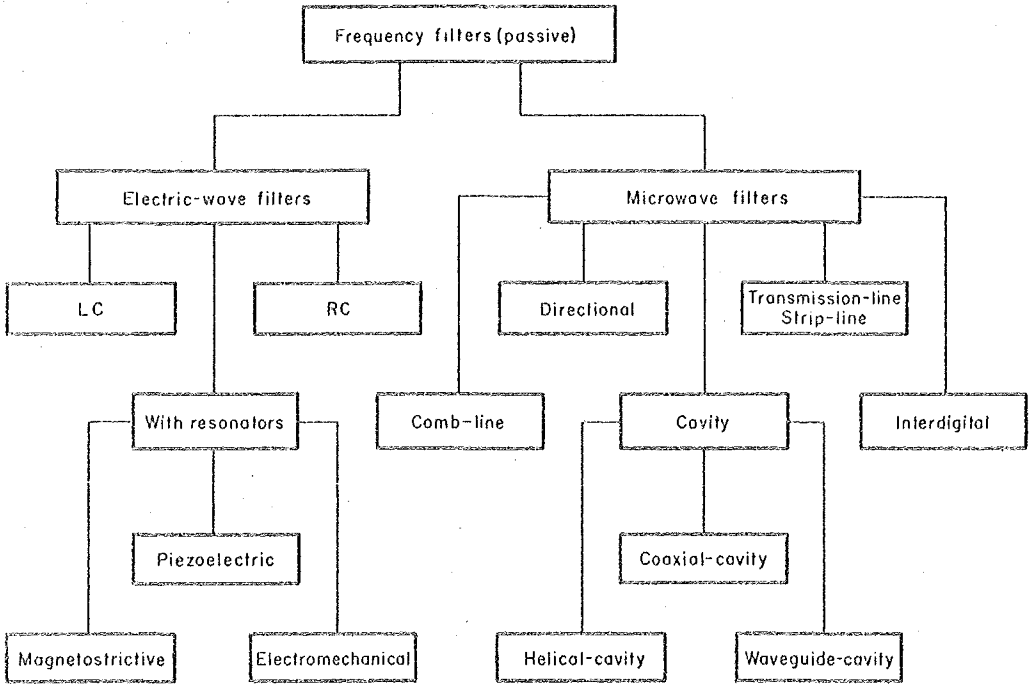


Fig. 3—Classification of passive frequency filters.

4. The band-reject filter (Fig. 7) is used in electronic equipment when a certain unwanted frequency or band of frequencies has to be rejected. Outside of the rejection band or stopband, all frequencies will pass without appreciable attenuation.

5. All-pass filters pass all frequency components of the input signal but introduce predictable phase shift for different components of the wave package. A short impulse on the input side of such a filter modifies itself into a longer frequency-modulated signal at the output. It is evident that the all-pass device can be called a filter only in a limited sense because in the frequency domain it does not discriminate between the amplitude of the various signals.

Filter Applications

The use of electric-wave filters in electronic equipment has increased as equipment has become more complex. Many subsystem operations rely upon filters.

Preselector Networks. Preselector networks are required at the input of all sensitive receivers. They separate the desired signal or signals from the unwanted signals. Because the desired signal is usually very low in amplitude, while the undesirable signals, including noise, may be of appreciably greater magnitude, the preselector network is required to have very low insertion loss for the desired signal and high attenuation for the undesirable signals. The attenuation of desirable signals reduces the signal-to-noise ratio. An increase of 1 db in insertion loss decreases the signal-to-noise ratio by almost 1 db. The preselector filter is usually tunable, covers a whole range of frequency and provides only a small part of the needed selectivity of the whole receiver.

I-F Filters. The next step of signal selection usually occurs in the i-f strip or i-f amplifier. This selection and signal discrimination is of very high quality, especially in communication receivers. The bandwidth of the i-f filter determines the quality of the system, including the ripples in the passband, the noise content, and also the sharpness of the separation between neighboring transmitted signals. Filters of this kind are usually designed in two or even three interstage blocks, separated by tubes or transistors.

SSB Filters. In contrast to the symmetrical preselector filter, the single-sideband (SSB) filter requires a nonsymmetrical attenuation response. Phase-difference networks are sometimes used instead of a filter to eliminate the unwanted sideband. In either case, the main purpose of the network is to suppress the unwanted sideband to such a degree that it does not contribute appreciably to amplitude distortion and instability in the received signal. Insufficient unwanted-sideband suppression and insufficient synchronism of the carrier frequency produce undesired beat frequencies.

Comb Filters. Where noise is prevalent or jamming is introduced, the extraction of a predetermined signal from a medium can be performed by optimum filters. In general the optimum filter is a device whose input consists of a mixture of signal and noise. The output of this filter is a signal closely approximating the desired signal. For a signal represented by a periodic series of pulses, such an optimal filter may be a comb filter consisting of a chain of narrow-band filters which pass discrete frequency components and discriminate against noise (noise usually has a continuous spectrum.) The most important application of comb filters is the extraction of Doppler-frequency-shift information for passing targets such as satellites, aircraft, and underwater missiles.

Multiplexing Filters. Filters can provide multiple use of a broad-spectrum beam between terminal stages of a radio relaying system. It is possible to create up to one thousand telephone channels in one microwave link. In the case of a wire-carrier or power-line communication network the frequency range extends from the audio band up to 200 kc. The use of coaxial lines widens the usable range of frequencies and allows more communication channels to be created. The purpose of multiplexing equipment, in general, is the channeling of voice communications, telegraph, telemetering or tele-

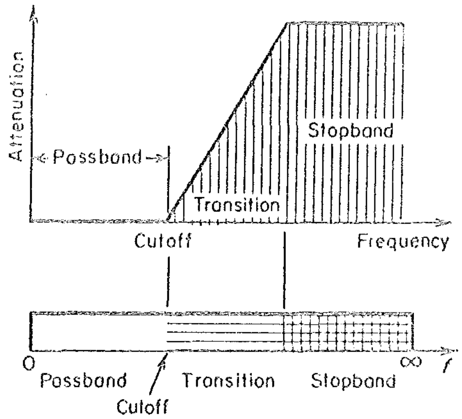


Fig. 4—Low-pass filter response.

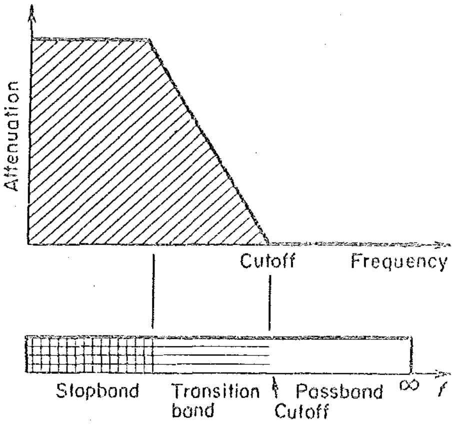


Fig. 5—High-pass filter response.

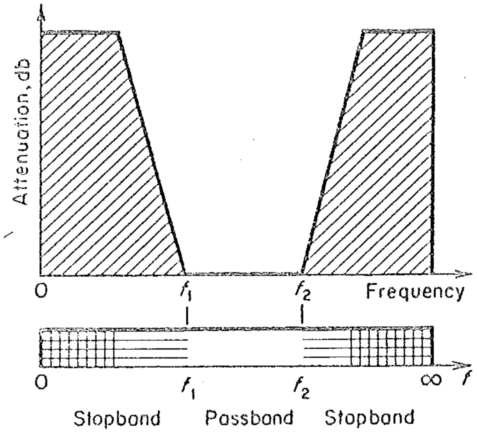


Fig. 6—Bandpass filter response.

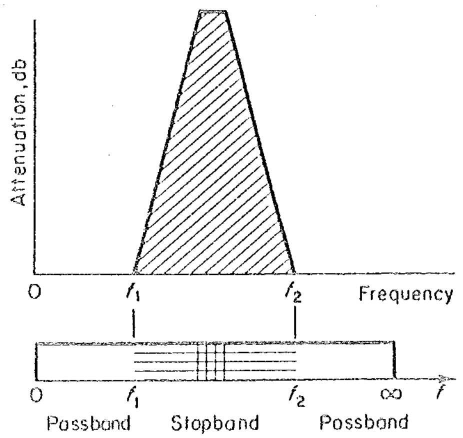


Fig. 7—Band-reject filter response.

control between distant points of road networks, pipe-line networks, power stations, etc. The major requirement of a multiplexing filter is to obtain the sharpest possible attenuation outside of the passband in order to suppress any crosstalk between the channels.

Anti-Jamming Filtering. Artificially created noise for jamming can completely destroy a radar target signal if no anti-jamming features are incorporated into a radar system. To improve target detectability, some special equipment features are needed and the narrow-band filter is the key component. The main requirement in this situation is that the filter, operating with pulsed signals, has both selectivity and the ability to minimize overshoot and ringing. To satisfy this requirement, the frequency-response curve of the filter usually has to be of a Bessel or Gaussian shape.

Matched Filters. The new science of correlation techniques and time-domain filtering is based upon the matched filter. Such filters are used for generation and detection of the famous chirp signals, which have been widely used in radar for target identification. A chirp signal is a long pulse having a frequency which changes continuously in one direction without reversal. All-pass filters are used to generate this type of pulse. Chirp signals can be used to transmit binary data since marks and spaces can be coded by corresponding ascending and descending frequency-modulated pulses.

Matched filters provide a spectrum-spreading technique, and can make effective use of any bandwidth and tolerate large distortion; they can be insensitive to noise, tones or spurious signals, resist jamming, operate with single-sideband-frequency translation or Doppler shift, reject impulse noise, provide good signal-to-noise ratio, and require no synchronization. Their areas of application include teletype, signaling field data, and various data-entry systems. Matched-filter correlation techniques are very useful in meteor-burst communication systems for minimizing the effect of multipath propagation and external interference.

Frequency Multipliers. Filters with a nonlinear reactance find application in all sorts of electronic equipment. In frequency multipliers, nonlinear elements and idlers are inserted between narrowband filters tuned to the fundamental and one of the harmonic frequencies (which has to be extracted). This arrangement can produce any signal harmonic related to the pilot source clock.

A chain of parameteric frequency multipliers can start with any low frequency which can be maintained constant.

The ideal network with one nonlinear reactor element transforms all the power of the fundamental input frequency into power at certain harmonics. Here the efficiency depends upon the familiar Q factor. Unavoidable conversion losses and the Q factor of nonlinear reactors in the harmonic generator have the same relationship as the insertion loss and Q factor of linear reactors in conventional filters.

Broadbanding Filters. A reactive network inserted between a transmitter and a narrow-band, low-frequency antenna can tremendously improve the bandwidth of the whole system. In the case of solid-state transmitters and high-speed binary transmission at low radio frequencies, broadbanding with the aid of filters is the only practical solution for many cases where already existing high-Q antennas are involved. The only consequence of this broadbanding is that the filter input impedance, with respect to the transmitter output, varies widely with frequency. This type of system imposes requirements on available power from the source; however, the efficiency of the transmitter for frequencies in the vicinity of the center frequency can still be as high as 90 per cent.

Impedance Transformation. Every bandpass filter is potentially an impedance transformer. No matter how the bandpass filter is developed, its input and output impedances can be made different from one another. The usual way to obtain the impedance transformation is to introduce Norton's ideal transformer, which consists of three elements in pi or T form. This transformation imposes some limitation on the transformation coefficient *n* (the step-down or step-up ratio). Some prototype filter sections may be transformed into sections with realizable elements for values of transformation ratio no greater than a certain maximum. The resulting network usually consists of fewer elements than the original prototype.

Filter Design Problems

Some basic relationships and definitions pertaining to filter design are given in Appendix 1.

There are two conflicting methods of filter design. The method originated by Zobel is known as the *image-parameter* method. The method originated by Norton and Bennett in 1931–32 is known as the *exact* or *polynomial-synthesis* method.

Image-Parameter Method. Image-parameter filter theory is based on the properties of a long transmission line. A simple network with lumped components is described in terms of this continuous structure. Several such elementary networks with equal characteristic terminal impedances are connected together to produce a chain of sub-networks which will possess a transmission constant equal to the sum of all the individual transmission constants of the elementary sections.

The properties of different sections are very easy to determine and for all practical cases are easy to tabulate. Calculation of the cascade of composite networks is, therefore, a relatively easy procedure.

The disadvantage of this method is that, in practice, it is the *effective* transmission constant rather than the image transmission constant that is of interest and importance. When one designs networks by the image-parameter method, the load impedance is assumed to be equal to the characteristic impedance of the network. In practice, constant-value ohmic resistors are used, rather than a variable image resistance. Many corrections will therefore be needed in order to determine the actual effective transmission coefficients. Another important disadvantage of this design method is that only a limited number of design parameters are available for choice. For example, the pole of effective attenuation can be chosen freely, but the number of zeros and their positions in the passband (which appear whenever the characteristic impedance of the filter and load impedance are equal) can then be modified only by switching from one class of impedance function to another. The choice of input-impedance class affects the class of attenuation because it changes the number of poles and zeros of the filtering function, *D*.

The fault of the image-parameter design is usually that the transmission is too flat and some discrimination is therefore lost. In practical terms this means that, for a particular level of passband ripple, only one image-parameter design is possible for each width of transition region.

The image-parameter method seems to be in disrepute among network theorists partially because of the cut-and-try method that is involved and partially because of the restricted freedom of design. The practical significance of this restriction, however, has not been adequately explored, and very little seems to have been done in applying the results of modern theory to improving older design procedure. If the Zobel design procedure could be brought up to date it might provide economical filters easy to design and adequate for most common applications.

Polynomial-Synthesis Method. The polynomial-synthesis method deals directly with effective parameters; it also provides an elegant solution to the approximation problem, but involves laborious computation for the determination of the element values. This procedure is now greatly simplified by tables, step-by-step design procedure, and design curves, but the theory behind these design tools is still in the province of the specialist. In more general synthesis procedures, a range of designs is possible in which greater stopband loss can be obtained at the expense of greater ripple factor. If, for example, a filter is designed by the polynomial method to have the same ripple as a Zobel filter of the same configuration, both filters will be identical.

Since the reflection coefficient is directly related to the insertion loss, it is usually necessary, if a low reflection coefficient is desired, to design the filter for flatter transmission than would otherwise be necessary. The polynomial filter has no practical limitation even for such an extremely severe requirement as reflection of 5 per cent or lower.

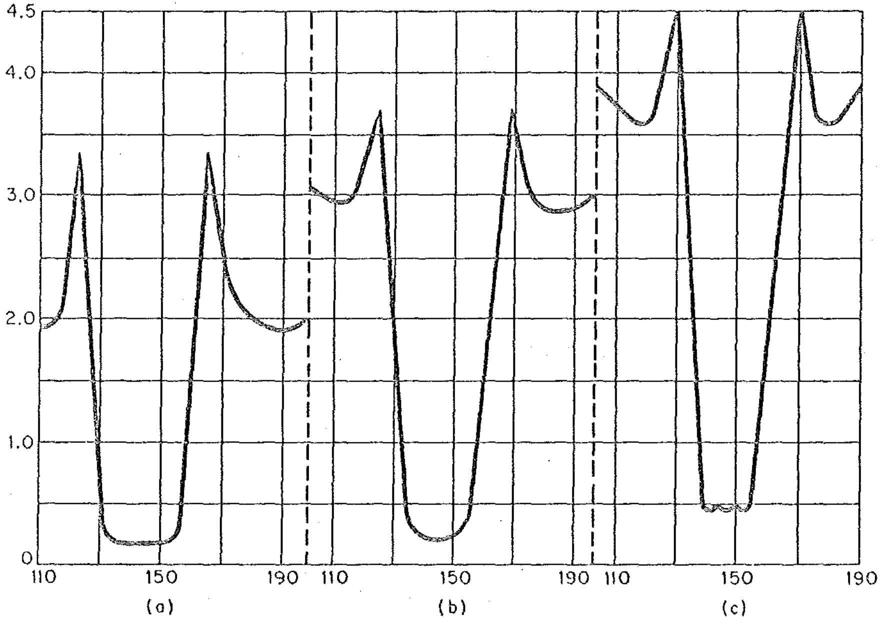
The use of the complex plane in polynomial-synthesis design is discussed in Appendix 2.

Comparison of Methods. In practice, either the synthesis or the image-parameter method of calculation may be used depending on the requirement in each specific case. In cases requiring the smallest number of elements to satisfy an exact performance specification, it is better to use the optimal design of the polynomial-synthesis method. This method certainly is worthwhile for mass production.

The synthesis procedure is always preferable when maximum stopband loss must be obtained at the expense of higher passband ripple, or where very flat transmission is needed to secure low reflections.

Figure 8(a) illustrates an attenuation curve for an *m*-derived, bandpass filter designed on the basis of image parameters. The accompanying schematic includes four coils and four capacitors and provides two peaks of attenuation outside of the passband. Figure 8(b) shows a filter of the same configuration designed according to polynomial-synthesis theory. In both cases the bandwidth and the quality factor of all components are the same. It is evident from this comparison that the second curve provides more attenuation outside of the passband. An explanation for this difference is the fact that, in the case of the ideal image-parameter filter, the bandpass region is much flatter.

As is well known, the losses add some rounding effect to the shape of the curve in the proximity of the cutoff. In most cases this rounding is very undesirable. Darlington and Nai-Ta-ming have developed an alternate design technique which is also based on polynomial synthesis, but in which the losses are taken into account and the rounding effect in the passband is effectively removed by predistortion. Figure 8(c) shows the same basic filter with the



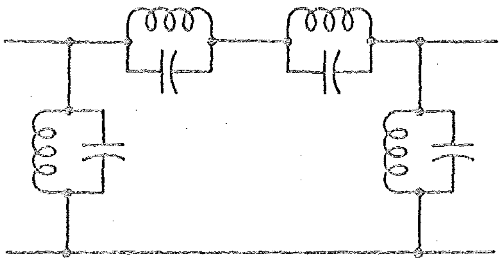


Fig. 8—Comparison of the amplitude responses of a bandpass filter designed by means of three different techniques. (a) Zobel filter, (b) polynomial filter, (c) predistorted polynomial filter.

design based on the predistorted technique. An insertion-loss ripple of 0.05 nepers or 0.43 db is present in the passband, and a certain amount of flat loss in the passband may be noted in the new design. In addition, the curve has a more rectangular attenuation shape. The difference between passband and stopband attenuation is 3 nepers, which is in accordance with the theory. From these curves the value of the predistortion technique is evident. The increase in passband attenuation of the predistorted filter does not diminish the practical value of the technique because in most applications the filter is followed by an amplifier to compensate for the loss.

Amplitude Response

There are many different shapes of amplitude-versus-frequency responses of filters which may be described by analytic functions. Also, there are established design procedures to approximate these types of response. The following are the main distinctive varieties of response.

1. Butterworth response (B) with "maximally flat" passband. Filters having this response are sometimes called power-term filters.
2. Chebyshev response (C) with "equal ripple" attenuation in the passband.
3. Inverse Chebyshev response (IC) with equal minima of attenuation in the stopband.
4. Chebyshev complete response (CC) with equal ripple attenuation in the passband and equal minima of attenuation in the stopband. Filters having this response are sometimes called eliptic integral filters or filters with Cauer parameters.
5. Gaussian response.
6. Bessel response with "maximally flat delay." Filters having this response are known as Thomson filters.
7. Equal-ripple-delay filter response.
8. Legendre filter response.
9. Synchronously tuned (multiple-pole) filter response.

The first four response types belong to the Chebyshev family.

Usually, gain is cheap while selectivity is very expensive. For this reason a tremendous effort is made to find the most rational way to design a network with the minimum number of expensive components while satisfying the fundamental requirements of attenuation, phase etc. The mathematical problem posed by these considerations consists primarily in finding a network whose transfer function fits some appropriate polynomials. In most cases the polynomials which provide the best and most economical solution are of the Chebyshev type.

The next three responses (5, 6, and 7) belong to the Gaussian family in the sense that they may all be conveniently used to satisfy phase and group-delay requirements. (See the section of this article entitled "Phase and Group Delay.") The Gaussian response is represented by the well known exponential formula. The importance of this response shape cannot be fully appreciated from the point of view of frequency discrimination since the rate of increase in attenuation for Gaussian, Bessel and equal-ripple-delay filters is very low. Gaussian-response filters should be regarded as compromise designs for pulsed systems where, except for the frequency discriminations, it is the truthful reproduction of the pulse shape that is important.

The amplitude response of an equal-ripple group-delay approximation depends upon a special design parameter, which is characterized by the height of the group-delay ripples. This filter, for a given number of poles, will produce different curves of attenuation which depend upon the magnitude of the group-delay ripples.

Figure 9 illustrates several of the typical amplitude discrimination shapes mentioned above.

In filter technology there are many other types of amplitude response which cannot be prescribed by specific parameters. The design of filters which supply such responses requires consideration of physical rather than mathematical factors. Among these filters are:

1. Single-sideband (type S), which is basically unsymmetrical (sharper on one side of the passband than on the other).

2. Extremely-narrow-bandpass filters designed to transmit virtually one frequency only.

3. Very-wideband response filters.

4. Filters with restricted or unrestricted attenuation in a restricted band or bands of frequencies.

Each of these types has different design techniques, different requirements for element values, and different component technology. An example is that of filters designed to reject unwanted frequencies on one side of the passband. Here the sharpness of the response curve on this particular side has to be much higher than on the opposite side of the passband where there are only limited attenuation requirements. The use of a symmetrical Chebyshev type for such an application would be wasteful of components and would complicate auxiliary problems such as insertion loss in the passband.

Very-narrow-band filters require an extremely high quality factor Q for resonators. Such designs use crystal resonators, electromechanical resonators, etc. Narrow-band filters are used for comb arrangements and pilot filters for SSB communication, and in modern frequency synthesizers using varactor multipliers. The quality of the filter depends exclusively upon the quality of filter components. Stability during temperature changes and under severe mechanical conditions is a factor which may have to be considered.

The Chebyshev approximation is impractical for very-narrow-bandpass filters because of the high-Q requirements for the components. Very-wideband filters, on the other hand, need high-Q resonators only when high rates of cutoff attenuation are required. Insertion loss is very low and does not present any problems for components in such wide-band applications. In order to realize bandpass filters (with a 100 per cent bandwidth, for example), a tandem combination of low- and high-pass filters is often used instead of direct bandpass synthesis. This technique can simplify the circuit and separate the problem of selectivity from the problem of insertion loss in the proximity of cutoff. It also avoids excessive requirements on inductive and other components used in the circuit.

The Chebyshev Family of Response Functions

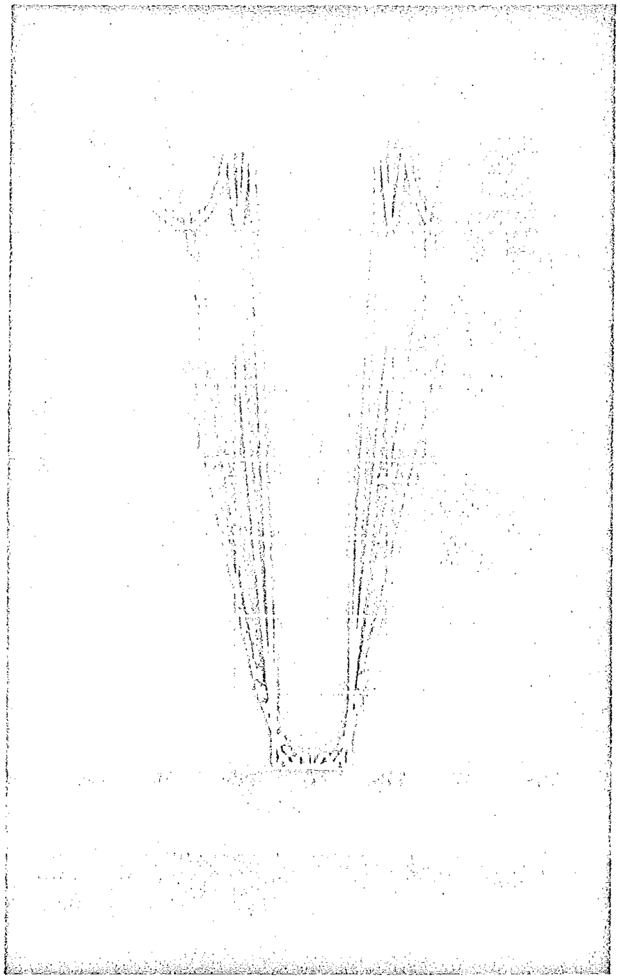
As indicated above, the Butterworth, Chebyshev, inverse Chebyshev, and Cauer-parameter filters are closely related to each other and can be developed along similar lines.

The Butterworth low-pass insertion-loss function is expressed by

$$A = 10 \log \left| 1 + \left(\frac{\omega}{\omega_c}\right)^{2n} \right| = 10 \log | 1 + \Omega^{2n} |$$
$$= 10 \log\left(1 + D^2\right) \text{ db}$$

or

$$a = \frac{1}{2} \ln(1 + \Omega^{2n}) = \frac{1}{2} \ln(1 + D^2) \text{ nepers}$$



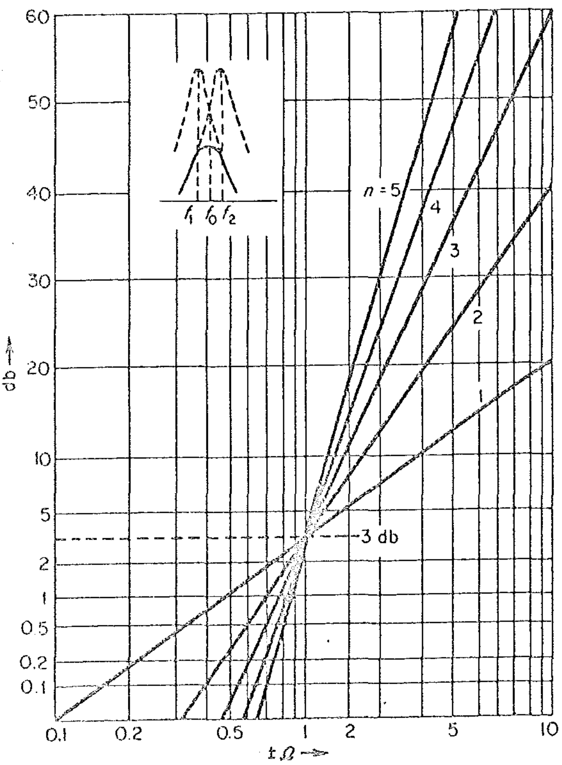


Fig. 10—Butterworth response shown on a logarithmic frequency scale $\left(\Omega = \frac{\omega}{\omega_o}\right)$.

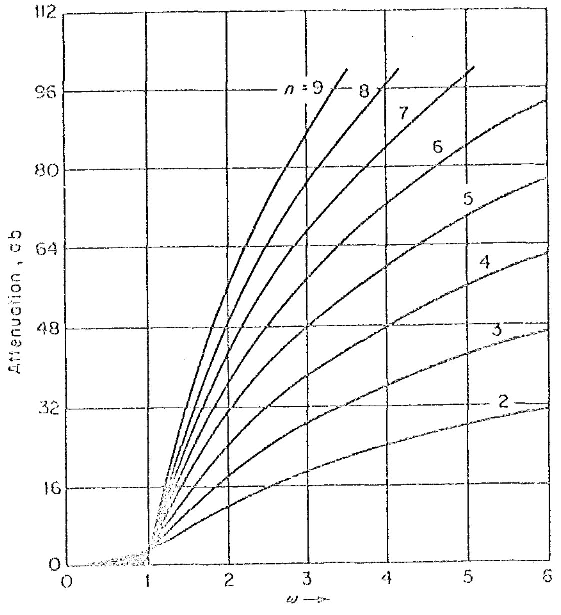


Fig. 11—Maximally flat response curves for Butterworth filters with values of n from 2 to 9.

where the filter discrimination factor $D = \Omega^n$ and the normalized frequency $\Omega = \omega/\omega_c$, the ratio of the given frequency ω to the cutoff frequency ω_c. The first ten Butterworth polynomials are given in Table I.

The insertion loss for these filters is 3 db at ω_c. The insertion-loss function has the flattest possible shape at the center of the passband and is a monotonically increasing function. Above the cutoff frequency the loss approaches a line drawn from ω_c which is increasingly linear on a logarithmic frequency scale at a rate of $6n$ db per octave. (See Fig. 10.)

Butterworth filters require n reactive elements (n appears as an exponent in the above equations). Maximally flat response curves for various values of n are given in Fig. 11. For $n = 2$ this shape corresponds to that obtained for the critically coupled condition of the familiar double-tuned circuit.

The Butterworth approximation is useful for many applications; however, its main advantage is its mathematical simplicity. The Butterworth function is unsuitable for applications which require uniform transmission of frequencies in the passband and sharp rise at cutoff. The response to a unit impulse input has overshoot which increases with increasing n, exceeding 11 per cent for $n > 4$.

Figure 12 shows an example of a low-pass filter response with an attenuation typical of the so-called Chebyshev filters where the abscissa is the normalized frequency Ω. The insertion-loss function for this response is

$$a = 10 \log |1 + \epsilon^2 C_n^2(\Omega)| = 10 \log(1 + D^2)$$

where $D = \epsilon C_n(\Omega)$, the parameter ϵ is the ripple factor and $C_n(\Omega)$ is the Chebyshev polynomial of the first kind and of order n. The first 12 orders of C_n are given in Table II. In the passband ($-1 \leq \Omega \leq 1$), the attenuation response varies between the values of zero and $a_{\max}$. The maximum passband insertion loss is $a_{\max} = 10 \log(1 + \epsilon^2)$. At frequencies slightly above Ω_c (the passband limit), the attenuation will surpass $a_{\max}$ for the first time. A transition range follows and the stopband begins with frequency Ω_s. Here the attenuation $a_{\min}$ is reached for the first time. If the filter is designed according to a Butterworth or Chebyshev polynomial, the attenuation curve will rise monotonically.

The Chebyshev response ($n = 2$) corresponds to that obtained with the overcoupled conditions of the familiar double-tuned passband circuit. In general, this shape has a number of ripples of equal height; this number is equal to the number of resonant circuits used. The equal-ripple filter for a given bandwidth has the greatest attenuation outside of the passband of any monotonic stopband or all-pole filter. The rate of increase depends not only upon the number of poles or resonators, but also upon a special design parameter, *i.e.*, the height of the ripples; the attenuation rate is higher for larger passband ripples.

The Chebyshev function is exceedingly useful in applications where the magnitude of the transfer function is of primary concern. This approximation gives more constant magnitude response throughout the passband but no improvement in decreasing the overshoot of the impulse response. The class of the Chebyshev functions is optimum in the sense that of all possible transfer functions with zeros at infinity (all-pole functions) it has the lowest complexity for yielding a prescribed maximum deviation in the passband and the fastest possible rate of cutoff outside the passband. As a consequence, the transition range for reaching a prescribed attenuation, $a_{\min}$, is a minimum, and the attenuation in the stopband is never less than this prescribed attenuation. No other polynomial possessing these optimum properties exists.

The Chebyshev polynomial includes the restriction that all the zeros of the transfer function lie in infinity. In other words, the reciprocal of the transfer function is required to be a polynomial. On the other hand, the Chebyshev rational functions are not so restricted. Rather, their transfer function takes the form

$$|Z_{21}|^2 = \frac{1}{1 + \epsilon^2 R_n^2(\Omega)}$$

and the attenuation becomes

$$\begin{aligned} a &= 10 \log | 1 + \epsilon^2 R_n^2(\Omega) | \\ &= 10 \log (1 + D^2) \end{aligned}$$

where $R_n(\Omega)$ is chosen so that a has an equiripple attenuation in the passband and the stopband. Here the filter discrimination factor $D = \epsilon R_n(\Omega)$. Depending upon whether it is even or odd, $R_n(\Omega)$ has one of two forms:

$$R_{2n}(\Omega) = \frac{A(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_3^2)\cdots(\Omega^2 - \Omega_{2n-1}^2)}{(\Omega^2 - \Omega_2^2)(\Omega^2 - \Omega_4^2)\cdots(\Omega^2 - \Omega_{2n}^2)}$$

or

$$R_{2n+1}(\Omega) = \frac{B(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_3^2)\cdots(\Omega^2 - \Omega_{2n-1}^2)}{(\Omega^2 - \Omega_2^2)(\Omega^2 - \Omega_4^2)\cdots(\Omega^2 - \Omega_{2n}^2)}$$

In the passband, $-1 \leq \Omega \leq 1$, $R_n(\Omega)$ must lie between the limits -1 and $+1$. In the stopband, $R_n(\Omega)$ should take the maximum possible absolute values for the given degree of n.

Parameters $\Omega_1 \cdots \Omega_{2n-1}$ are always in the passband, while $\Omega_2 \cdots \Omega_{2n}$ are in the stopband. Moreover, the following relation also holds:

$$\Omega_1 \Omega_2 = \Omega_{2n-1} \Omega_{2n} = \Omega_s \Omega_c$$

where Ω_c is the passband ripple bandwidth and Ω_s corresponds to the first frequency attenuated by $a_{\min}$. In other words, the poles of R_n are the reciprocals of its zeros. The integer n determines the complexity of the function; specifically, it is equal to the number of Ω^2 zeros (or poles or a suitable combination of the two types of critical frequencies) that must be specified. Because of the reciprocal relationship between the zeros and the poles of the function its value at any Ω_1 in the range $0 < \Omega < 1$ is the reciprocal of its value at $1/\Omega_1$ in the range $1 < \Omega < \infty$. If the critical frequencies can be found so that rational function has equiripples in the passband, it will automatically have equiripples in the stopband. In Fig. 12, $\Omega_c = 1$ is used as the actual cutoff frequency.

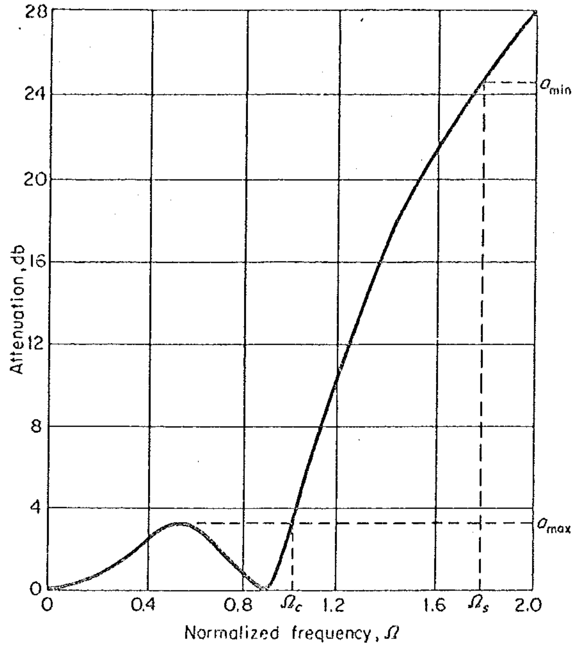


Fig. 12—Typical Chebyshev attenuation response for a low-pass filter.

[illegible]

1	$1 + s$
2	$1 + 1.4142s + s^2$
3	$1 + 2.0000s + 2.000s^2 + s^3$
4	$1 + 2.6131s + 3.4142s^2 + 2.6131s^3 + s^4$
5	$1 + 3.2361s + 5.2361s^2 + 5.2361s^3 + 3.2361s^4 + s^5$
6	$1 + 3.8637s + 7.4641s^2 + 9.1416s^3 + 7.4641s^4 + 3.8637s^5 + s^6$
7	$1 + 4.4940s + 10.0978s^2 + 14.5920s^3 + 14.5920s^4 + 10.0978s^5 + 4.4940s^6 + s^7$
8	$1 + 5.1528s + 13.1371s^2 + 21.8462s^3 + 25.6884s^4 + 21.8462s^5 + 13.1371s^6 + 5.1528s^7 + s^8$
9	$1 + 5.7588s + 16.5817s^2 + 31.1634s^3 + 41.9864s^4 + 41.9864s^5 + 31.1634s^6 + 16.5817s^7 + 5.7588s^8 + s^9$
10	$1 + 6.3925s + 20.4317s^2 + 42.8021s^3 + 64.8824s^4 + 74.2334s^5 + 64.8824s^6 + 42.8021s^7 + 20.4317s^8 + 6.3925s^9 + s^{10}$

[illegible]

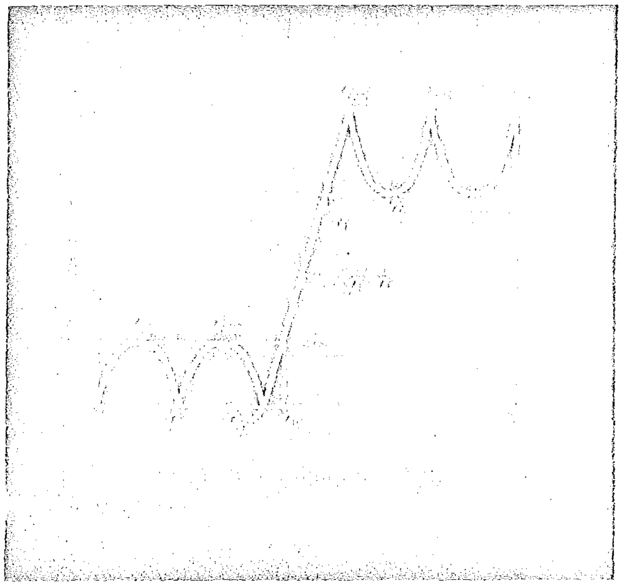
C_0	1
C_1	Ω
C_2	$2\Omega^2 - 1$
C_3	$4\Omega^3 - 3\Omega$
C_4	$8\Omega^4 - 8\Omega^2 + 1$
C_5	$16\Omega^5 - 20\Omega^3 + 5\Omega$
C_6	$32\Omega^6 - 48\Omega^4 + 18\Omega^2 - 1$
C_7	$64\Omega^7 - 112\Omega^5 + 56\Omega^3 - 7\Omega$
C_8	$128\Omega^8 - 256\Omega^6 + 160\Omega^4 - 32\Omega^2 + 1$
C_9	$256\Omega^9 - 576\Omega^7 + 432\Omega^5 - 120\Omega^3 + 9\Omega$
C_{10}	$512\Omega^{10} - 1280\Omega^8 + 1120\Omega^6 - 400\Omega^4 + 500\Omega^2 - 1$
C_{11}	$1024\Omega^{11} - 2816\Omega^9 + 2816\Omega^7 - 1232\Omega^5 + 220\Omega^3 - 11\Omega$
C_{12}	$2048\Omega^{12} - 6144\Omega^{10} + 6912\Omega^8 - 3584\Omega^6 + 840\Omega^4 - 72\Omega^2 + 1$

From the preceding discussion it is clear that there are four parameters involved in the specification of the optimum filter:

1) the maximum attenuation in the passband, a_{max};
2) the maximum gain in the stopband (or minimum attenuation, a_{min});
3) the transition interval; and
4) the complexity of the function denoted by n.

Any three of these may be given in a practical problem. Minimizing the fourth is automatically accomplished by use of the appropriate Chebyshev rational function.

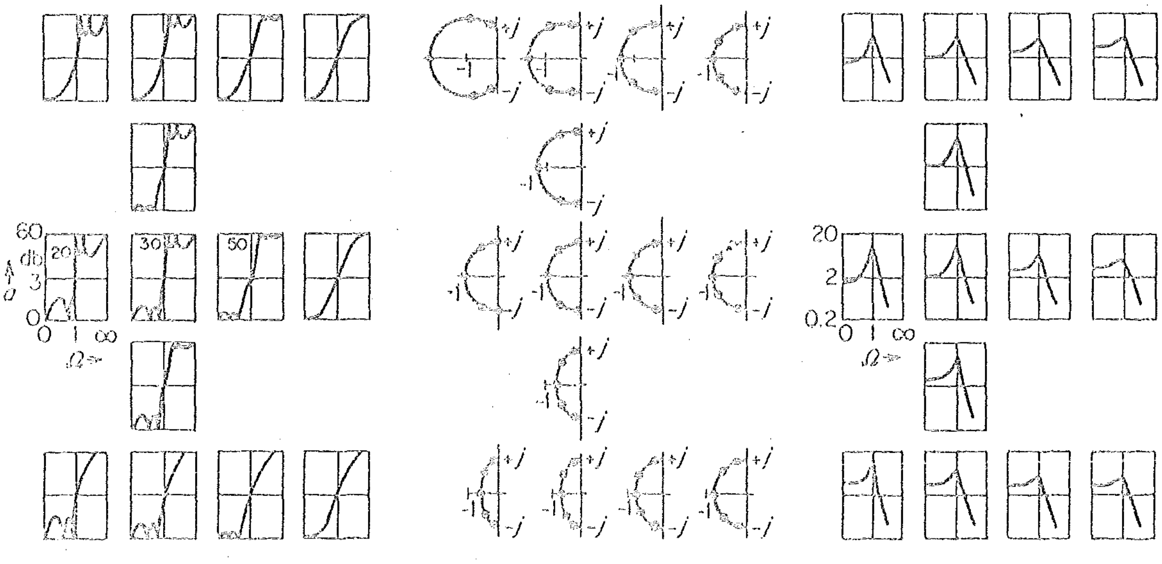
The universal normalized Chebyshev response is shown in Fig. 13. The transition interval between passband and stopband is represented by Δa, which can be simply related to the transmission function. All design parameters, such as value and position of a_{max} and a_{min}, as well as frequencies Ω_c and Ω_s are under the control of the designer.



If the stopband attenuation a_{min} is increased to infinity, the value of Ω_s will then have to go to infinity and the defined restricted stopband will be compressed and moved to infinite frequency, in which case the Chebyshev rational function reduces to a Chebyshev polynomial. To design a low-pass prototype filter which exhibits any response of the Chebyshev family one need not go through the complete synthesis procedure. Tables of the element values for normalized low-pass filters are available. The properties of the Chebyshev filters are tabulated in Fig. 14. The responses for inverse Chebyshev filters are shown in the first row of Fig. 14(a). The corresponding locus of the transfer-function poles, shown in the first row of Fig. 14(b), changes from the deformed half circle to the half circle which is characteristic of Butterworth filters.

In the middle row of Fig. 14, the responses for filters with Chebyshev approximation in the passband and stopband are shown. The amplitudes of ripples and amount of attenuation guaranteed in the stopband are changing in definite steps from one extreme value to another. The corresponding transfer-function poles remain on a circle.

In intermediate cases when the ripples in the passband and ripples in the stopband are arbitrary (more ripple may be accepted in the passband in order to get more attenuation in the stopband), the locus of poles is deformed into an ellipse.



(a) (b) (c)

Fig. 14—Properties of the family of Chebyshev filters: (a) attenuation shapes; (b) locus of the POLES zeros of transmission; (c) normalized group delay versus Ω. Each attenuation shape in (a) corresponds to a similarly [illegible]-of-transmission locus in (b) and to a group-delay characteristic in (c).

In the case of an arbitrary design with small ripples in the passband and guaranteed minimum in the stopband, the half-circle locus is deformed as shown in the second row of Fig. 14. In this case, the half axis of the "circle" along the real negative direction is longer than the axis along the imaginary direction.

The last row of Fig. 14 illustrates the degeneration of typical Chebyshev responses when limiting conditions are gradually applied. The filter with no ripples in the passband and no poles of attenuation outside of it degenerates into a power-term filter with maximally flat amplitude response and with the zeros of filtering located on a semicircle.

Phase and Group Delay

A specific amplitude response, given by a and D vs frequency, does not describe the complete transmission property of a filter. Indeed, the filter discrimination factor is somewhat superficial because it describes only the gain and loss of the network. The phase characteristic which is included in the effective transmission factor, H, becomes very important when considering radar and communication systems, especially those using pulsed signals.

The plot of zeros and poles for the Chebyshev family of filters indicates that when a curve enters into the transition frequency region, it is sharper when the transfer-function poles are concentrated closer to the imaginary s-plane axis. In the proximity of the poles the effective phase angle changes more rapidly and the group time delay has high peaks.

A flat group delay is desirable because this signifies that all frequencies will be delayed the same amount while going through the filter. If the various frequencies are not delayed equally, dispersion results, and the output for a pulsed input does not retain its identity.

Figure 14(c) shows curves of the normalized group delay; it may be observed that relatively constant group delay in the passband can be reached only when one avoids using sharp attenuation curves. A transmission function of the Chebyshev family is thus optimum only from the point of view of the attenuation requirement but not from the point of view of the group delay.

The phase response of the ideal Gaussian filter is linear, and no overshoot will be produced as a result of rapid signal changes. Realizable all-pole Gaussian magnitude filters with a finite number of elements yield nonsymmetrical pulse responses because the phase is not sufficiently linear. The trailing edge of the output pulse is longer than the leading edge.

The phase response in finite-pole Bessel filters is more linear than that of the Gaussian magnitude type with an equal number of poles. The skirt of selectivity of the passband is sharper for the Bessel filter, but the attenuation cutoff for both filters is not very great. Both types have very poor attenuation characteristics (especially with a wide passband), but from the point of view of group-delay distortion or phase characteristics their pulse responses, in comparison with Chebyshev or Butterworth, are remarkably good.

For purposes of comparison, the group delay of Bessel, Gaussian, Chebyshev, and Butterworth filters of second degree are plotted in Figs. 15, 16, 17 and 18. It should be noted that the group delay of the Bessel filter is flat at the center of the passband, while the Gaussian curve drops far off center frequencies and the Butterworth has a pronounced peak at the cutoff frequency.

Although the Bessel filter phase characteristic is greatly superior to that of the Chebyshev filter, its disadvantages preclude its use in most cases. The Bessel insertion-loss response in decibels versus frequency approximates a parabolic curve, so that its

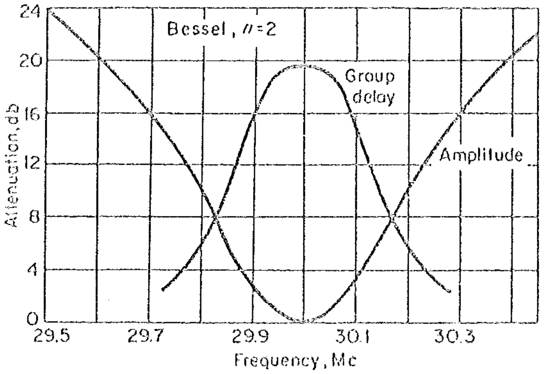


Fig. 15—Bessel amplitude and group-delay response.

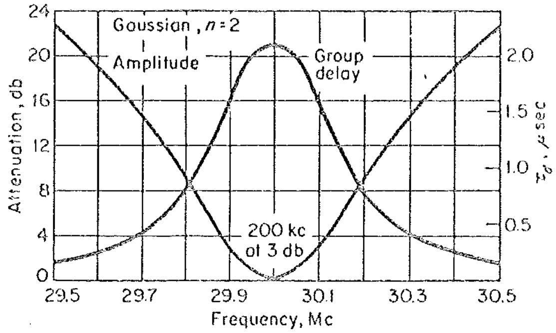


Fig. 16—Gaussian amplitude and group-delay response.

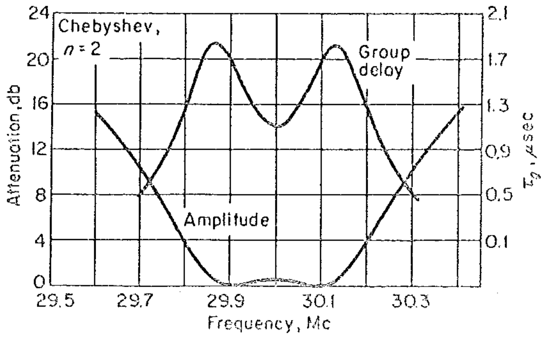


Fig. 17—Chebyshev amplitude and group-delay response.

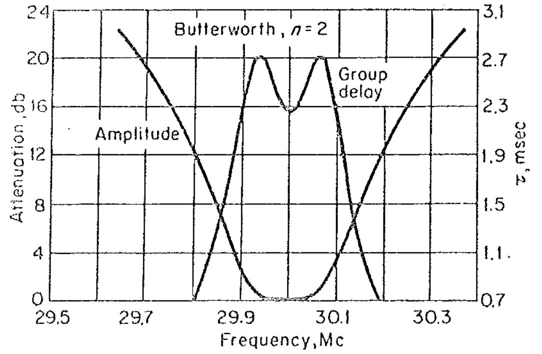


Fig. 18—Butterworth amplitude and group-delay response.

voltage standing-wave ratio (VSWR) increases rapidly as the frequency deviates from center frequency. Over most of the 3-db bandwidth the VSWR and reflection coefficient are high. As a result, reflection interactions with a slightly mismatched load or generator may seriously affect the otherwise good phase-shift response of this type of filter. For a given selectivity, Bessel filters need more elements than the Chebyshev type. In fact, no matter how many elements are used, this type of filter cannot provide appreciably greater selectivity than a parabolic insertion-loss response curve. The multiple resonator structure for Bessel filters is highly unsymmetrical, with a very large variation in the coupling elements from one end to the other. This large variation makes the design difficult, and values of the couplings are more likely to be in error relative to each other.

The impulse response of an equal-ripple delay filter is quite symmetrical relative to the mean value of group delay. The impulse response of a Bessel filter is not as symmetrical as that of the equal-ripple filter, although it shows more symmetry around the center, and that of the Gaussian filter, even for higher-order approximations, is the least symmetrical of the three. A plot of the impulse responses for these three filters ($n = 3$) is given in Fig. 19. For a larger number of poles (more complicated filters), the impulse responses become more and more similar, but with increasing complexity the equal-ripple delay response gives a more symmetrical pulse response. In the case of the equal-group-delay filter, the estimated overshoot will be comparable to the Bessel filter, but the steady-state amplitude response is somewhat better.

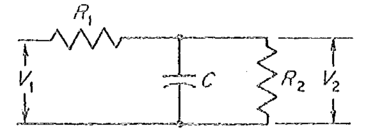
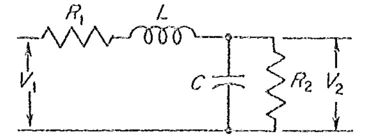
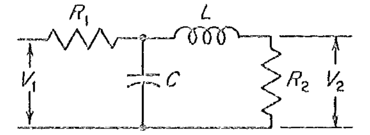
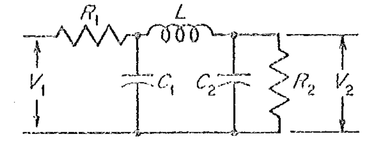
Simple Polynomial Filters. The conditions for Butterworth and Chebyshev response for the simple polynomial filters shown in Table III are derived in Appendix 3.

Zobel Filters. In image-parameter theory, the insertion loss is usually computed as the sum of the image attenuation, the reflection loss and the interaction loss. This method is useful for computing the loss of a filter whose constants have been determined and whose design is complete. It is not a convenient method for showing how the insertion loss varies with the choice of parameters. It is possible, however, to obtain complete insertion-loss formulas for multi-section image-parameter filters in a form suitable for study. A discussion of attenuation of Zobel filters is given in Appendix 4.

Coupling

Neither image-parameter theory nor modern synthesis theory requires a specific concept of coupling in order to design the filter or to explain its physical operation. So-called "half sections" or "full sections" can be connected together if they have ap-

Table III—Simplest Polynomial Filters

Order	Network	Transfer Function
First		$1 + j\omega C \frac{R_1R_2}{R_1+R_2} = 1 + j\omega CR$
Second		$1 + j\omega\left(CR_p + \frac{L}{R_s}\right) - \omega^2 LCK$
Second		$1 + j\omega\left(CR_p + \frac{L}{R_s}\right) - \omega^2 LCK'$
Third		$1 + j\omega\left[(C_1+C_2)R_p + \frac{L}{R_s}\right] - \omega^2\frac{L}{R_s}(C_1R_1 + C_2R_2)$ $-j\omega^3 LC_1C_2R_p$

propriate characteristic impedances. In a discussion of the polynomial filter, the term coupling loses its sense because filter components are a result of the basic design procedure. Coupling in modern filters refers only to the parasitic effects of one component upon another. The electronic engineer sometimes uses the degree of coupling to describe the particular response as over-coupled, under-coupled, critically or optimally coupled, and transitionally coupled, but even these descriptive terms are being replaced by Chebyshev or Butterworth responses.

The term coupling was re-introduced in network synthesis theory by Milton Dishal. There is no physical coupling between coil and capacitor in the sense of connection, but the numerical value of a coupling term has an effect on the bandwidth similar to that of the familiar coupling coefficient. The element values for ladder networks may be expressed so that every reactive component of each element of the filter is related to the reactive component of the immediately preceding element and to a definite bandwidth (such as the 3-db down value.) The numerical results are the normalized coefficients of coupling. Structures which may be normalized in terms of coupling include not only the bandpass configuration, where coupling is used in its original physical sense, but also the low-pass, high-pass, and band-reject structures. For the low-pass filter of Fig. 20, coupling is defined by the ratios

$$\frac{\Omega_{12}}{\Omega_{3\text{db}}} = k_{12}; \quad \frac{\Omega_{23}}{\Omega_{3\text{db}}} = k_{23}; \quad \frac{\Omega_{34}}{\Omega_{3\text{db}}} = k_{34}$$

where

$$\Omega_{12} = \frac{1}{\sqrt{C_1 L_2}}; \quad \Omega_{23} = \frac{1}{\sqrt{L_2 C_3}}; \quad \Omega_{34} = \frac{1}{\sqrt{C_3 L_4}}$$

In most high-frequency designs, especially the microwave type, the use of the normalized value of k may simplify the adjustment procedure since the numerical value of k could be directly applied to the tuning of the actual filter.

Coefficient of Coupling. For a magnetically resonant circuit, the value of the coefficient of coupling can be obtained from the theory of transformers:

$$k = \frac{M}{\sqrt{L_p L_s}}$$

Analogously, one can define the coefficient of coupling for the four other types of two-pole filters mentioned above.

In general, the degree of coupling or the coupling coefficient is the ratio of the reactance of the coupling element to the geometric mean of the open-circuit impedances from opposite terminals.

The coupling, K, will now be defined as

$$K = \frac{y}{\sqrt{g_1 g_2}}$$

where y is the coupling admittance and g_1, g_2 are the input and output short-circuit admittances of the filter at resonance frequency. From

$$\frac{g_1}{\omega_m C_I} = d_1 = \frac{1}{Q_1}; \quad \frac{g_2}{\omega_m C_{II}} = d_2 = \frac{1}{Q_2}$$

it may be seen that

$$K = \frac{k}{\sqrt{d_1 d_2}} = k\sqrt{Q_1 Q_2}$$

Multi-Pole Filters

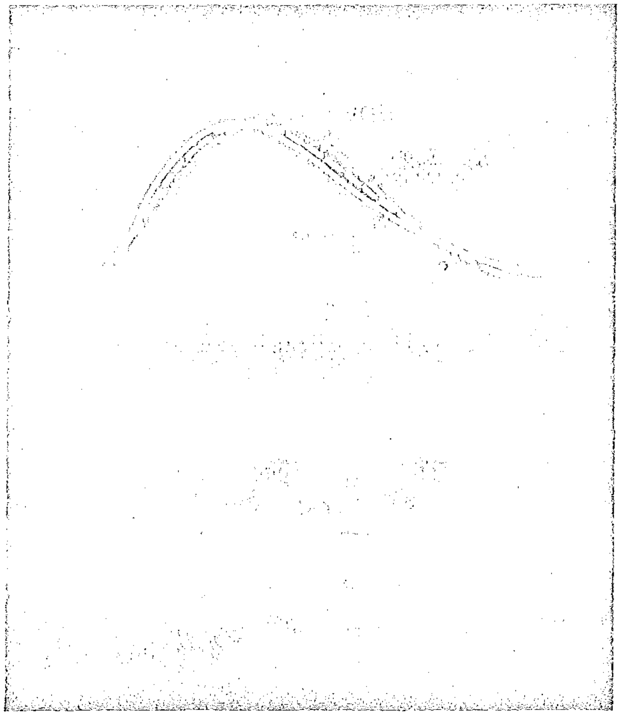
The design of multi-pole filters generally follows either of two different methods. In the synchronously tuned method, all the resonant circuits are tuned to the desired center frequency at each node. The symmetrically detuned method (staggered tuning) has one or more resonant circuits detuned symmetrically about center frequency. Strong coupling is equivalent to strong resonant detuning.

Both methods may arrive at the same amplitude-frequency response, but the absolute value of the output at center frequency is generally greater for the synchronously tuned filter. This is the reason for realizing only synchronously tuned filters in practice, especially when the network consists of more than two poles.

In order to understand the physical concept of coupling and selectivity, it is worthwhile to concentrate one's attention on simple bandpass filters in the form of a double-tuned circuit such as that generally used between amplifier stages and similar circuits.

A two-pole bandpass filter can produce the same amplitude-frequency response as a two-stage tuned amplifier with a single resonant circuit in each stage. Moreover, the response of several stages of single-tuned circuits in cascade can be replaced by a single filter with an equal number of resonant circuits, which may be coupled by either capacitive or inductive reactances; the method of coupling may alternate from each resonant circuit to the next as required by the situation.

Two basic assumptions will be used here. First, we will assume that coupling reactance is independent of frequency; the change in reactance is so small that, within the limits of the frequencies wherein the



filter is operating, the change can be completely disregarded. Secondly, we will assume that each resonant circuit is coupled only with the circuit adjacent to it.

The general schematic of the filter with two antiresonant circuits is shown in Fig. 21. It can be shown that five different ways of coupling produce essentially the same responses and that all of these may be represented by the schematic of Fig. 21.

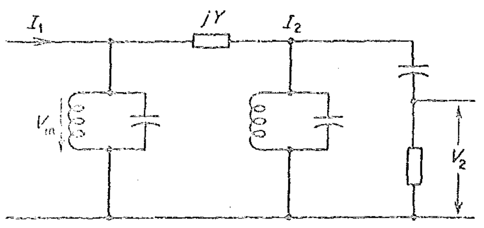
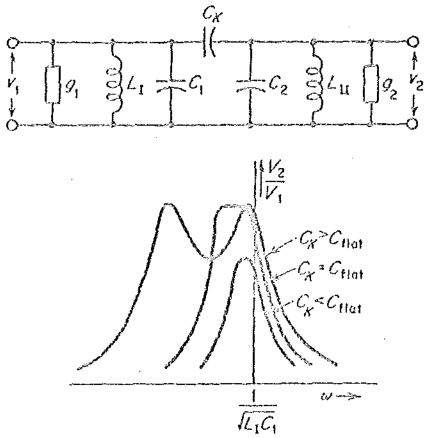


Fig. 21—General representation of a two-pole filter.



$$C_1 = C_I - C_K \approx C_I \qquad C_I = C_1 + C_K \approx C_1$$

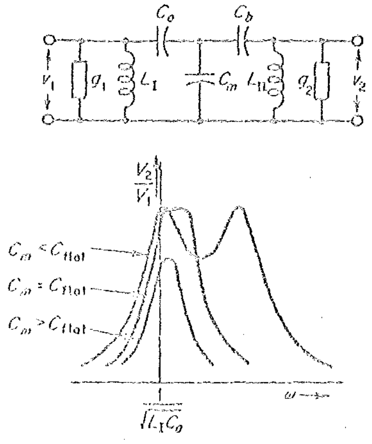
$$C_2 = C_{II} - C_K \approx C_{II} \qquad C_{II} = C_2 + C_K \approx C_2$$

$$g_1 = d_1 \omega_m C_I \qquad k^2 = \frac{C_K^2}{(C_1 + C_K)(C_2 + C_K)} = \frac{C_K^2}{C_I C_{II}}$$

$$g_2 = d_2 \omega_m C_{II}$$

$$C_K = \frac{y}{\omega_m} \qquad y = \omega_m C_K$$

Fig. 22—Two-pole filter with series capacitive coupling.



$$C_a = \frac{C_I C_{II} - C_K^2}{C_{II} - C_K} \approx C_I \left(1 + \frac{C_K}{C_{II}}\right) \approx C_I \qquad C_I = \frac{C_a(C_b + C_m)}{C_a + C_b + C_m} \approx C_a$$

$$C_b = \frac{C_I C_{II} - C_K^2}{C_I - C_K} \approx C_{II}\left(1 + \frac{C_K}{C_I}\right) \approx C_{II} \qquad C_{II} = \frac{C_b(C_a + C_m)}{C_a + C_b + C_m} \approx C_b$$

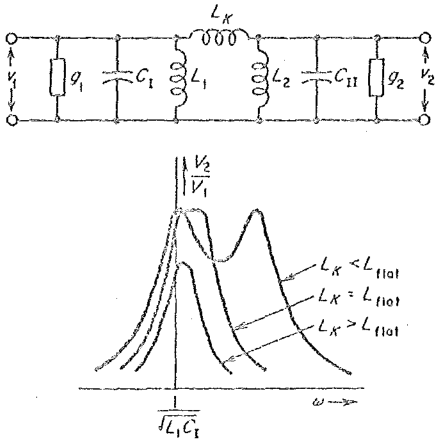
$$C_m = \frac{C_I C_{II} - C_K^2}{C_K} \approx \frac{C_I C_{II}}{C_K} \qquad y = \omega_m \frac{C_a C_b}{C_a + C_b + C_m} \approx \omega_m \frac{C_a C_b}{C_m}$$

$$g_1 = d_1 \omega_m C_I$$

$$g_2 = d_2 \omega_m C_{II} \qquad k^2 = \frac{C_a C_b}{(C_a + C_m)(C_b + C_m)} \approx \frac{C_a C_b}{C_m^2}$$

$$C_K = \frac{y}{\omega_m}$$

Fig. 23—Two-pole filter with shunt capacitive coupling.



$$L_1 = \frac{L_K L_I}{L_K - L_I} \approx L_I \left(1 + \frac{L_I}{L_K}\right) \approx L_I \qquad L_I = \frac{L_1 L_K}{L_1 + L_K} \approx L_1$$

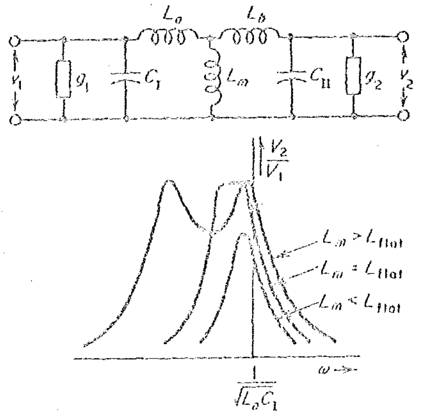
$$L_2 = \frac{L_K L_{II}}{L_K - L_{II}} \approx L_{II} \left(1 + \frac{L_{II}}{L_K}\right) \approx L_{II} \qquad L_{II} = \frac{L_2 L_K}{L_2 + L_K} \approx L_2$$

$$g_1 = d_1 \omega_m C_I \qquad k^2 = \frac{L_1 L_2}{(L_1 + L_K)(L_2 + L_K)} \approx \frac{L_1 L_2}{L_K^2}$$

$$g_2 = d_2 \omega_m C_{II}$$

$$L_K = -\frac{1}{\omega_m y} \qquad y = -\frac{1}{\omega_m L_K}$$

Fig. 24—Two-pole filter with series inductive coupling.



$$L_a = \frac{L_K L_I (L_K - L_{II})}{L_K^2 - L_I L_{II}} \approx L_I - L_m \approx L_I \qquad L_I = \frac{L_a L_b + L_b L_m + L_m L_a}{L_b + L_m} \approx L_a$$

$$L_b = \frac{L_K L_{II}(L_K - L_I)}{L_K^2 - L_I L_{II}} \approx L_{II} - L_m \approx L_{II} \qquad L_{II} = \frac{L_a L_b + L_b L_m + L_m L_a}{L_a + L_m} \approx L_b$$

$$L_m = \frac{L_K L_I L_{II}}{L_K^2 - L_I L_{II}} \approx \frac{L_I L_{II}}{L_K} \qquad k^2 = \frac{L_m^2}{(L_m + L_a)(L_m + L_b)} \approx \frac{L_m^2}{L_a L_b}$$

$$g_1 = d_1 \omega_m C_I$$

$$g_2 = d_2 \omega_m C_{II} \qquad \omega_m y = \frac{-L_m}{L_a L_b + L_b L_m + L_m L_a} \approx \frac{L_m}{L_a L_b}$$

$$L_K = -\frac{1}{\omega_m y}$$

Fig. 25—Two-pole filter with shunt inductive coupling.

Figs. 22 and 23 illustrate methods of capacitively coupling two resonant circuits. In Figs. 24 and 25, methods of inductively coupling the same circuits are shown. The three capacitors in Fig. 22 can be transformed into the T configuration of Fig. 23 by using the well known delta-star transformation. A similar transformation may be used for the inductive coupling networks in Figs. 24 and 25. Formulas to facilitate such transformations are shown in Fig. 26.

A two-pole filter with magnetic coupling between the first and second resonant circuits is shown in Fig. 27. From transformer theory, we know that two different values of the coupling coefficient can result. This results from the fact that the sign for mutual inductance, M, can be positive or negative and depends upon the polarity of the secondary winding. The value of y in Fig. 21 can therefore be either positive or negative. Very often, instead of pure inductive, pure capacitive, or even pure magnetic coupling, we have a mixture of couplings. This may be due to such factors as the presence of unavoidable distributed capacitance between the coils. Even in such cases, the general equivalent schematic in Fig. 21 is still valid since one type of reactance will generally predominate in each specific frequency range. When y is resonant in the proximity of the passband, one no longer has a two-pole filter, and this case will therefore not be considered here.

Formulas are shown in Figs. 22, 23, 24, 25, and 27 for the calculation of element values. Also shown are the corresponding amplitude-frequency response curves. From these curves one can see in which direction the passband deviates from resonance. The effects of coupling on the center frequency and insertion loss of the filter are also evident. With increased value of coupling, insertion loss becomes lower. In the proximity of critical coupling, the bandwidth becomes larger and, finally, with a further increase in value of coupling, ripples in the passband will appear. In the case of the magnetic coupling shown in Fig. 27, the center frequency remains constant. In the case of the capacitive coupling of Fig. 22 and the parallel inductive coupling of Fig. 25, the center frequency goes down, and in the two remaining cases it goes up.

Filter Transformations

As was previously indicated, low-pass filters, through appropriate frequency and network transformations, can provide information for bandpass, high-pass and band-reject filters.

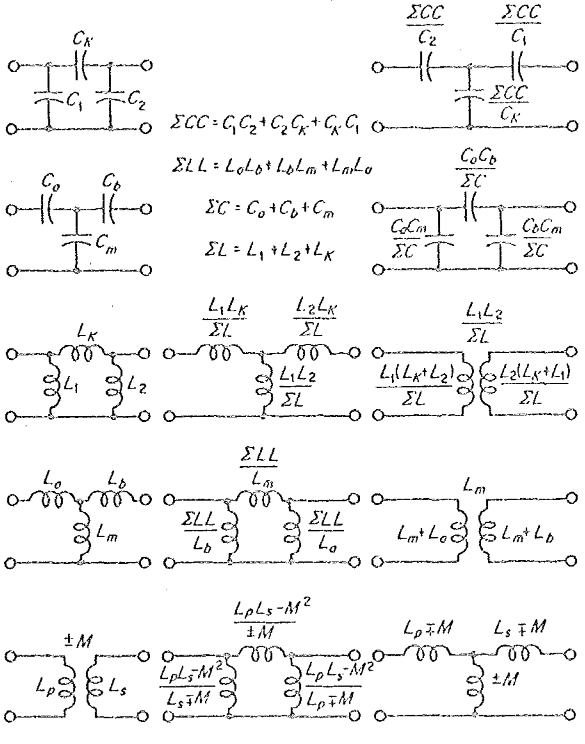
To transform low-pass filters into high-pass filters, it is necessary to use the frequency transformation

$$\Omega_{\rm HP} = \frac{1}{\Omega_{\rm LP}}$$

The normalized cutoff frequency of the high-pass filter is equal to that of the low-pass filter.

$$\Omega_{c({\rm HP})} = \Omega_{c({\rm LP})} = 1$$

The resulting transformation of elements is given in the first two lines of Table IV.



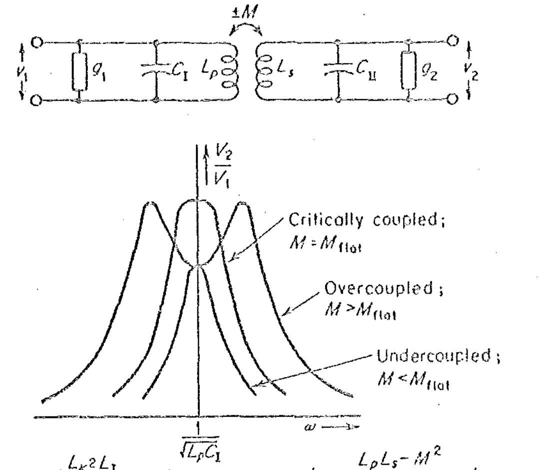
$$\Sigma CC = C_1C_2+C_2C_K+C_KC_1$$

$$\Sigma LL = L_oL_b+L_bL_m+L_mL_o$$

$$\Sigma C = C_o+C_b+C_m$$

$$\Sigma L = L_1+L_2+L_K$$

Fig. 26—Equivalent schematics for various coupling configurations. Element values are given in a form suitable for performing transformations from one configuration to another.



$$L_p = \frac{L_K{}^2L_{\rm I}}{L_K{}^2-L_{\rm I}L_{\rm II}} \approx L_{\rm I} \qquad L_{\rm I} = \frac{L_pL_s-M^2}{L_s} \approx L_p$$

$$L_s = \frac{L_K{}^2L_{\rm II}}{L_K{}^2-L_{\rm I}L_{\rm II}} \approx L_{\rm II} \qquad L_{\rm II} = \frac{L_pL_s-M^2}{L_p} \approx L_s$$

$$\pm M = \frac{L_KL_{\rm I}L_{\rm II}}{L_K{}^2-L_{\rm I}L_{\rm II}} \approx \frac{L_{\rm I}L_{\rm II}}{L_K} \qquad k^2 = \frac{M^2}{L_pL_s}$$

$$g_1 = d_1\omega_mC_{\rm I}$$

$$\omega_mY = \frac{\mp M}{L_pL_s-M^2} \approx \frac{\mp M}{L_pL_s}$$

$$g_2 = d_2\omega_mC_{\rm II}$$

$$\pm L_K = \frac{1}{\omega_mY}$$

Fig. 27—Two-pole filter with magnetic coupling.

With the frequency transformation

$$\Omega = a\left(\gamma - \frac{1}{\gamma}\right)$$

where γ is the new normalized frequency, the frequency characteristics of the low-pass filter will be transformed into bandpass characteristics. The normalized frequency of the low-pass filter, Ω_ν, corresponds to two frequencies in the bandpass characteristics γ_ν and $\frac{1}{\gamma_\nu} = \gamma_{-\nu}$, which are geometrically symmetric since

$$\gamma = 1 = \gamma_\nu \gamma_{-\nu}$$

The transformation of the cutoff frequency can be made with the use of the expression

$$\Omega_c = 1 = a(\gamma_c - \gamma_{-c})$$

from which the constant of transformation, a, can be determined.

If the upper and lower cutoff frequencies f_c and f_{-c} are normalized with respect to the reference frequency f_b, where

$$f_b = \sqrt{f_c f_{-c}}$$

then

$$\gamma_c = \frac{f_c}{f_b} = \sqrt{\frac{f_c}{f_{-c}}} = \frac{1}{\gamma_{-c}}$$

The normalized frequencies in the stopband such as Ω_s can be obtained in a similar way,

$$\Omega_s = a(\gamma_s - \gamma_{-s})$$

and the relationship between the passband and the stopband limits is

$$\Omega_s = \frac{\gamma_s - \gamma_{-s}}{\gamma_c - \gamma_{-c}}$$

The normalized bandpass frequency γ_ν can be found from values of Ω_ν belonging to the low-pass filter

$$\gamma_{\pm\nu} = \pm\frac{\Omega_\nu}{2a} + \sqrt{\left(\frac{\Omega_\nu}{2a}\right)^2}$$

Low-pass to bandpass network-element transformations are summarized in Table IV.

Similar relationships are valid for the low-pass to bandstop transformation. They are given by the equations

$$\Omega = \frac{1}{a\left(\gamma - \frac{1}{\gamma}\right)}$$

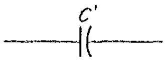
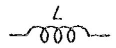
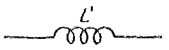
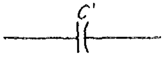
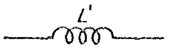
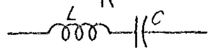
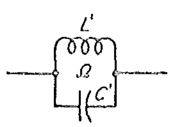
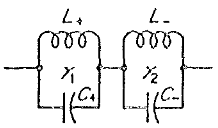
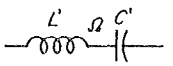
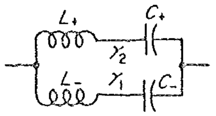
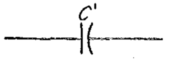
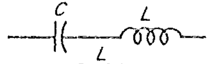
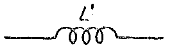
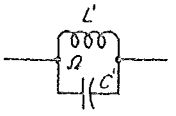
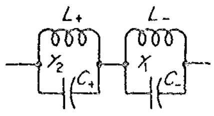
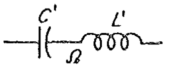
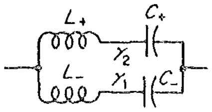
and

$$\gamma_{\pm\nu} = \pm\frac{1}{2a\Omega_\nu} + \sqrt{\left(\frac{1}{2a\Omega_\nu}\right)^2}$$

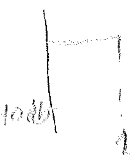
where $\nu = 1, 2$. The set of formulas and the elements involved in this transformation are given in Table IV.

The network components given in Table IV, obtained after transformation, and also the normalized frequencies Ω_ν, γ_ν, may be used to translate a network to a desired frequency range (by choice of

Table IV—Filter Transformations

Transformation	Low-pass		Schematic	Value
Low-pass to high-pass		→		$L = \frac{1}{C'}$
		→		$C = \frac{1}{L'}$
Low-pass to bandpass		→		$C = \frac{1}{L} = aC' \quad \gamma = 1$
		→		$L = \frac{1}{C} = aL' \quad \gamma = 1$
		→		$C_+ = \frac{1}{L_-} = aC'(1 - \gamma_1^2)$ $C_- = \frac{1}{L_+} = aC'(1 + \gamma_1^2)$
		→		$L_+ = \frac{1}{C_-} = aL'(1 + \gamma_1^2)$ $L_- = aL'(1 + \gamma_2^2)$ $\gamma_{2,1} = \sqrt{\left(\frac{\Omega}{2a}\right)^2 + 1} \pm \frac{\Omega}{2a}$
Low-pass to bandstop		→		$C = \frac{1}{L} = \frac{C'}{a} \quad \gamma = 1$
		→		$L = \frac{1}{C} = \frac{L'}{a} \quad \gamma = 1$
		→		$C_+ = \frac{1}{L_-} = \frac{a}{L'}(1 + \gamma_1^2)$ $C_- = \frac{1}{L_+} = \frac{a}{L'}(1 + \gamma_2^2)$
		→		$L_+ = \frac{1}{C_-} = \frac{a}{C'}(1 + \gamma_1^2)$ $L_- = \frac{1}{C_+} = \frac{a}{C'}(1 + \gamma_2^2)$ $\gamma_{2,1} = \sqrt{\left(\frac{1}{2a\Omega}\right)^2 + 1} \pm \frac{1}{2a\Omega}$

$t_{LP}(f) = t_{BP}\left(\frac{f - 10^6}{10^4}\right)$



f_b) and a desired impedance level (by choice of R_b).

For low-pass and high-pass filters $f_b = f_c(\Omega_c)$. For bandpass and band-reject filters,

$$f_p = f_c f_{-c}$$

$$L_b = \frac{R_b}{2\pi f_b}$$

$$C_b = \frac{1}{2\pi f_b R_b}$$

Realization and Narrow-Band Approximation

The direct, conventional low-pass to bandpass transformation, while correct theoretically, is not always justified practically. The element values may be too small or too large. The parasitic capacitance to ground cannot be taken into account and therefore may distort the response. The node between a capacitor and a coil in a series arm becomes very sensitive to stray capacitance at some frequencies, and the quality of the series arm has to be very high in order to produce a low level of insertion loss in the passband. It is therefore desirable to simplify the network realization in order to remove the selectivity from the series arm, and to substitute added selectivity in the parallel arms.

The conventional transformation for a third-order low-pass filter to a bandpass realization requires three coils and three capacitors. Essentially the same kind of response curve can be realized, however, with three parallel resonators coupled together by mutual inductance, by capacitors, by inductances, or by inductances and capacitances as shown in Fig. 28.

Variation of the coupling reactances with frequency causes the response of the filter to be nonsymmetrical, although a close approximation of this effect may be taken into account. The original schematic with three coils and three capacitors will theoretically guarantee the ideal polynomial response. The schematic with coils or capacitors in the series arm is only a narrow-band approximation and realizes the transformed theoretical low-pass filter response only for a limited bandwidth. Nevertheless, good accuracy is maintained over a wide range (20 per cent) of center frequency. The design, almost by necessity, must use coupling circuit terminology, especially for microwave-frequency filters where the resonators are realized in the form of cavities.

For wideband filters, especially at low frequencies where a lumped-element technique could be used, the direct transformation is useful. It does not create too much of a problem except that the number of bulky and expensive coils (especially at very low frequencies) will complicate the realization.

M. Dishal has devised a synthesis procedure for the dissipative (non-ideal) bandpass filter. [2] His method is equally good for wideband and narrowband types and has proven to be practical for the design engineer. Dishal uses a 3-db bandwidth normalization and normalized coupling. The minimum number of resonators may be obtained from equations or from curves. After the number of resonators is determined, the components can be found. To obtain the first resonator for the narrow-band type of schematic shown in Fig. 28, the Q factor at the first node is selected. This means that the source resistance determines the first-node inductance and capacitance. For a very small value of percentage bandwidth, the ratio of coupling to shunt elements is approximately reciprocal to percentage bandwidth and is very small. Physically, this means that capacitor coupling is not always realizable and one must sometimes employ mutual inductive coupling. After having determined the number of resonators, type of coupling and input Q, the remainder of the circuit elements are rigidly determined by a step-by-step design procedure with known values of loaded Q, k, and M.

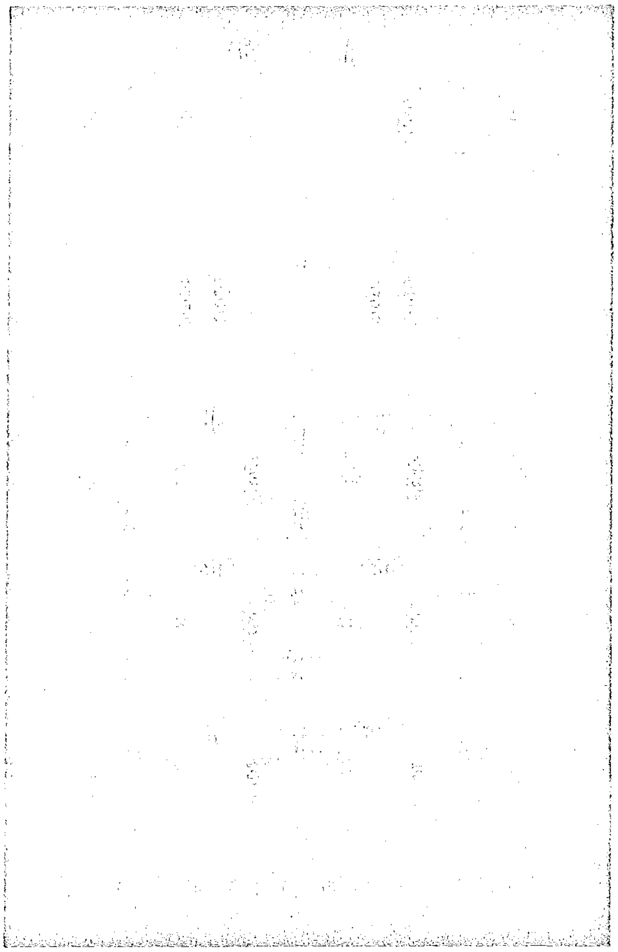
To obtain the results predicted by theory, the unloaded Q of each element must be greater than a certain minimum. For internal reactances of low-pass filters, this minimum Q is q_{min}; for internal resonators of bandpass circuits it is $q_{min}(f_o/b\omega_{3db})$.

A design example illustrating the use of Dishal's method is given in Appendix 5.

Physical Problems of Filter Design

Since coils and capacitors are still the most commonly used passive network elements, and we are concerned with filters handling a large amount of reactive power, our attention must be directed towards utilizing the best reactive element. This may be a coil with a ferramic core, a coaxial resonator, a helical resonator, a waveguide or a crystal resonator, depending upon the application.

The importance of the physical aspect of power



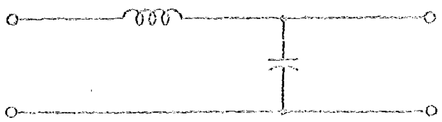


Fig. 29—Simple low-pass filter.

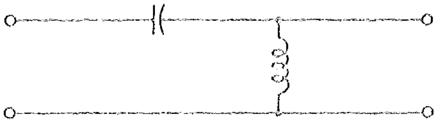


Fig. 30—Simple high-pass filter.

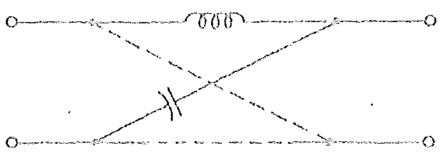


Fig. 31—Elementary lattice structure.

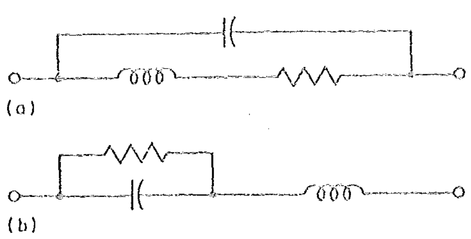


Fig. 32—Equivalent circuits for an inductor (a) and a capacitor (b) which include the effects of impurity elements.

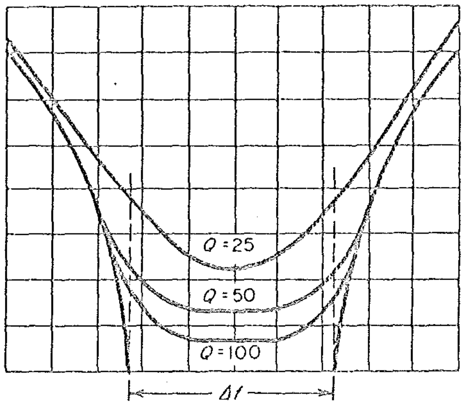


Fig. 33—Attenuation characteristics of a bandpass filter as a function of the quality factor of the circuit.

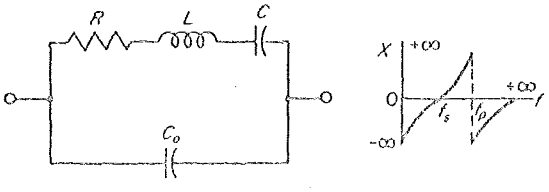


Fig. 34—Crystal resonator equivalent circuit and reactance curve.

transformation inside a reactive network should be emphasized. For each watt of power dissipated inside of the filter, the reactive power may be as high as 10,000 volt-amperes. This ratio of dissipative to reactive power requires that the Q of the components be at least 10,000. Even with such a large Q, half of the input power will be lost at the 3-db point. Good-quality conventional coils can be made with practical values of Q no higher than 500. The problem, therefore, is to concentrate reactive power in a few high-quality elements and to make these filter elements convert most of the reactive power concentrated in the region of the cutoffs.

When the amount of reactive power circulating inside a two-terminal network is known, the value of reactance below or above resonator frequency can be determined. This value is proportional to the algebraic difference between magnetic and electric power. The sum of the reactive power in the reactive dipole, on the other hand, determines the sharpness of the reactance variation of the same network. For a reactive four-terminal network, the total reactive power is proportional to the group delay. The group delay can therefore be interpreted as a measure of the efficiency of energy transmission. This efficiency is defined by the ratio of transmitted real power to the reactive energy required by the real-power transmission. In minimum phase networks (practically all types of filters), the attenuation and phase are interdependent. Thus it can be concluded that, in minimum phase filters, the required conversion of reactive power may be found from the attenuation curve or curve of selectivity. In the region of the cutoffs the converted reactive power assumes large values. If the response curve is very sharp, the network must balance this reactive power so that magnetic and electric power will circulate (exchange) inside the filter and not reach the load.

These considerations provide a physical explanation of the difficulty encountered by the designer when he trys to obtain good matching near cutoff frequency. The absolute value of the difference between electric and magnetic power is an appropriate criterion for good matching.

Physical Elements of the Filter. The main elements of a filter are reactances: lumped capacitance and lumped inductance. It is possible to design some filters by using only capacitors and resistors. This combination is especially useful in the case of active networks or active filters. Regular passive filters require both types of reactance.

The simplest low-pass filter has two arms as shown in Fig. 29. The series arm has an inductor and the shunt arm a capacitor. In the case of a simple high-pass filter, the series arm consists of a capacitor and the shunt arm consists of an inductor (Fig. 30). An example of the simplest bandpass filter is shown in Fig. 31.

To a first approximation, lumped inductance and capacitance can be considered as pure reactances, but closer investigation reveals that losses and reactive impurities are also present. The ordinary inductor at relatively low frequencies is wound on a magnetic core (powder iron or ferramic); other inductors are simply single-layer solenoids wound on a nonmagnetic coil form. In both cases, although the losses are very low in comparison with the value of reactance, they cannot be neglected, especially when one designs a very-narrow-band filter or desires a very sharp response curve.

Losses tend to decrease the rate of attenuation rolloff, increase the attenuation within the passband, and, in certain cases, prohibit the realization of very-narrow-bandpass filters. The parasitic effect of distributed capacitance across the coil or series lead inductance in capacitors is damaging in another respect. In the low-pass filter, parasitics will create the effect of a parallel resonant circuit instead of a coil. The filter may thus provide unexpected rejection at certain frequencies in the stopband, or even in the passband, if the self-resonant frequency of such a coil is sufficiently low. Parasitic reactance in the lumped components produces distortion of the amplitude response and may destroy the network response altogether if not neutralized or properly taken into consideration.

Equivalent circuits for an inductor and a capacitor are shown, with impurities taken into account, in Fig. 32. The conventional measure of the quality of any reactance is the quality factor Q which describes how many times the reactance of a coil or capacitor is greater than the resistance. The most common value of Q in the conventional inductor at radio frequencies is 50–300. The Q factor for the capacitor at the same frequency is usually higher: 500–5000. The higher these values are, the better the filters that can be designed. Figure 33 illustrates the effect of the Q factor on the shape of the response of a bandpass filter.

The demand for quality factor in ordinary lumped components, especially coils, has intensified research to find some substitute for the inductor and capacitor. Historically, the first and most successful substitute was the piezolectric crystal; next were magnetostrictive components and electromechanical devices. Lumped elements are the oldest filter elements, and they remain the most widely used at low frequencies.

Experience proves that, for example, a very good bandpass filter can be made when its components have a Q factor not less than 20 to 25 times $f_o/\Delta f$ where f_o is the center frequency of the filter and Δf is the bandwidth (passband). For the same bandwidth and attenuation characteristic, the filter with the lowest center frequency will need the lowest quality factor. If, for example, $f_o = 10{,}000$ cycles and $\Delta f = 3000$ cycles, the Q factor cannot be less than 66. When an element that meets these specifications is used, a very good bandpass filter can be designed for a commercial telephone signal. If the frequency $f_o = 150{,}000$ cycles and $\Delta f = 3000$ cycles, the Q factor has to be greater than 1000. Even the best coils with ferramic core material cannot provide a Q factor greater than 600, and, therefore, a conventional element cannot be used in such a filter.

Crystal Filters

The equivalent schematic of a piezoelectric resonator is shown in Fig. 34. There are three reactances in this first-approximation schematic. L and C are the motional parameters and C_o is a capacitance across the crystal which can be considered a parasitic parameter. The piezoelectric resonator has a very high Q factor, at least several thousand, and it is very stable with time and temperature conditions. With such resonators, filters having any desired selectivity and extremely small bandwidths can be designed and built.

The piezoelectric crystal exhibits some unpredictable and undesirable modes of oscillation, which is a very serious limitation of ordinary crystals. Some of these spurious modes are very close to the fundamental frequency. Another limitation of the device is that the ratio between the parasitic capacitance and motional capacitance cannot be made less than a certain value. This value determines the possible bandwidth of the filter. In the low-frequency range (up to 200 kc), this bandwidth can be no more than about 10 per cent. At higher frequencies the percentage bandwidth will be less and less, and at 30 Mc it can be only a fraction of one per cent even

when all supplementary measures are taken into consideration.

Because of such factors as spurious oscillation and small physical crystal size, crystal resonators do not operate satisfactorily at very high frequencies. Coaxial and helical resonators are generally used at these frequencies.

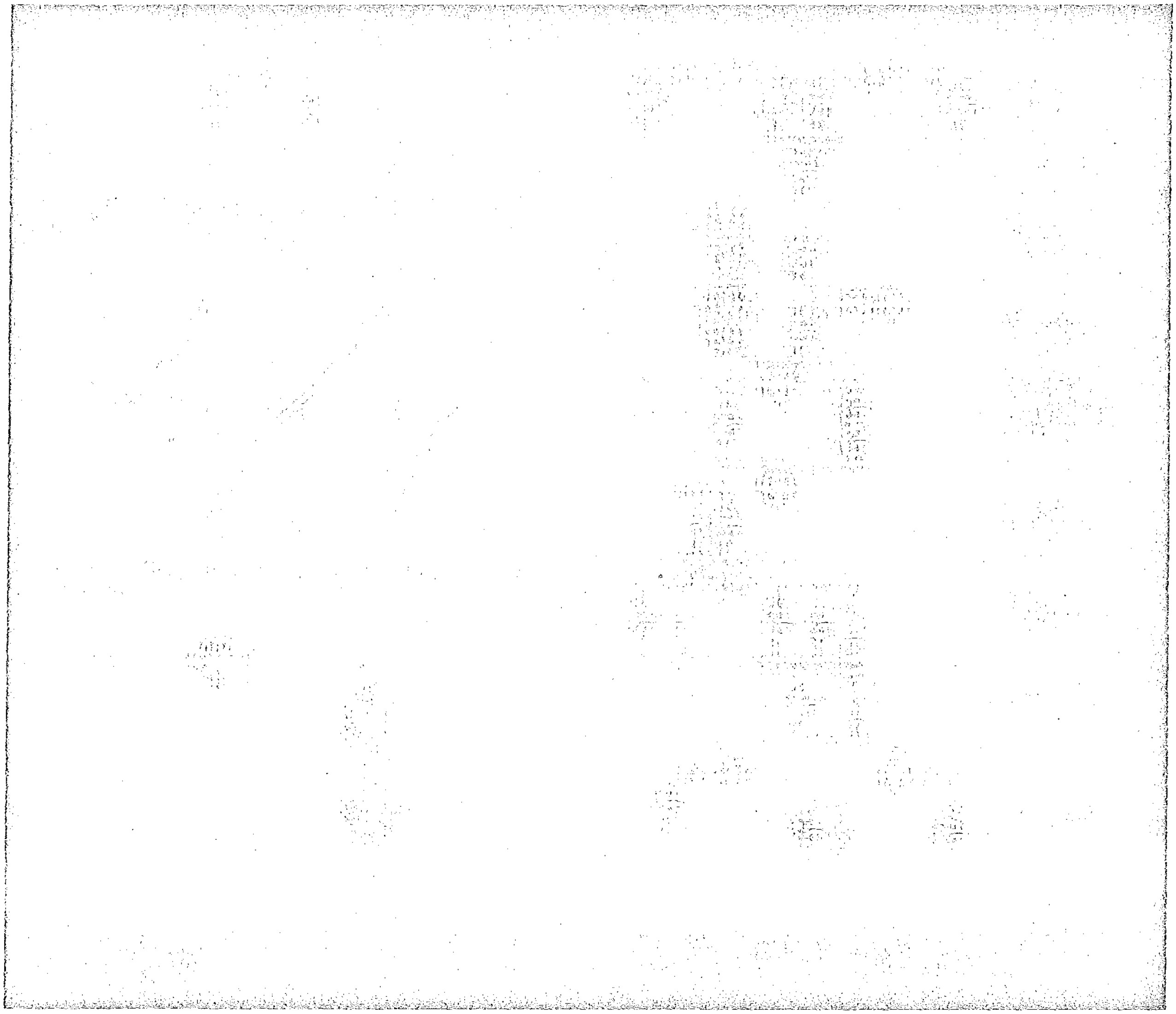
The most efficient use of the quartz crystal is obtained under the following conditions:

- The quartz crystal dissipates the largest part of the reactive power converted at cutoff frequency.
- A high-quality stable quartz resonator in a ladder schematic produces the critical (and sharpest) attenuation response pole.
- The bandwidth of the network involved is as large as possible.

Simultaneous satisfaction of these three requirements was impossible for a long time, but Fig. 35 illustrates a solution now available. Here, a filter used for carrier communication has been designed by using network transformations. The section shown in Fig. 35(a) produces a sharp pole at f_1 and another pole, less sharp, at f_2. The transformed network using a quartz crystal is shown in Fig. 35(b). Both networks are identical if certain conditions are satisfied. Pole f_2 [Fig. 35(a)] must be placed so that the input impedance from the right side of the network becomes zero at frequency f_1. In other words, the series combination of the two parallel networks (L_2C_2 and L_3C_3) must have a series resonance at f_1. In addition, the input impedance from the right side of Fig. 35(b) must have a pole at frequency f_2; *i.e.*, the series and parallel reactances of f_2 must produce a pole.

In Fig. 35(a) the sharpest pole is produced by L_1C_1. In Fig. 35(b), the sharpest pole is produced by $L_3'C_3'$. The calculated ratio C_2/C_3 is very large, and additional capacitance may be added, if desired, since the capacitance ratio of an x-cut quartz crystal is only about 140. The capacitance ratio $C_p/C_s;(C_2/C_3)$ is, to a first approximation, inversely proportional to the distance between the pole frequencies f_1 and f_2. The distance between f_1 and f_2 could be several times larger than that shown in Fig. 35(a) since a practical ratio between parallel and series capacitance is about 200. With increasing bandwidth the ratio becomes more unfavorable; for a bandwidth of 60 per cent the ratio is still good, but between 60 and 90 per cent the ratio C_p/C_s is so small that the quartz can no longer be used.

A typical arrangement for inserting a crystal in a

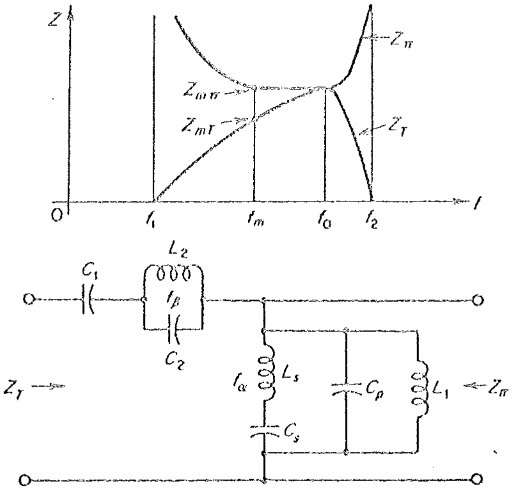


ladder filter is shown in Fig. 36; Fig. 37 is the corresponding reactance diagram. The crystal provides a trap for the energy around its resonant frequency, f_c. This results in zero transmission through the filter at f_c (the peak of attenuation in the filter). The same crystal will also provide two parallel resonances, one below f_c, f_a, and the other above f_c, f_s. Since f_s is in the stopband, its presence will tend to decrease the attenuation. The addition of one parallel resonance in the series arm of the ladder filter can eliminate the effect of one parallel resonance in the crystal. The high-quality series resonance of the crystal can then be used for the filter without being deteriorated by undesirable phenomena.

Figure 38 shows an equivalent circuit which accounts for a piezoelectric resonator. The problem of realization reduces itself to that of modifying a conventional LC network in such a way that the schematic of Fig. 38 is obtained.

A design procedure which finally results in the desired modification is given in Fig. 39. Here the original schematic is represented in ladder form. Two sections with different peaks of attenuation (f_c and f_s) are to be connected. The design procedure follows the steps shown in Fig. 39. The element across the line in the center of the final network is a nonconventional dipole which produces an unusually sharp peak of attenuation.

A filter section which produces a peak of attenuation on the high-frequency side of the passband is



$C_1 = C_{k1} \cdot \frac{2a}{1-a^2} (\Omega_\beta + a)$

$C_2 = C_{k1} \cdot \frac{\Omega_\beta^2 - a^2}{1-a^2} \cdot \frac{\Omega_0 + a}{\Omega_0 - \Omega_\beta}$

$L_2 = L_{k1} \cdot \frac{1-a^2}{(\Omega_\beta + a)^2} \cdot \frac{\Omega_0 - \Omega_\beta}{\Omega_0 + a}$

$C_p = C_k \cdot \frac{1-a^2}{4a^2} \cdot \frac{\Omega_\beta - a}{\Omega_\alpha - a} \cdot \frac{\Omega_0 - a}{\Omega_0 + a}$

$C_p/C_s = \frac{(\Omega_\alpha + a)^2}{4a^2} \cdot \frac{\Omega_0 - a}{\Omega_\alpha - \Omega_\beta} \cdot \frac{\Omega_\beta - a}{\Omega_0 - \Omega_\alpha}$

$L_s = L_k \cdot \frac{1}{\sqrt{1-a^2}} \cdot \frac{(\Omega_\alpha - a)^2}{\Omega_\alpha - \Omega_\beta} \cdot \frac{\Omega_\alpha + a}{\sqrt{1-\Omega_\alpha^2}}$

$L_1 = L_k \cdot \frac{4a^2}{1-a^2} \cdot \frac{\Omega_\alpha + a}{\Omega_\beta + a}$

$\frac{Z_{mT}}{Z_{m\pi}} = \frac{(\Omega_0 - \Omega_\alpha)(\Omega_0 + a)}{\Omega_0^2 - 1}$

$\Omega_0 = \frac{\Omega_\alpha \sqrt{1-a^2} + a \cdot \sqrt{1-\Omega_\alpha^2}}{\sqrt{1-a^2} - \sqrt{1-\Omega_\alpha^2}}$

$a = \frac{f_2 - f_1}{f_2 + f_1}$

$\Omega_\alpha = a \frac{\eta_\alpha^2 + 1}{\eta_\alpha - 1}$

$\Omega_\beta = a \frac{\eta_\beta^2 + 1}{\eta_\beta - 1}$

$\eta_\alpha = \frac{\omega_\alpha}{\omega_m} = \frac{f_\alpha}{\sqrt{f_1 f_2}}$

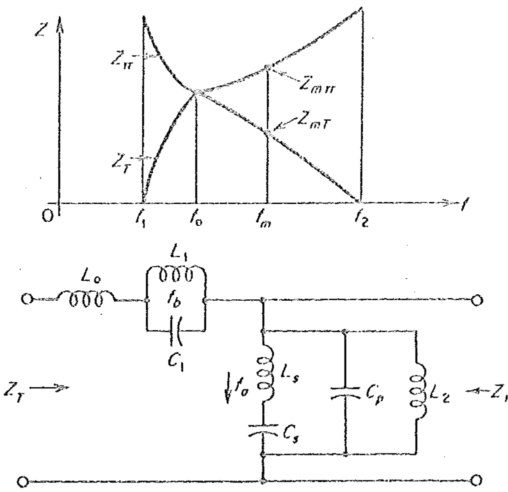
$\eta_\beta = \frac{\omega_\beta}{\omega_m}$

$\Delta\omega = 2\pi(f_2 - f_1)$

$C_k = \frac{1}{\omega_m^2 L_k}$ $C_{k1} = \frac{1}{\Delta\omega Z_{mT}}$

$L_k = \frac{Z_{m\pi}}{\Delta\omega}$ $L_{k1} = \frac{1}{\omega_m^2 C_{k1}}$

Fig. 40—Filter section, with equivalent crystal resonator, having peak attenuation on the high-frequency side of the passband.



$L_0 = L_{k1} \cdot \frac{1-a^2}{2a} \cdot \frac{1}{|\Omega_0| + a}$

$L_1 = L_{k1} \cdot \frac{1-a^2}{|\Omega| + a} \cdot \frac{|\Omega_0| - |\Omega_b|}{\Omega_b^2 - a^2}$

$C_1 = C_{k1} \cdot \frac{(|\Omega_b| + a)^2}{1-a^2} \cdot \frac{(|\Omega_0| + a)}{|\Omega_0| - |\Omega_b|}$

$C_p = C_k \cdot \frac{1-a^2}{4a^2} \cdot \frac{|\Omega_0| + a}{|\Omega_a| + a}$

$C_p/C_s = \frac{(|\Omega_a| - a)^2}{4a^2} \cdot \frac{|\Omega_0| + a}{|\Omega_a| - |\Omega_b|} \cdot \frac{|\Omega_b| + a}{|\Omega_0| - |\Omega_a|}$

$L_s = L_k \cdot \frac{1}{\sqrt{1-a^2}} \cdot \frac{(|\Omega_a| + a)^2}{|\Omega_a| - |\Omega_b|} \cdot \frac{|\Omega_a| - a}{\sqrt{1-\Omega_a^2}}$

$L_2 = L_k \cdot \frac{4a^2}{1-a^2} \cdot \frac{|\Omega_a| - a}{|\Omega_b| - a} \cdot \frac{|\Omega_0| + a}{|\Omega_0| - a}$

$\frac{Z_{mT}}{Z_{m\pi}} = \frac{(|\Omega_0| - a)(|\Omega_a| - |\Omega_a|)}{\Omega_0^2 - 1}$

$\Omega_0 = \frac{a\sqrt{1-\Omega_a^2} + |\Omega_a| \sqrt{1-a^2}}{\sqrt{1-\Omega_a^2} - \sqrt{1-a^2}}$

$a = \frac{f_2 - f_1}{f_2 + f_1}$

$\Omega_a = a \frac{\eta_a^2 + 1}{\eta_a - 1}$

$\Omega_b = a \frac{\eta_b^2 + 1}{\eta_b - 1}$

$\eta_a = \frac{\omega_a}{\omega_m}$

$\eta_b = \frac{\omega_b}{\omega_m} = \frac{f_b}{\sqrt{f_1 f_2}}$

$\Delta\omega = 2\pi(f_2 - f_1)$

$C_k = \frac{1}{\omega_m^2 L_k}$ $C_{k1} = \frac{1}{\Delta\omega Z_{mT}}$

$L_k = \frac{Z_{mT}}{\Delta\omega_1}$ $L_{k1} = \frac{1}{\omega_m^2 C_{k1}}$

Fig. 41—Filter section, with equivalent crystal resonator, having peak attenuation on the low-frequency side of the passband.

shown in Fig. 40, along with the appropriate design formulas and response curves. Similar information on a filter section which produces a peak of attenuation on the low-frequency side of the passband is given in Fig. 41. ■

Appendix 1—Basic Filter Relationships

The usual operating conditions for a filter are shown in Fig. 1-1 where R_1 and R_2 are pure resistances. A relation between the power P_2 dissipated in the load resistance R_2 and the output voltage is

$$P_2 = \frac{V_2^2}{R_2}$$

The maximum power P_m that can be delivered by the signal source

$$P_m = \frac{V_o^2}{4R_1} \qquad (1)$$

leads to the definition of the effective transmission factor H:

$$H = \sqrt{\frac{P_m}{P_2}} \qquad (2)$$

The natural logarithm of H is called the effective transmission constant: $g = \ln H$. Its real part, $\ln |H|$, is the effective attenuation a, and its imaginary part $j \sphericalangle H$, is the effective phase angle b.

$$\begin{aligned} g &= a + jb = \ln |H| + j \sphericalangle H \\ &= \ln \frac{V_o}{2V_2} + \ln \sqrt{\frac{R_2}{R_1}} \qquad (3) \end{aligned}$$

$$H = e^g$$

The attenuation a is expressed here in nepers. The analogous expression in decibels is

$$A = 20 \log_{10} |H|, \text{db}$$

The effective attenuation can never become negative. The voltage insertion loss is equal to the effective attenuation when $R_2 = R_1$.

A relation between the input impedance Z of the four-terminal network and the resistance R, which represents the source impedance, defines the input reflection factor ρ.

$$\rho = \frac{R - Z}{R + Z} \qquad (4)$$

A non-zero reflection factor signifies physically that the maximum deliverable power P_m is not being transmitted through the network to the output since a portion Pr is being reflected. Thus, the definition of the reflection constant, g_r, is

$$g_r = a_e + jb_e = \ln \frac{1}{\rho} = \ln \sqrt{\frac{P_m}{P_r}} \qquad (5)$$

where a_e is echo attenuation and b_e is the corresponding phase angle.

Let us assume that input impedance of the filter is Z and the source impedance connected to the filter is Z_o (Fig. 1-2). When $Z = R = 1$, the voltage V across the filter will be $V = V_o/2$ where V_o is the source voltage. For this condition $V - V_o/2 = 0$. If $Z \neq R$, then

$$V = \frac{V_o}{2} - \left(V - \frac{V_o}{2}\right)$$

where

$$V - \frac{V_o}{2} \neq 0$$

and we have the ratio

$$\frac{\frac{V_o}{2}}{V - \frac{V_o}{2}} = \frac{V_o}{2V - V_o} = \frac{V_o}{2V_o \frac{Z}{1 + Z} - V_o}$$

$$= \frac{1 + Z}{2Z - 1 - Z} = \frac{Z + 1}{Z - 1} = \frac{1}{\rho}$$

The ratio between the reflected power and the power dissipated in the load resistance (P_r/P_2) is used to define a characteristic factor (g_k) where

$$g_k = a_k + jb_k = \ln \frac{1}{\rho} = \ln \sqrt{\frac{P_r}{P_2}} \qquad (6)$$

Here a_k is the characteristic attenuation, and b_k is the characteristic phase angle.

Equations (2), (5) and (6) are related as follows:

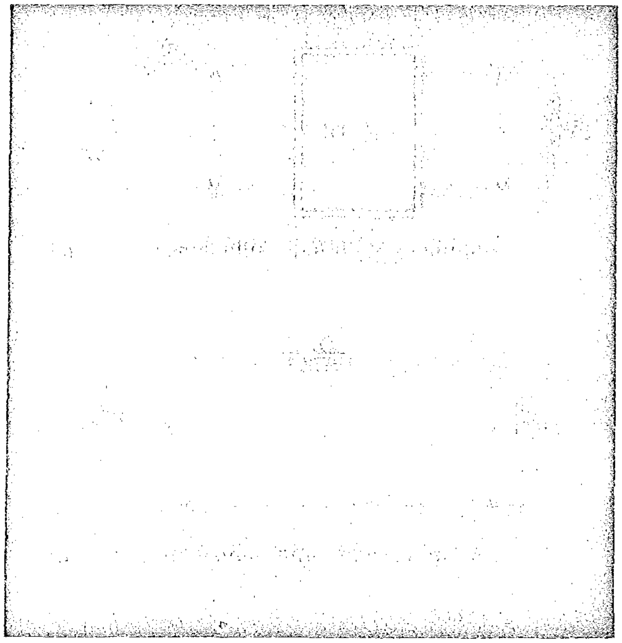


Table 1 — [illegible] Function D

a, nepers	$D = \sqrt{e^{2a} - 1}$
0.000	0.000
0.005	0.101
0.010	0.142
0.015	0.175
0.00	0.00
0.05	0.32
0.10	0.47
0.15	0.59
0.20	0.70
1.00	2.53
1.25	3.34
1.50	4.37
1.75	5.67
2.00	7.3
2.50	12.1
3.00	20.1
3.50	33.1
4.00	54.6

[illegible]

ρ, per cent	a_e, nepers	A_e, decibels
1	4.6	39.9
2	3.9	33.8
3	3.5	30.3
4	3.2	27.7
5	3.0	26.0
8	2.5	21.7
10	2.3	19.9
15	1.9	16.5
20	1.6	13.85
25	1.35	11.7
50	0.55	4.77

$$|H|^2 = \left|\frac{P_m}{P_2}\right| = e^{2a}$$

$$\left|\frac{1}{\rho}\right|^2 = \left|\frac{P_m}{P_r}\right| = e^{2a_e}$$

$$|D|^2 = \left|\frac{P_r}{P_2}\right| = e^{2a_k}$$

$$|H\rho|^2 = e^{2a-2a_e} = |D|^2 = e^{2a_k}$$

where $a - a_e = a_k$ is in nepers, and D is the filter discrimination factor.

It is seen that the effective attenuation a is equal to the sum of the echo attenuation a_e and the characteristic attenuation a_k.

$$\begin{aligned} a &= a_e + a_k, \text{nepers} \\ A &= A_e + A_k, \text{decibels} \end{aligned} \tag{7}$$

If the filter consists of reactive circuit elements, no power can be dissipated within the four-terminal network. Since R_1 and R_2 are assumed to be pure resistances, all power values under consideration are effective values, and the difference between the maximum deliverable source power P_m and the reflected power P_r must be equal to P_2, the power which passes through the network and arrives at the output load. According to Eq (7),

$$|P_m| - |P_r| = |P_2|$$

$$|P_m| = |P_2| + |P_r| = |P_m| \cdot (e^{-2a} + e^{-2a_e})$$

For filters with lossless elements, every value of effective attenuation a corresponds to a certain value of echo attenuation according to the relation:

$$\begin{aligned} e^{-2a} + e^{-2a_e} &= \left|\frac{1}{H}\right|^2 + |\rho|^2 = 1 \\ 1 + |H\rho|^2 &= |H|^2 \cdot \end{aligned} \tag{8}$$

from which it follows that

$$|H|^2 = 1 + |D|^2 \tag{9}$$

In order to avoid the use of absolute-value notation, we will equate $|D|^2 = D^2$ where D is the filter discrimination factor.

$$D = \sqrt{e^{2a} - 1} = \sqrt{\frac{P_m - P_2}{P_2}}$$

Table A gives the numerical relationship between attenuation a, in nepers, and the filter discrimination factor D. D is a function of effective attenuation and increases from 0 to ∞ when the value of attenuation grows in the same direction (from 0 to ∞).

$$a = \ln\sqrt{1 + D_2}$$

or

$$A = 10\log_{10}(1 + D^2) \tag{10}$$

Equation (7) also yields

$$a = \ln\sqrt{1 - \rho^2} = \ln\sqrt{1 - e^{-2a_e}}$$

or

$$A = -10\log_{10}(1 - \rho^2) \tag{11}$$

Table B gives the numerical relationship between values of ρ, a_e and A_e.

The effective transmission factor H and the filter discrimination factor D are both functions of the frequency f, or the angular frequency $\omega = 2\pi f$. Both H and D are needed for the determination of quantities such as impedance and admittance which fully characterize the four-terminal network and from which the circuit element values can be derived.

Appendix 2—Use of the Complex Frequency Plane in Filter Design

Real networks operate with real voltages and currents which are functions of time, but the idealized mathematical models involved in polynomial synthesis operate with fictitious currents and voltages which are neither real nor functions of time. In such models, these parameters are functions of the complex variable s. In order to obtain real currents and voltages from complex values, certain mathematical transformations have to be used. While the mathematical model has no resemblance to reality, it greatly simplifies prediction about the behavior of the actual networks.

A point $s = \sigma + j\omega$ in the S plane may be represented by the product of a sinusoid and an exponential $u = e^{\sigma t} \cdot \sin\omega t$. If σ is negative, the amplitude of u decreases exponentially with time; if σ is positive, u increases exponentially. This behavior can serve as a description of the complex frequency s. A complex response function $R(s)$ of any network can be described as the ratio of any current or voltage in a network to any other, and all response functions are thus ratios of polynomials in s (complex frequencies).

$$R(s) = \frac{P(s)}{Q(s)} \tag{12}$$

where $P(s)$ and $Q(s)$ are polynomials.

Any polynomial can be written in terms of its roots.

$$\begin{aligned} P(s) &= as^n + bs^{n-1} + cs^{n-2} + \cdots + es + f \\ &= a(s - r_1)(s - r_2)(s - r_3)\cdots(s - r_n) \end{aligned}$$

where $r_1, r_2, r_3, \cdots, r_n$ are the roots of $P(s)$. The polynomials in Eq (12) can be factored and written in the following form:

$$R(s) = A\,\frac{(s - p_1)(s - p_2)\cdots(s - p_p)}{(s - q_1)(s - q_2)\cdots(s - q_q)}$$

where p_1, p_2, etc. are the roots of P_s, and q_1, q_2, etc. are the roots of $Q(s)$. The constant factor A (the ratio of the coefficients) serves only to change the amplitude of $R(s)$ but has no bearing on the form of the response. Whenever $s = p_k$, $R(s)$ becomes equal to zero; these specific values of s are called zeros of the response. Whenever

$s = q_k$, $R(s)$ becomes infinite; these particular values of s are called poles of the response. No matter how complicated the response function is, it can be expressed uniquely in terms of its poles, zeros and a constant multiplier. The number of poles and zeros is exactly equal if the poles and zeros at infinity are included. The simplest possible filter has only a single pole and hence only a single zero.

The pole-zero concept is very useful, because it permits the response to be expressed in analytical form. For a passive network the poles of the transfer function must always lie to the left of the imaginary axis. They cannot lie *on* the imaginary axis except as an approximation. Poles on the imaginary axis do not physically exist for passive networks; they would correspond to a tank circuit with infinite quality factor. Only in an active network can a pole lie on the imaginary axis.

If the filter consists of a finite number of lumped linear elements, it can be expressed with the help of three real polynomials, E, P, and F.

$$H = \frac{E}{P};\ Z_{\text{in}} = \frac{E+F}{E-F}$$

Here H is the coefficient of effective transmission and Z_{in} is the normal input impedance. Furthermore,

$$e^{2a} = 1 + D^2$$

where $a = \ln H$ is the effective attenuation of the filter, and $D = \frac{F}{P}$ is the filter discrimination function.

For a reactive network the polynomials are related in the following way:

$$E(s)E(-s) = P(s)P(-s) + F(s)F(-s) \qquad (13)$$

If polynomials P, E, F are known, it is possible to find all the elements of the network, and to determine its schematic arrangement.

The problem of synthesis consists in finding a filtering system which has the minimum number of elements essential for effective attenuation but which does not have more than the given $a_{\max}$ in the passband nor less than the given $a_{\min}$ in the stopband. This problem can be separated into three parts:

1. Finding the best approximation of the given requirements in the form of a rational function $D = F/P$.
2. Determining the polynomial $E(s)$ by Eq (13).
3. Determining the filter system according to the polynomials P, E, F.

The roots of $E(s)$ equal the roots of the system's characteristic equation and correspond to the frequencies of the free oscillations which may occur in a charged network during transition processes. Therefore, we will call $E(s)$ the characteristic polynomial of the network. Since in a passive quadripole network the free oscillations should be damped, the real part of the roots of $E(s)$ should be negative. This condition makes it possible to determine the polynomial $E(s)$ by the known roots of the right-hand part of Eq (13). All the roots with a negative real part correspond to polynomial $E(s)$, and those within a positive real part correspond to polynomial $E(-s)$. Thus, the whole problem boils down to finding the roots of the right hand part of Eq (13). When the number of pairs of complex roots exceeds three, calculations become very complicated.

Appendix 3—Simplest Polynomial Filters

The first filter network shown in Table III is known as a first-order filter.

The effective transmission factor H, which defines the behavior of this filter network, is

$$H = 1 + j\omega C \frac{R_1 R_2}{R_1 + R_2}$$

or simply

$$H = 1 + j\omega CR$$

The presence of j in this expression shows that there is a phase shift introduced by the use of the filter.

The effective attenuation in decibels is

$$A = 20 \log |H| = 10 \log(1 + \omega^2 C^2 R^2)$$

Second-Order Filters. For the second-order filter shown in Table III, the quantity H will be

$$H = 1 + j\omega\left(C \frac{R_1 R_2}{R_1 + R_2} + \frac{L}{R_1 + R_2}\right) - \omega^2 LC \frac{R_2}{R_1 + R_2}$$

This expression can be made to look simpler by substituting

$$\frac{R_1 R_2}{R_1 + R_2} = R_p$$

$$R_1 + R_2 = R_s$$

and

$$\frac{R_2}{R_1 + R_2} = K$$

Thus

$$H = 1 + j\omega\left(CR_p + \frac{L}{R_s}\right) - \omega^2 LCK$$

To obtain the expression for effective attenuation we must eliminate j by multiplying by the complex conjugate:

$$|H|^2 = 1 + \omega^2\left[\left(CR_p + \frac{L}{R_s}\right)^2 - 2LCK\right] + \omega^4 L^2 C^2 K^2$$

The effective attenuation is

$$A = 10 \log |H|^2$$

The transmission factor H can be more compactly expressed in the form

$$|H|^2 = 1 + \alpha\omega^2 + \beta\omega^4$$

where α and β may be chosen so that we get a Butterworth or Chebyshev response as desired. Mathematically, a Butterworth response is simply $1 + \omega^{2n}$ which means that for $n = 2$ the value of α must be set equal to 0. This being the case,

$$A = 10 \log(1 + \omega^4 L^2 C^2 K^2)$$

where

$$CR_p + \frac{L}{R_s} = 2LCK$$

Third-Order Filters. Adding one more capacitor to the previous circuit results in the third-order filter (see Table III), for which the transmission factor H becomes

$$H = 1 + j\omega\left[(C_1 + C_2)R_p + \frac{L}{R_s}\right] - \omega^2 \frac{L}{R_s}(C_1 R_1 + C_2 R_2) - j\omega^3 LC_1 C_2 R_p$$

The absolute value can be found by the same procedure we used before.

$$|H|^2 = 1 + \omega^2\left\{\left[(C_1 + C_2)R_p + \frac{L}{R_s}\right]^2 - \frac{2L(C_1R_1 + C_2R_2)}{R_s}\right\}$$

$$+ \omega^4\left\{\frac{L^2(C_1R_1 + C_2R_2)^2}{R_s^2} - 2LC_1C_2\left[(C_1 + C_2)R_p + \frac{L}{R_s}\right]R_1\right\}$$

$$+ \omega^6L^2C_1^2C_2^2R_p^2$$

$$H^2 = 1 + \alpha\omega^2 + \beta\omega^4 + \gamma\omega^6$$

Here α and β must be set equal to 0.

The response of the third-order Butterworth filter is then expressed by

$$A = 10\log(1 + \omega^6L^2C_1^2C_2^2R_p^2)$$

Chebyshev Polynomials. The equation for the all-pole (transfer function) filter always takes the general form

$$|H|^2 = a_0 + a_1\omega^2 + a_2\omega^4 + \cdots + a_n\omega^{2n}$$

where n is the order of the network and a_1, a_2, a_n depend on the resistances, capacitances and inductances. Most low-pass filters have no attenuation at zero frequency; in such cases the term a_o is unity.

For the second-order filter, the Chebyshev polynomial to consider is that of the fourth order:

$$C_{2n}(\Omega) = C_4(\Omega) = 8\Omega^4 - 8\Omega^2 + 1$$

For this polynomial it is noted that at

$\Omega = 0$	$C_4(0) = 1$
$\Omega = 1$	$C_4(1) = 1$
$\Omega = 0.71$	$C_4(0.71) = -1$
$\Omega > 1$	$C_4(\Omega > 1)$: increases rapidly

For all positive values of Ω less than unity, the approximating function

$$1 - t + tC_4(\Omega) \qquad \left(\text{where } t = \frac{\epsilon^2}{1 + \epsilon^2}\right)$$

will lie between 1 and $1 - 2t$, and the maximum attenuation in the passband is

$$A = \pm 10\log(1 - 2t)$$

If the value of the ripple $a_{\max}$ is 1.25 db, the corresponding t is 0.125. The approximation function becomes

$$1 - 0.125 + 0.125(8\Omega^4 - 8\Omega^2 + 1) = 1 - \Omega^2 + \Omega^4$$

Using $|H|^2$ for the second-order filter as given above, and taking $R_2 = \infty$, so that $K = 1$, we have

$$|H|^2 = 1 + \omega^2[(CR_1)^2 - 2LC] + \omega^4L^2C^2$$

Comparing this with

$$1 - \Omega^2 + \Omega^4$$

we see that for identity the following relations must hold:

$$\Omega^4 = \omega^4L^2C^2$$

$$\Omega^2 = (2LC - C^2R_1^2)\omega^2$$

$$\omega^2LC = (2LC - C^2R_1^2)\omega^2$$

$$L = CR_1^2$$

The third-order filter is related to the Chebyshev polynomial of the sixth order, $C_6(\Omega)$. For $\Omega = 1$, $C_6(\Omega) = -1$, so that we consider

$$1 + t + tC_6(\Omega)$$

which oscillates between 1 and $1 + 2t$. Therefore, we have

$$1 + 18t\Omega^2 - 48t\Omega^4 + 32t\Omega^6$$

For $t = 1/16$, which corresponds to ±0.25 db, the condition for Chebyshev response becomes

$$\omega^6L^2C_1^2C_2^2R_1^2 = 2\Omega^6$$

$$2\omega^4LC_1C_2(C_1 + C_2)R_1 = 3\Omega^4$$

$$\omega^2(C_1 + C_2)^2R_1^2 = \Omega^2$$

We can see that, even in its simplest forms, the equations become very cumbersome.

Appendix 4—Attenuation of Zobel Filters

Insertion-loss formulas for Zobel filters are obtained with the following equations. For one, two, or three pi sections,

$$\pm D = -\Omega\left[(1 - \Omega^2)\frac{R}{Z_\pi} - \frac{Z_\pi}{R}\right] \tag{14}$$

$$\pm D = -\Omega(2 - 4\Omega^2)\left[(1 - \Omega^2)\frac{R}{Z_\pi} - \frac{Z_\pi}{R}\right] \tag{15}$$

$$\pm D = -\Omega(3 - 4\Omega^2)(1 - 4\Omega^2) \cdot\left[(1 - \Omega^2)\frac{R}{Z_\pi} - \frac{Z_\pi}{R}\right] \tag{16}$$

Equation (14) describes the filtering function for a single pi section while Eqs (15) and (16) refer to two and three pi sections. In each case, $2 \geq Z_\pi/R \geq 1$.

The corresponding expressions for T sections are

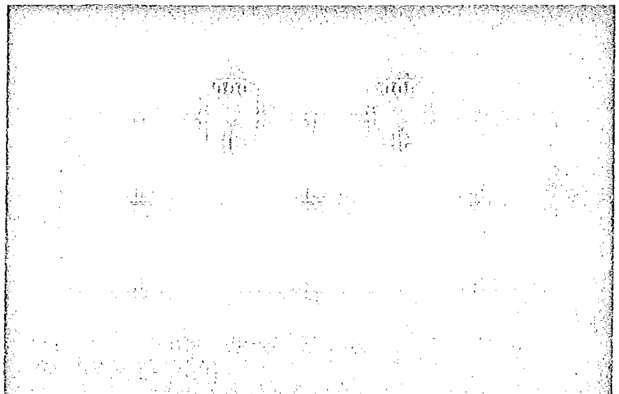
$$\pm D = \Omega\left[(1 - \Omega^2)\frac{Z_T}{R} - \frac{R}{Z_T}\right] \tag{17}$$

$$\pm D = \Omega(2 - 4\Omega^2)\left[(1 - \Omega^2)\frac{Z_T}{R} - \frac{R}{Z_T}\right] \tag{18}$$

$$\pm D = \Omega(3 - 4\Omega^2)(1 - 4\Omega^2) \cdot\left[(1 - \Omega^2)\frac{Z_T}{R} - \frac{R}{Z_T}\right] \tag{19}$$

Equation (17) gives the filtering function for a single T section; Eqs (18) and (19) give the functions for two- and three-section T filters.

These expressions describe the peaks and valleys of attenuation in essentially the same way as the familiar curves of the Chebyshev filter with one important exception. The values of the ripples are not equal.



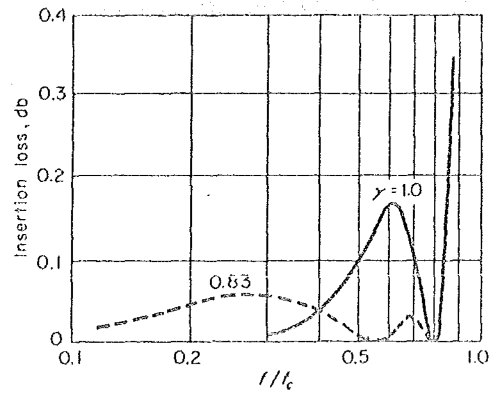


Fig. 4-2—Passband insertion loss of the low-pass Zobel filter with two sections.

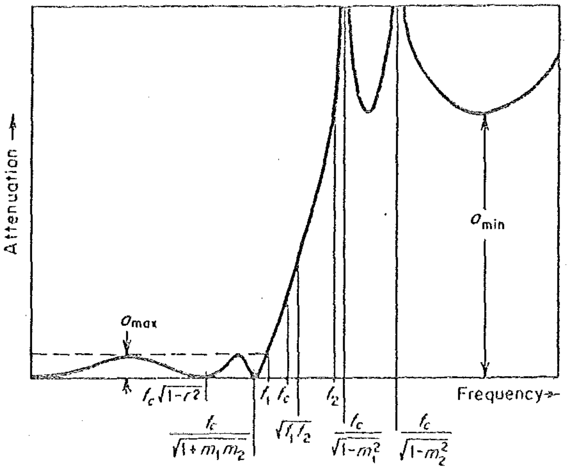


Fig. 4-3—Typical filter specification and Zobel design parameters.

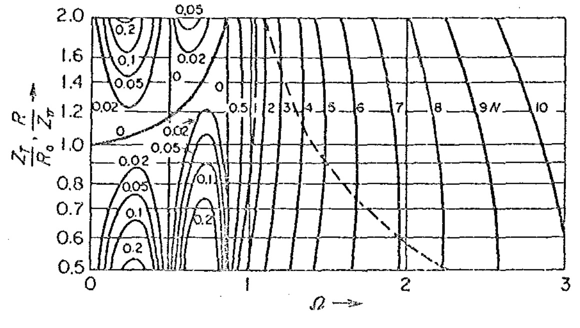


Fig. 4-4—Effective attenuation of the Zobel filter as a function of frequency and design resistance.

For the case of the two *m*-derived sections shown in the circuit of Fig. 4-1, Saraga gives the following formulas: [1]

$$U = \frac{X_x}{R} = -r\,\frac{m_1 + m_2}{1 + m_1m_2}\cdot\frac{\Omega}{\Omega^2 - \dfrac{1}{1 + m_1m_2}} \qquad (20)$$

$$V = \frac{X_y}{R} = -r\,\frac{1 + m_1m_2}{m_1 + m_2}\cdot\frac{\Omega^2 - \dfrac{1}{1 + m_1m_2}}{\Omega(\Omega^2 - 1)} \qquad (21)$$

Here R is the design resistance of the filter sections, R_1 is the resistance of the generator and load, m_1 and m_2 are the m values of the two Zobel sections, X_xX_y are the series and diagonal arms of the equivalent lattice, $r = R/R_1$, and $\Omega = f/f_c$, where f_c is the cutoff frequency.

The insertion loss in terms of the lattice reactances is given by

$$A = 10\log\left[1 + \left(\frac{UV}{U - V}\right)^2\right] = 10\log(1 + D^2) \qquad (22)$$

The quantity D can be computed directly from Eqs (20) and (21).

$$D = \frac{-(m_1 + m_2)(1 + m_1m_2)}{r(1 - m_1{}^2)(1 - m_2{}^2)} \cdot \frac{\Omega[\Omega^2 - (1 - r^2)]\left[\Omega^2 - \dfrac{1}{1 + m_1m_2}\right]}{\left(\Omega^2 - \dfrac{1}{1 - m_1{}^2}\right)\left(\Omega^2 - \dfrac{1}{1 - m_2{}^2}\right)} \qquad (23)$$

Equations (22) and (23) give the location of the poles and zeros. The two peaks of attenuation appear in familiar form as the two factors in the denominator or when $U = V$ in Eq (22). It should be noted that for $r < 1$ there are two finite zeros in the passband, one determined by the product of the m values and the other determined solely by r (the ratio of the design resistance to the terminating resistance). For unity ratio, when load resistance and design resistance are equal, the second passband zero moves back to zero frequency. The expression for D shows that if either m_1 or m_2 is given the value of unity, then the corresponding peak of attenuation moves to infinite frequency. For two constant k sections, both m's are equal to unity and the passband zero determined by the m's occurs at 0.707 times the cutoff frequency. The most interesting possibility indicated by Eq (23) is that of controlling the passband ripple by proper choice of the design resistance (relative to the terminations). The zero determined by r can be located freely without regard to the other design constants at the point in the passband which gives the minimum ripple. This is the point which makes the two ripple peaks of equal amplitude. In Fig. 4-2 the solid curves show the single passband zero and single ripple occuring when the load and source have the same resistance. The dashed curve shows what happens when the terminated resistances are increased 20 per cent, corresponding to $r = 0.833$ for this particular design. A second passband zero is brought in at $1 - r^2$ or at 0.553 times the cutoff frequency. This reduces the ripple amplitude by more than 3 to 1. As the termination resistance is made greater than the design resistance, the two ripples can be made equal and the amplitude reduction is then more than 4 to 1. The passband ripple can be controlled by selecting the terminating resistance in much the same way that the stopband peaks and valleys can be controlled by choice of the m values.

The filter specification is shown in Fig. 4-3 in terms of maximum passband ripple and minimum stopband loss. The optimum design in this case (for a given width of the transition region) is the one for which the two m values are selected to give equal stopband minimums (valleys) and the design resistance is selected to give equal passband ripples. The cutoff is indicated as f_1 and the start of the stopband is f_2.

In the case of the Zobel filter with three constant K sections,

$$D = (3\Omega + 16\Omega^3 + 16\Omega^5)r - (3\Omega - 19\Omega^3 + 32\Omega^5 - 16\Omega^7)\frac{1}{r}$$

$$D = \Omega(3 - 4\Omega^2)(1 - 4\Omega^2)\left[(1 - \Omega^2)\frac{1}{r} - r\right] \qquad (24)$$

When one of the factors in Eq (24) disappears, the value of D vanishes, and with it vanishes the affective attenuation.

In the case of a pi-section filter, one zero will be at

$$(1 - \Omega^2)\frac{1}{r} - r = 0$$

For the T-section filter one zero will be at

$$(1 - \Omega^2)r - \frac{1}{r} = 0, \text{ or } R_1 = R\sqrt{1 - \Omega^2}$$

The second zero will be at $\Omega = 0$ independent of the loading resistance. The third zero will occur when

$$\Omega = \sqrt{\frac{1}{4}} = \pm 0.5$$

and

$$\Omega = \sqrt{\frac{3}{4}} = \pm 0.87$$

also independent of load resistance.

It is possible with the aid of Eq (24) to calculate attenuation as a function of normalized frequency for different values of load and design impedances.

From Fig. 4-4, it is evident that, for the lower part of the passband, the best ratio for Z_T/R is 1.1. In the upper part of the passband the best ratio is 1.5. For practical reasons, only a single value of this ratio can be used.

Appendix 5—Synthesis Procedure for Non-Ideal Bandpass Filters

Let us assume that the following filter specification is given:

Input and output impedance = 50 ohm (voltage source).
Passband is maximally flat (no ripples).
1 db bandwidth = 8.4 mc = bw_p (passband width)
40 db bandwidth = 35 mc = bw_s (stopband width)
Center frequency, $f_o = \sqrt{f_1 f_2} = 100$ Mc

A solution follows:

1. $bw_s/bw_p = 35/8.4 = 4.17 = \Delta f_{40\text{db}}/\Delta f_{1\text{db}}$ (the response form factor)
2. voltage ratio in the passband = 1.122/1
3. voltage ratio V_p/V_s corresponding to 40 db = 100/1

The number of resonators required is 3.54 (calculated from Cited Reference [3]), so four resonators will be used, as shown in Fig. 5-1. The number of resonators can also be found with tables if the amount of ripple (in V_p/V) is known, and the 3-db bandwidth is determined from the appropriate lower set of curves of Cited Reference [3], pp. 193–198. With known $bw_{3\text{db}}$, the next step is to find $bw_s/bw_{3\text{ db}}$, and, using the same ripple factor, to find the attenuation in the stopband (bw_s) from the upper set of curves which belong to the filter of given complexity.

The ratio $bw_p/bw_{3\text{db}} = 0.84$ corresponds to $bw_{3\text{db}} = 10$ Mc. For the ratio $bw_{40\text{db}}/bw_{3\text{db}} = 3.5$ and zero db ripple, we find that four resonators will provide approximately 40 db rejection.

The necessary constants for a network having no peaks of attenuation and a transfer function with four poles have been tabulated. [3] The following constants for the case of equal resistive terminations are obtained from that tabulation:

$$q_{2,3} = 26;\ q_1 = q_4 = 0.776$$
$$k_{12} = k_{34} = 0.840;\ k_{23} = 0.542$$

Note that the filter has a physical symmetry relative to the center of the structure which is characteristic of all filters of the Chebyshev family. Minimum unloaded Q of the two internal resonators will be

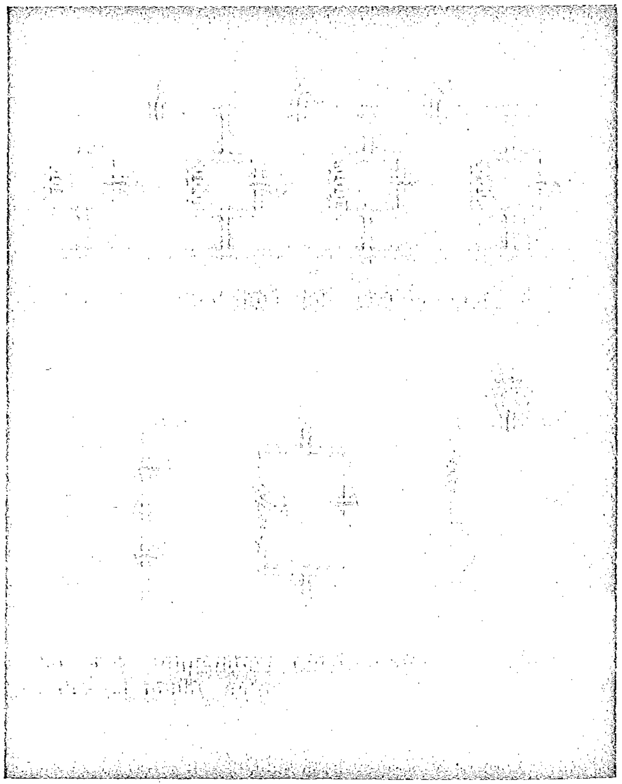
$$Q_{\min 2,3} = q_{2,3}\frac{f_o}{\Delta f_{3\text{db}}} = 260$$

To estimate insertion loss in the passband of the filter to be constructed the following procedure has to be used:

- Measure the actual Q of the resonators on the Q meter.
- Obtain the value of $q_{\min}$ from the table. [3]
- Calculate $Q_{\min} = q_{\min}f_o/\Delta f_{3\text{db}}$.
- Determine the coefficient, $u = Q/Q_{\min}$.
- Using the curves given by Fubini and Guilliman, determine the insertion loss for the number of resonators used. [4]

In the problem above the coefficient $u = 10$, and from the curve the insertion loss is found to be 0.9 db.

The required Q_1 of the first resonator may be obtained by choosing such values of the nodal inductance and capacitance (in the case of capacitive coupling) that the generator resistance produces the desired Q_1. The alternative method is to use the transforming circuit to couple the nonresonant generator to the first node. For most applications this technique is advisable because it allows a choice of values of L and C that are easily realized in practice. A reasonable value of inductance for the filter designed to operate at 100 Mc is about 0.075 μh. This value will be used for all coils, which will, by the same token, require a nodal capacity of 33.8 pf in all cases. If capacitive coupling is used, these elements and the shunt capacitors can be calculated as follows.



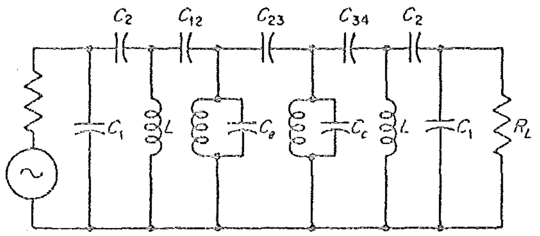


Fig. 5-3—Final four-pole filter design.

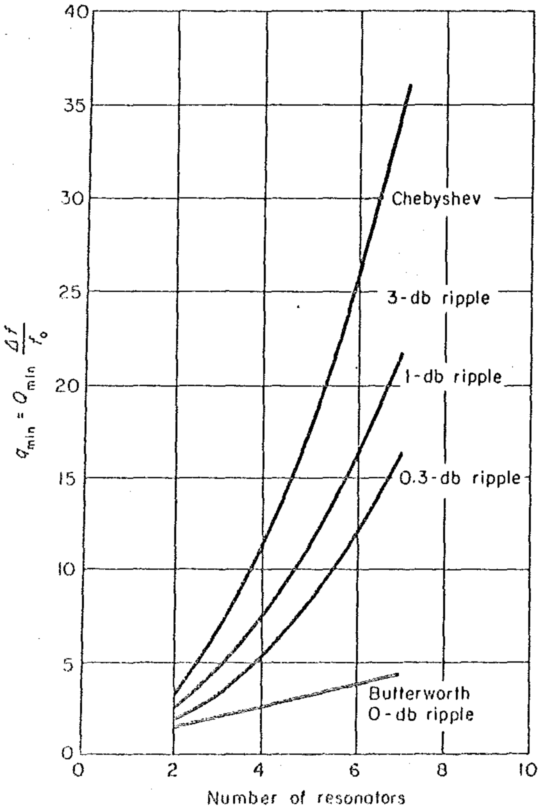


Fig. 5-4—Relative minimum unloaded Q for Butterworth and Chebyshev filters. Q_{min} is the unloaded Q for which the loss is infinite. $\Delta f/f_o$ is the relative bandwidth.

$$C_{12} = k_{12}(\Delta f_{3db}/f_o)\sqrt{C_I C_{II}}$$

(The C's with subscript Roman numerals designate node capacitors.) C_{12} = 0.84(10/100)33.8 = 2.48 pf. The first nodal capacitor is set at 33.8 so $C_a = C_I - C_{12} = 31.32$.

$$C_{23} = k_{23}(\Delta f_{3db}/f_o)\sqrt{C_{II} C_{III}} = 1.83$$

where $\sqrt{C_I C_{II}} = \sqrt{C_{II} C_{III}}$ and so on; $C_I = 33.8$ pf.

$$C_b = 33.8 - 1.83 - 2.48 = 29.49 \text{ pf}$$
$$C_{34} = C_{12} = 2.48 \text{ pf}$$
$$C_c = C_b = 29.49 \text{ pf}$$
$$C_d = C_a = 31.32 \text{ pf}$$

The circuit design for the filter calculated above is shown in Fig. 5-1. The design is considered complete except that the proper transformation of the generator impedance and the first resonator has to be done to produce the required quality factor in the first resonator. The same transformation has to be performed with the last resonator. The most practical transformation for the above case is shown in Fig. 5-2. The shunt capacitor may be used to absorb the distributed capacitance usually associated with the input and output circuit. The transforming circuit can be calculated in the following fashion:

$$\frac{C_1}{C_2} = \sqrt{\frac{R_x}{R_1}}$$

$$\frac{C_1 C_2}{C_1 + C_2} = C_a$$

where R_x is the transform value of R_1 required to produce the specified Q. Incidental coil dissipation (neglecting capacitive losses) is accounted for in the following way:

$$\frac{1}{R} = \frac{1}{R_x} + \frac{1}{Q_L X_L}$$

In this equation R is the value of nodal shunt resistance required to obtain the specific Q_{out}. The output transformation is similar to the input transformation, and the final circuit of the completed filter is shown in Fig. 5-3. Fig. 5-4 shows the relative minimum unloaded Q for Butterworth and Chebyshev filters.

Cited References

[1]. "Insertion Loss and Insertion Phase Shift of Multi-Section Zobel Filters with Equal Image Impedances," W. Saraga, *P.O. Electrical Engineering*, Vol 39, January 1947, pp 167–172.

[2]. "Design of Dissipative Band-Pass Filters Producing Exact Amplitude Frequency Characteristics," M. Dishal, *Proc. IRE*, Vol 37, September 1949, pp 1050–1069.

[3]. *Reference Data for Radio Engineers*, 4th ed., ITT Corp., New York (1956).

[4]. "Minimum Insertion Loss Filters," Fubini and Guillemin, *Proc. IRE*, Vol 47, January 1959.

Bibliography

"Theory and Design of Electric Wave Filters," O. Zobel, *Bell Syst. Tech. J.*, January 1923.

"Synthesis of Parameter Symmetrical Filter," A. Grossman, *Proc. IRE*, April 1957.

Reference Data for Radio Engineers, 4th ed., ITT Corp., New York (1956).

"Direct Coupled Resonator Filters," S. Cohn, *Proc. IRE*, February 1957.

"Realization of Filters with Helical Components," A. Zverev and H. Blinchikoff, *IRE Trans. PGCP*, September 1961.

"Filters in Electronics," A. Zverev, *Westinghouse Engineer*, March 1963.

"Microwave Filters," E. Torgow, ELECTRO-TECHNOLOGY, April 1961.

Network Analysis and Synthesis, L. Weinberg, McGraw-Hill Book Co., New York (1962).

"Network Transformations in Ware Filter Design," A. Zverev and H. Blinchikoff, *Electronics*, June 26, 1959.

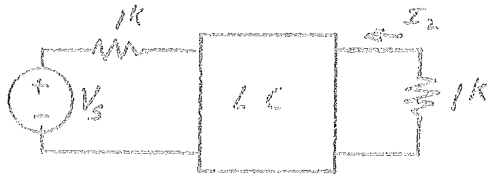
EE 593 NETWORK SYNTHESIS II Jan. 11, 1973

Homework Problems

16-A1

Realize the high-pass transfer admittance whose squared magnitude is given by

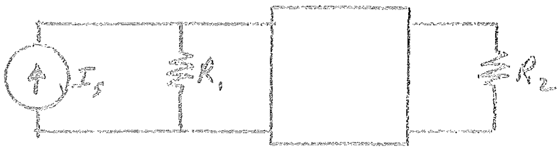
$$\left| -Y_{S2}(j\omega) \right|^2 = \frac{1}{4} \frac{\left(\frac{\omega}{30}\right)^6}{1 + \left(\frac{\omega}{30}\right)^8}$$



as a lossless ladder network terminated at both ends in a 1000-Ohm resistance.

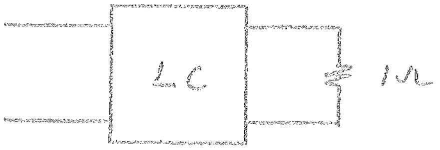
16-A2

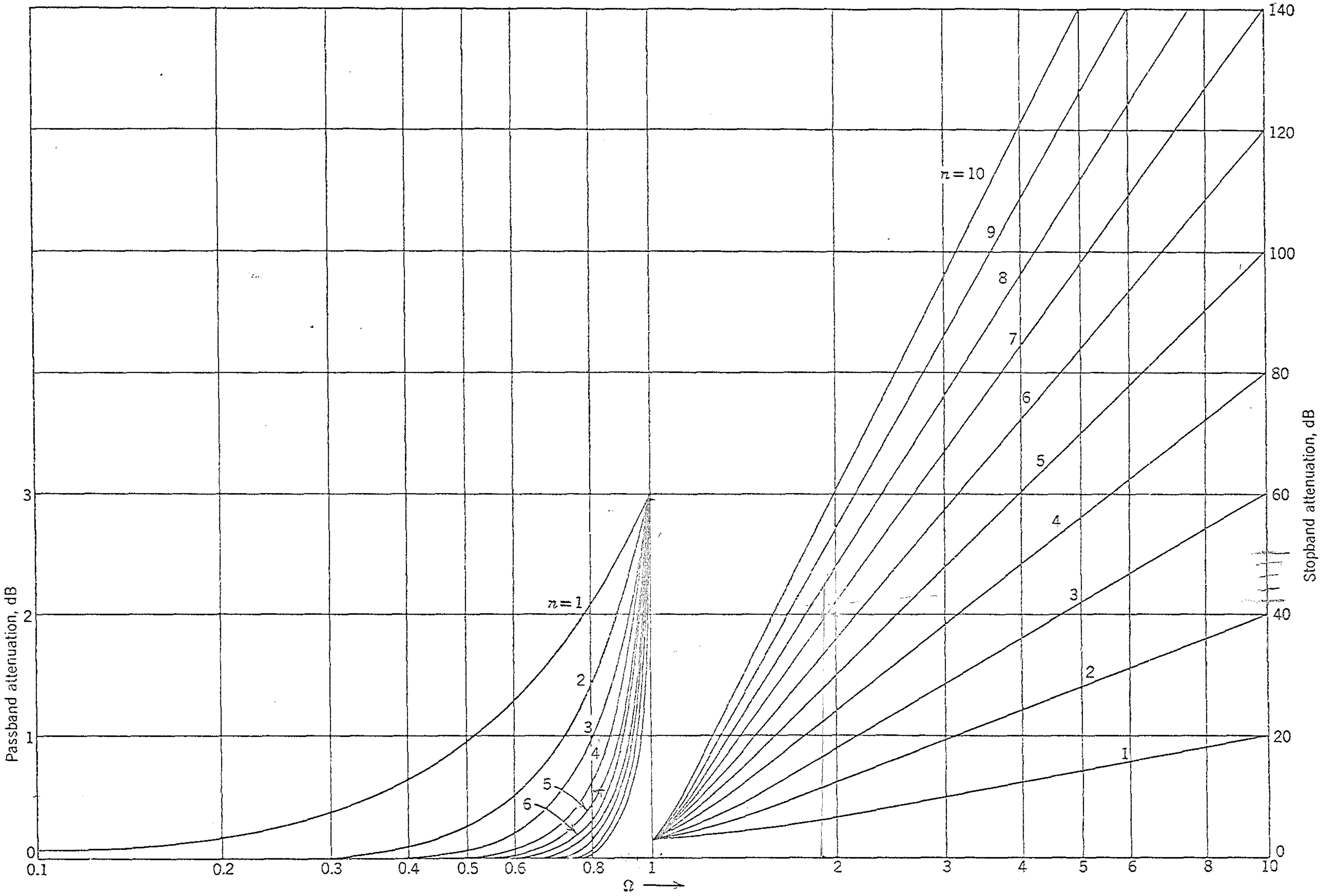
Given $$F(s) = \frac{s^2}{(s+1)^2}$$



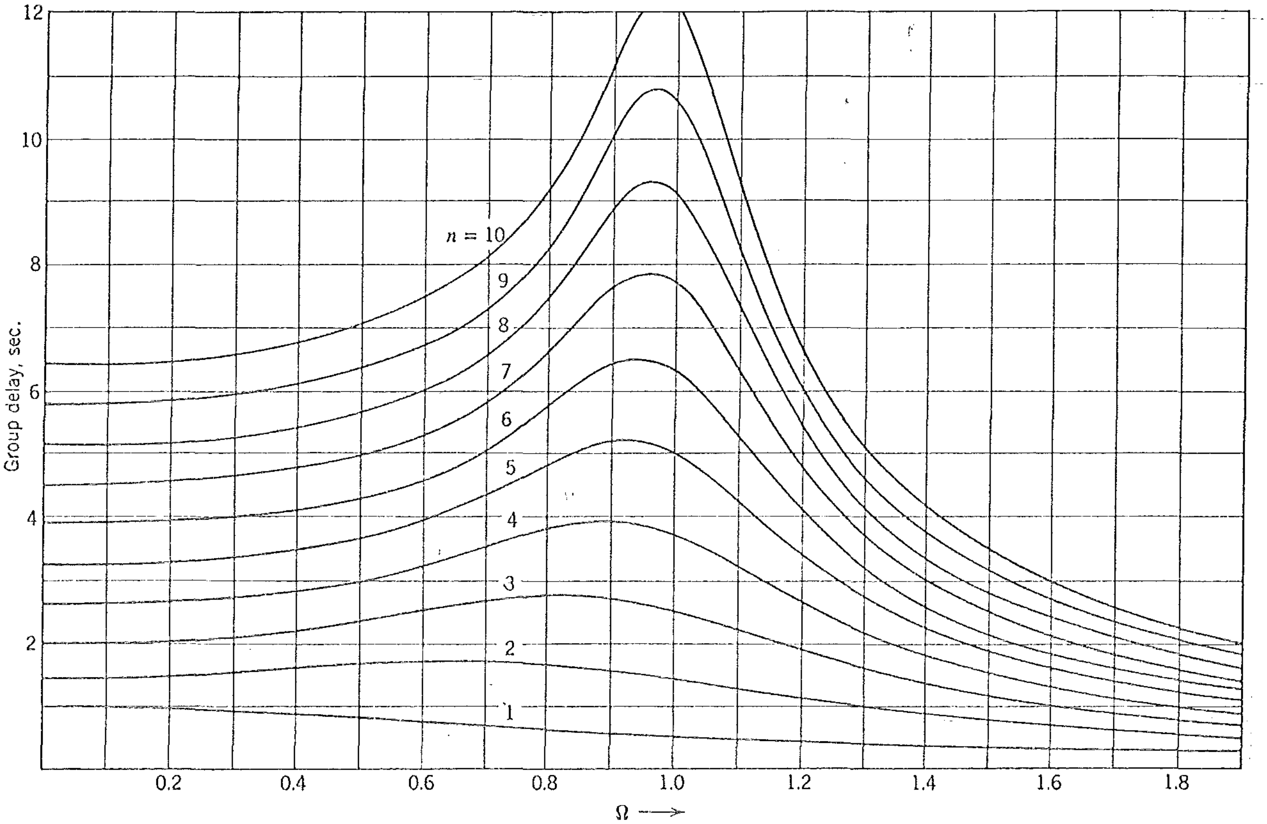
a) If this is the transmission function $t = \frac{2\,V_2}{I_s}$ of a lossless quadripole terminated in a 1-Ohm resistance at both ends, find the network.

b) If this is the transfer impedance $Z_{12} = \frac{V_2}{I_1}$ of a lossless quadripole terminated in a load resistance of 1-Ohm, find the network.

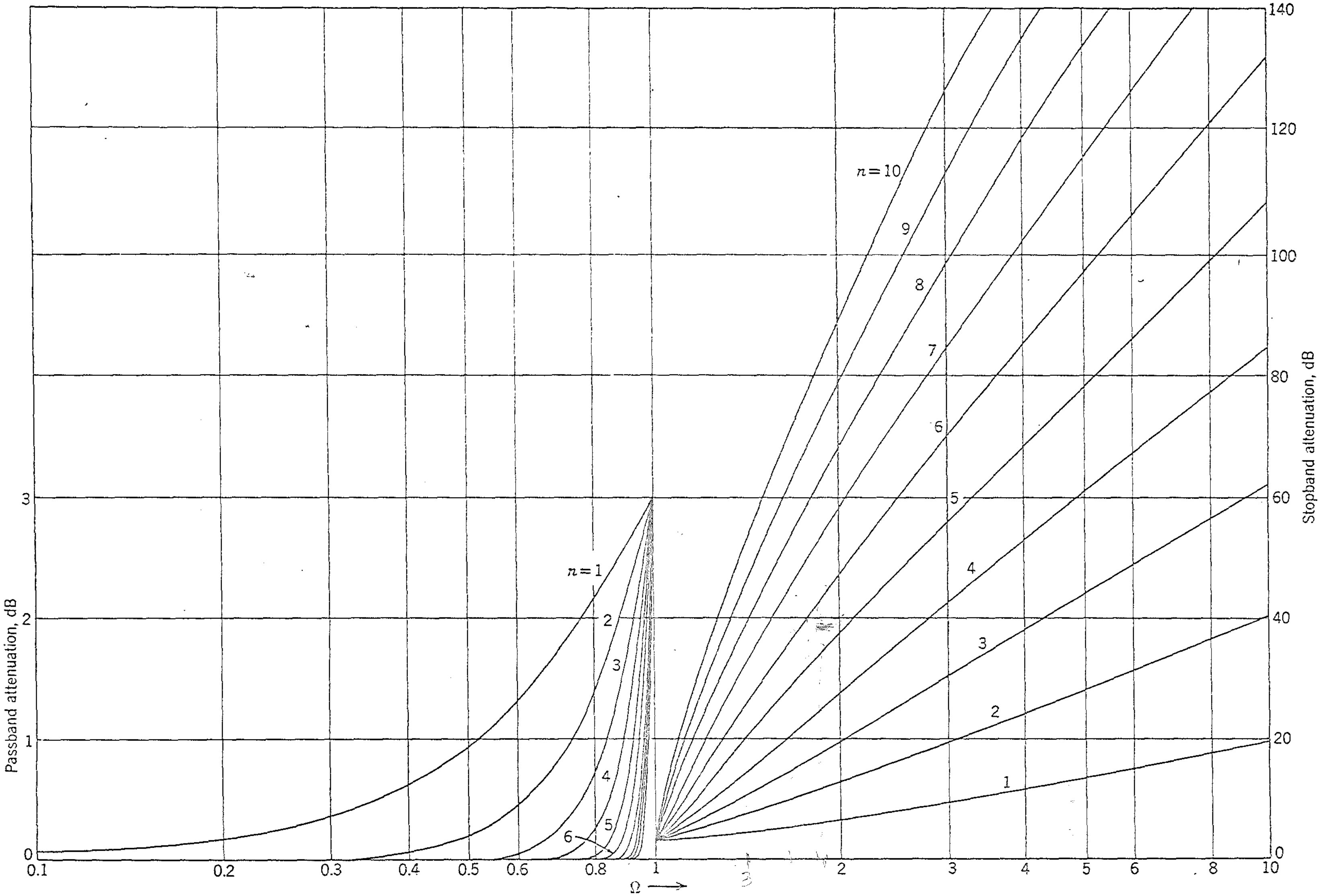




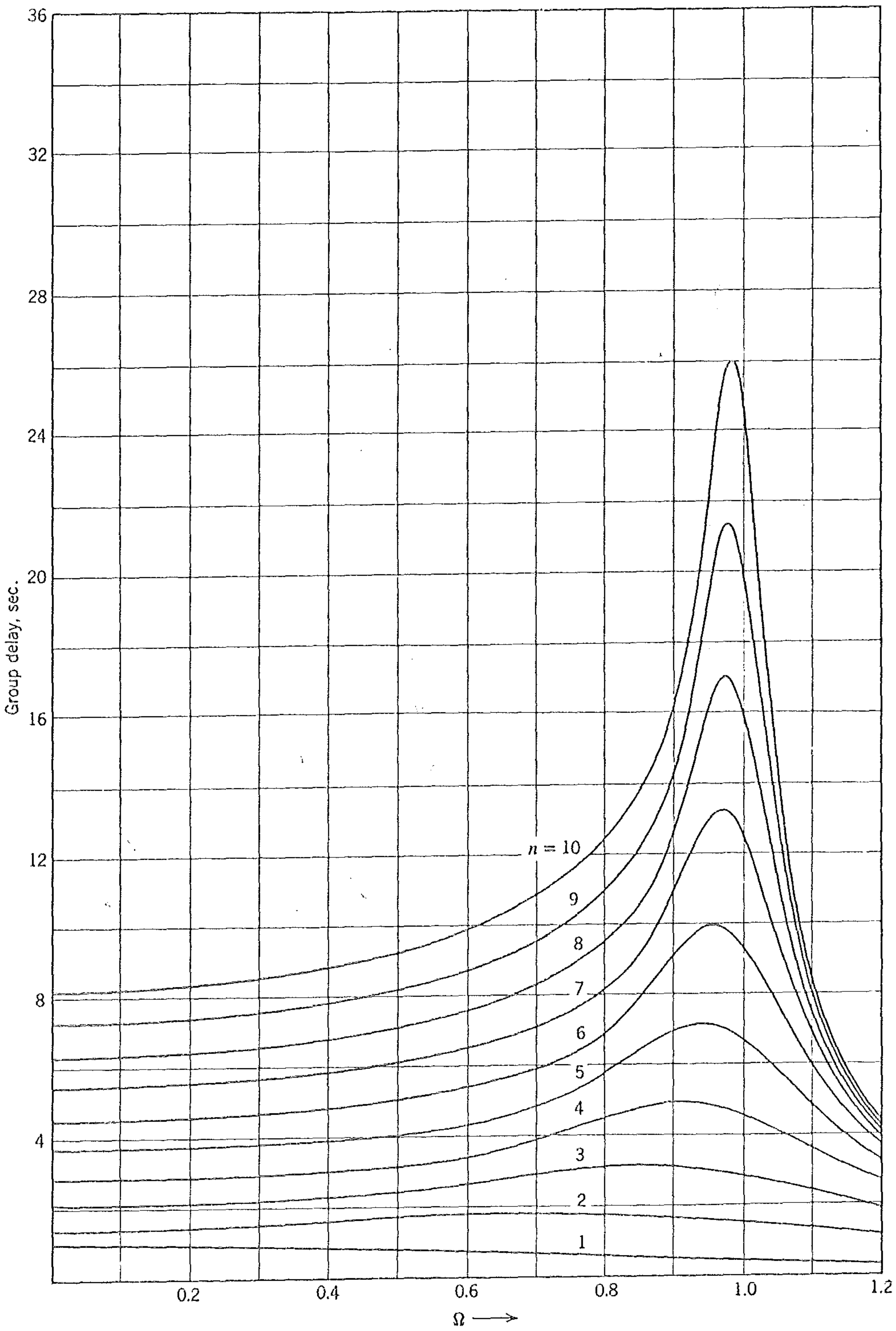
Curve 1. Attenuation characteristics for Butterworth filters.



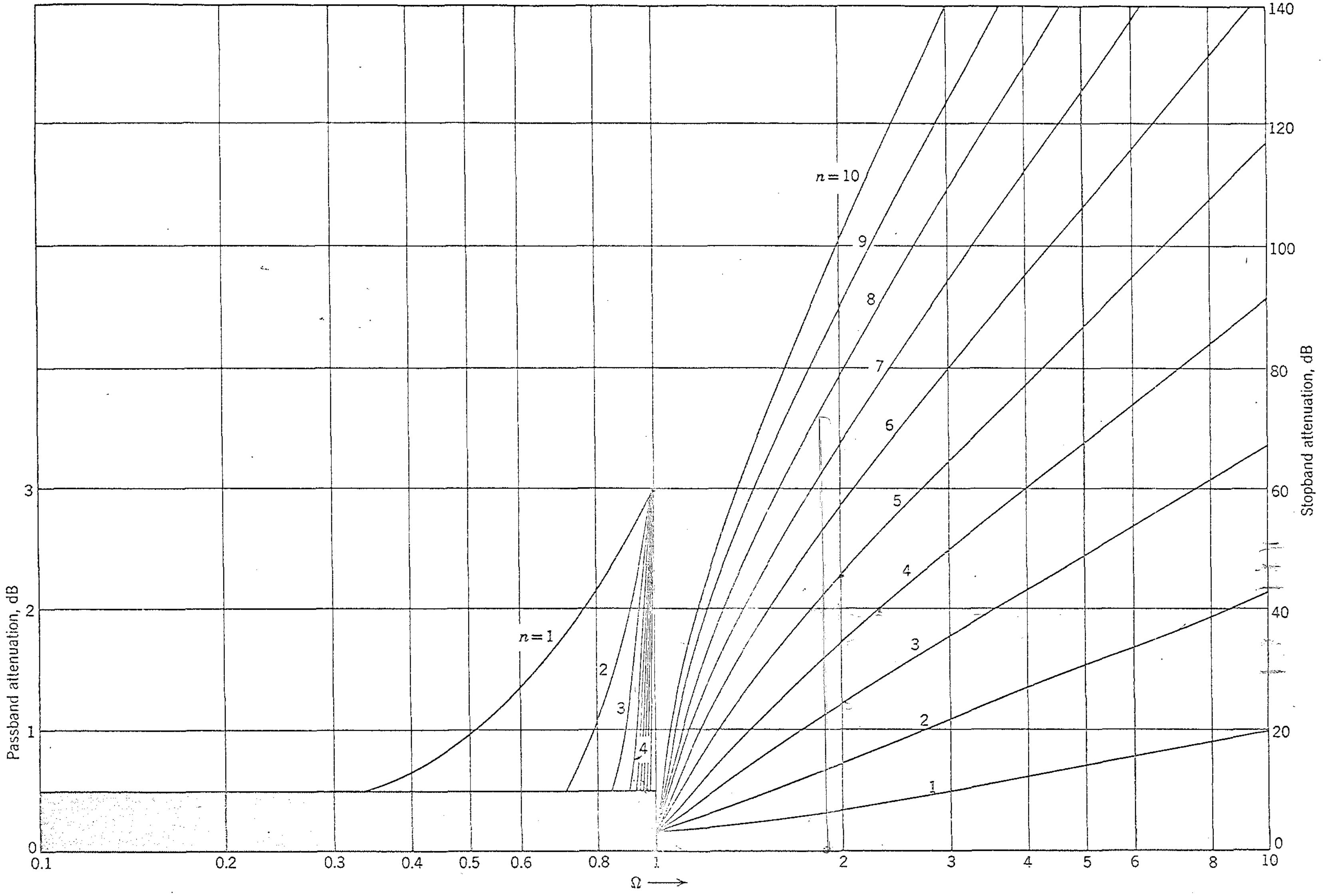
Curve 2. Group-delay characteristics for Butterworth filters.



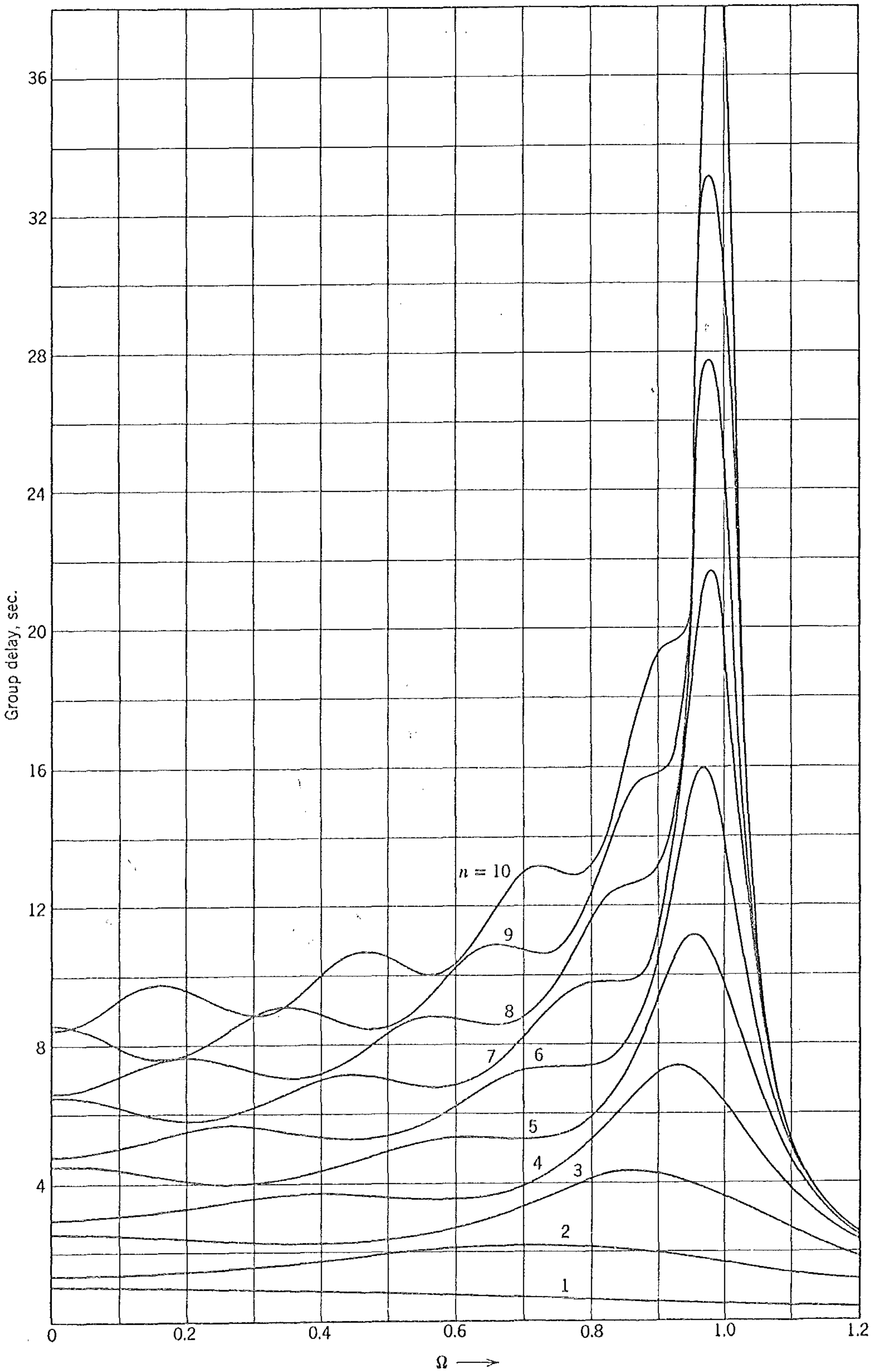
Curve 3. Attenuation characteristics for Chebyshev filter with 0.01 dB ripple.



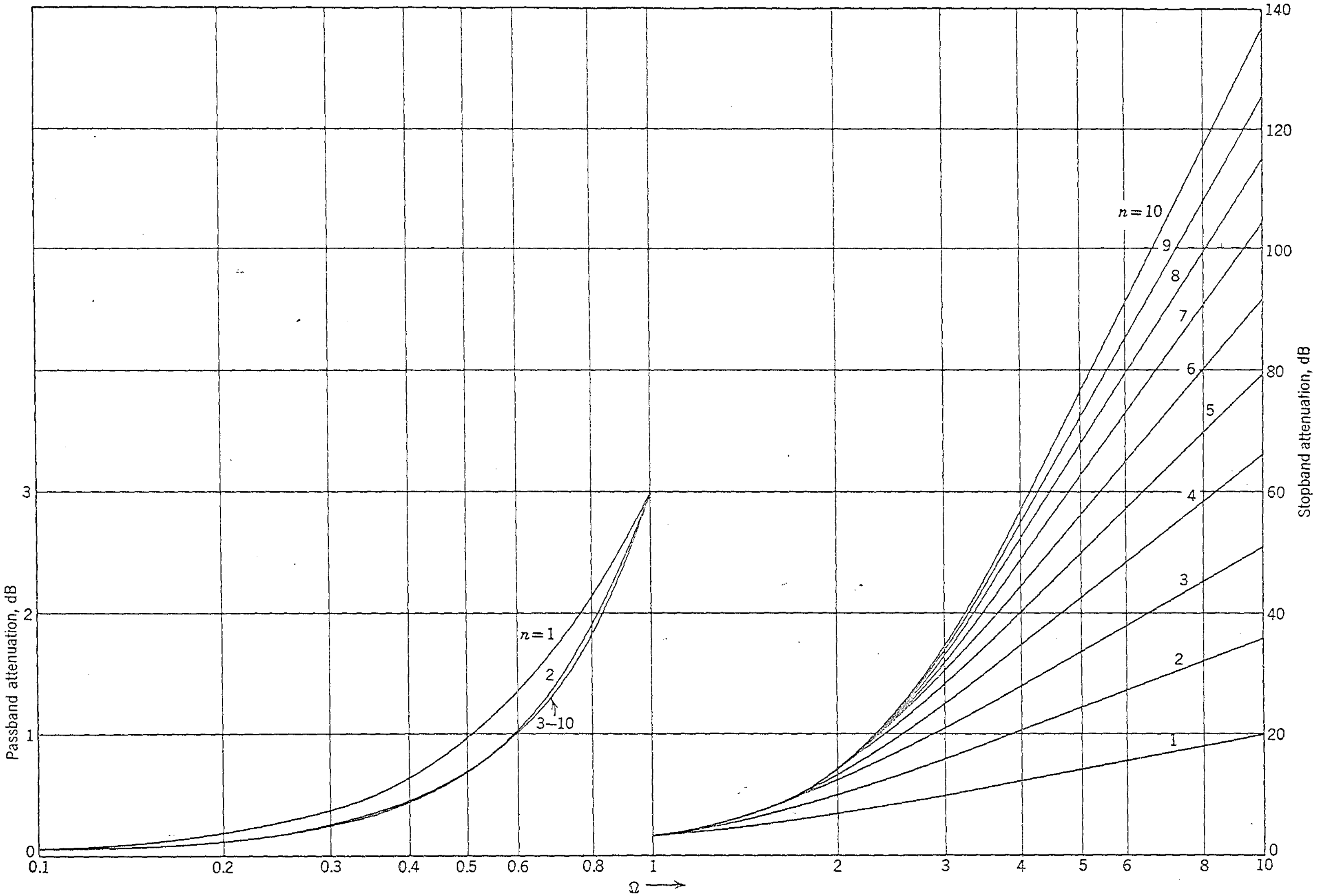
Curve 4. Group-delay characteristics for Chebyshev filter with 0.01 dB ripple.



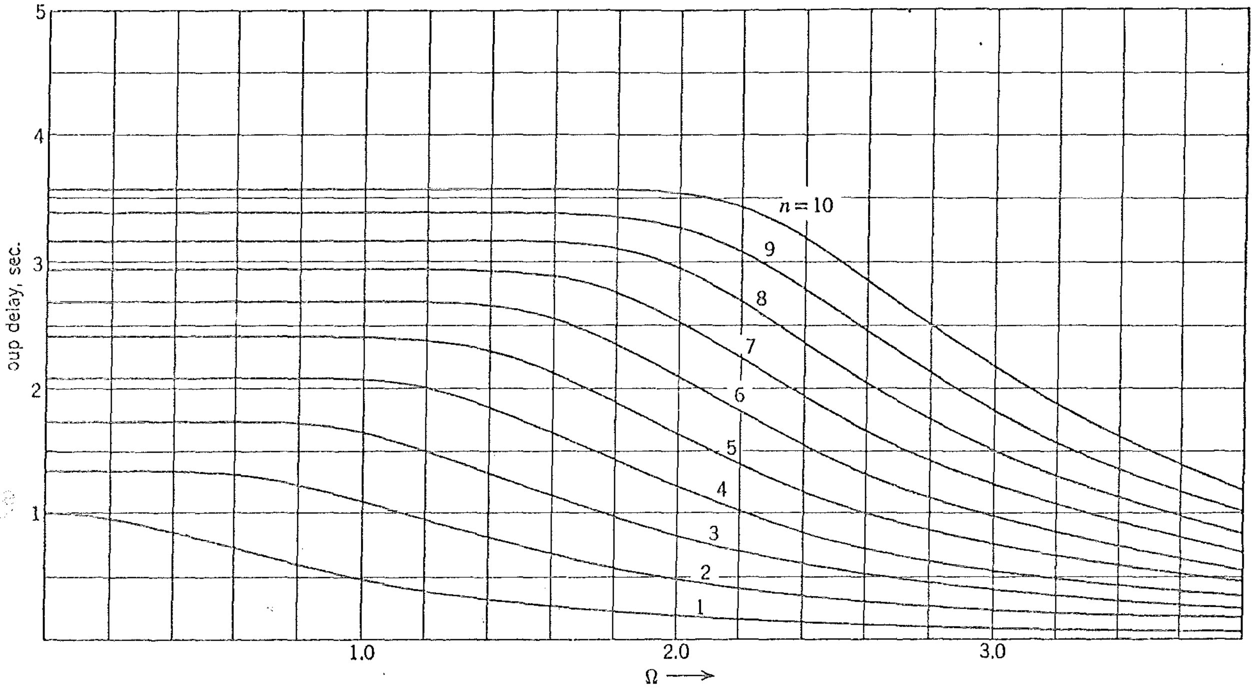
Curve 7. Attenuation characteristics for Chebyshev filter with 0.5 dB ripple.



Curve 8. Group-delay characteristics for Chebyshev filter with 0.5 dB ripple.



Curve 11. Attenuation characteristics for maximally flat delay (Bessel) filters.

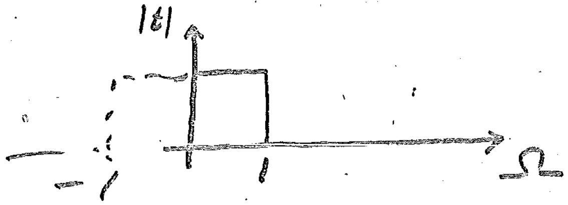


Curve 12. Group-delay characteristics for maximally flat delay (Bessel) filters.

16.7 Frequency Transformations (p. 479)

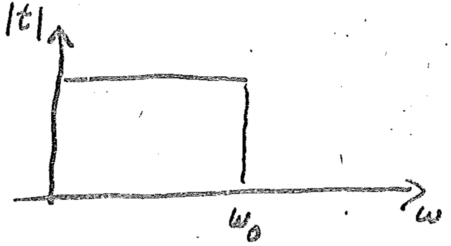
Let $\mathcal{S}$ correspond to the normalized L.P. filter, and

$\Omega = \mathrm{Im}\{\mathcal{S}\}$



Let s correspond to the actual filter, and

$\omega = \mathrm{Im}\{s\}$

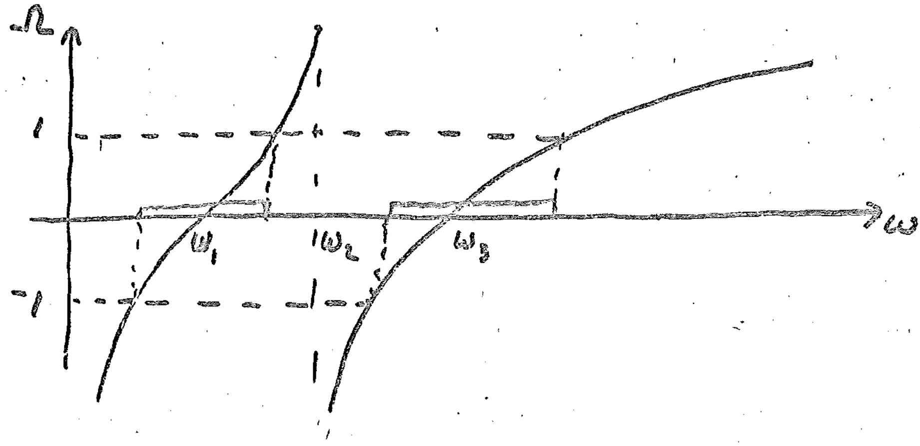


For simple frequency scaling $\mathcal{S} = \frac{s}{\omega_0}$, $\Omega = \frac{\omega}{\omega_0}$

Reactance Transformations

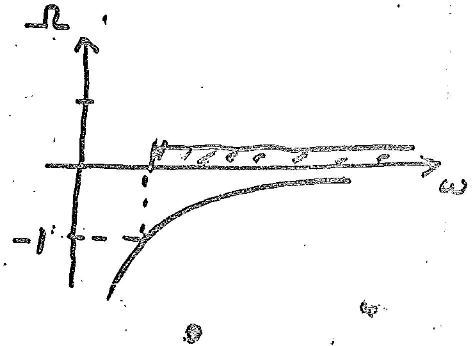
$\Omega = Z_{LC}$ function of ω

Example: $\Omega = \dfrac{(\omega_1^2-\omega^2)(\omega_3^2-\omega^2)}{\omega(\omega_2^2-\omega^2)}$



LP → HP

$$S = \frac{1}{s} \quad , \quad \Omega = -\frac{1}{\omega}$$



Design Procedure for HP Filter

1. HP specification ⟹ LP spec. (normalized)

2. HP attenuation requirements ⟹ L.P. attenuation

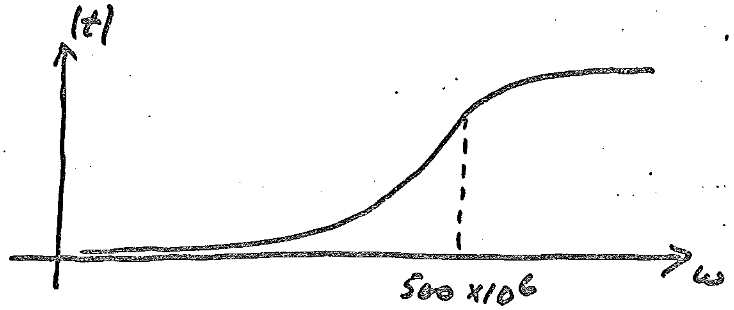
3. Realize the normalized LP filter.

4. Replace L by C $\quad C_i = \frac{1}{L_i}$

   Replace C by L $\quad L_j = \frac{1}{C_j}$

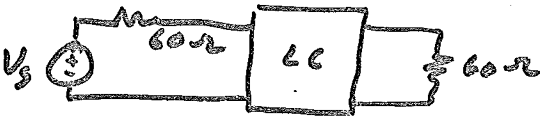
5. Remove the frequency and impedance normalization

Example:



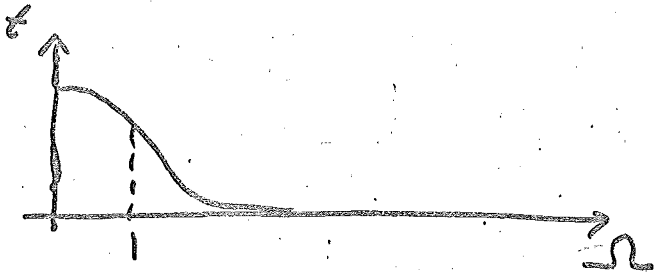
75 MHz

$150\pi \approx 500 \times 10^6$



Assume that a 3rd order Butterworth filter will be adequate

LP spec



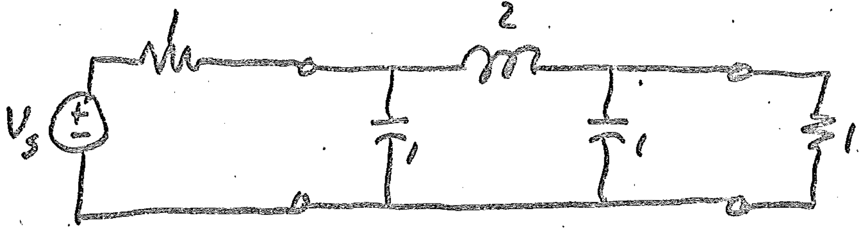
From Example 1 p. 435

$$|t(j\Omega)|^2 = \frac{1}{1+\Omega^6}$$

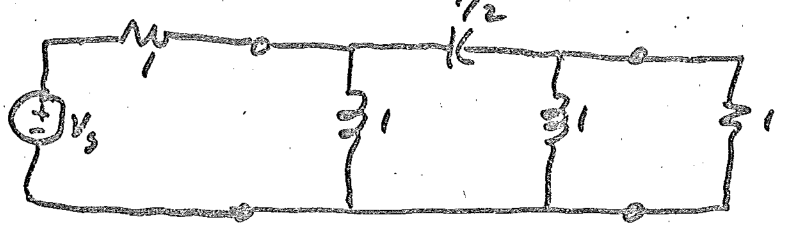
$$|\rho_1(j\Omega)|^2 = \frac{+\Omega^6}{1+\Omega^6}$$

$$\rho_1(s) = \frac{+s^3}{s^3+2s^2+2s+1}$$

$$z_{11} = R_1\frac{1-\rho_1}{1+\rho_1} = \frac{2s^2+2s+1}{2s^3+2s^2+2s+1}$$



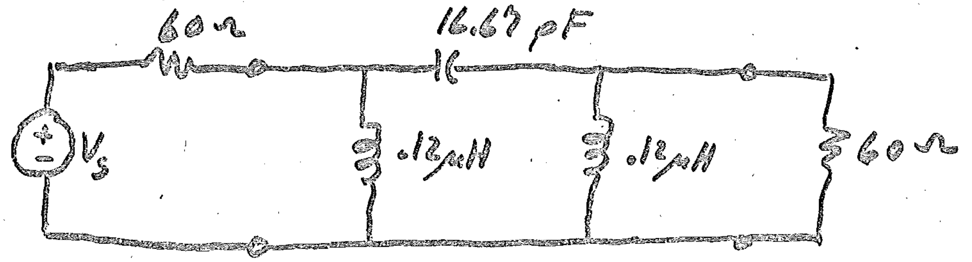
Change to HP



$L' = L\frac{60}{500\times10^6}$

$C' = C\frac{1}{60\times500\times10^6}$

Change to $\omega_0 = 500\times10^6$ and $R = 60\,\Omega$



For computation purposes the LP prototype can be used as follows:

Suppose we are interested in attenuation at $\omega = 200 \times 10^6$ rad/sec.

From $$\Omega = -\frac{\omega_0}{\omega} = -\frac{500}{200} = -2.5$$

$$\Omega^2 = 6.25$$

$$\Omega^6 = 244.14$$

$$|t(j\Omega)|^2 = \frac{1}{1+\Omega^6} = \frac{1}{1+244.14} = 4.08 \times 10^{-3}$$

$$dB = 10 \log_{10} |t(j\Omega)|^2 = \underline{\underline{-23.9\ dB}}$$

is the ~~atten~~ gain at $\omega = 200 \times 10^6$ rad/sec

* * *

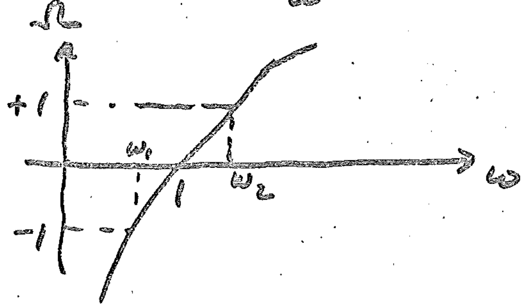
$\underline{LP - BP}$

$$S = \frac{s^2 + \overset{1}{\cancel{\omega_0^2}}}{s}$$

$$\Omega L = \frac{\omega^2 - 1^{\omega_0^2}}{\omega} L = \omega L - \frac{L\omega_0^2}{\omega}$$

$$- \frac{1}{\frac{1}{L\omega_0^2}\omega}$$

$$\Omega = \frac{\omega^2 - 1^{\omega_0^2}}{\omega}$$



when $\Omega = \pm 1$

$$\omega^2 \mp \omega - 1^{\omega_0^2} = 0$$

$$\omega = \frac{\pm 1 \pm \sqrt{1 + 4^{\omega_0^2}}}{2}$$

$$\omega_2 = \frac{1}{2} + \sqrt{\frac{1}{4} + 1^{\omega_0^2}}$$

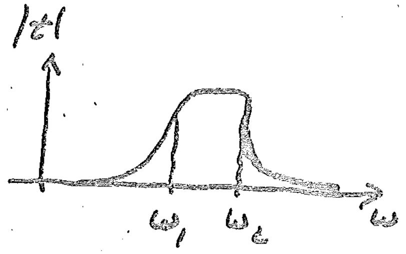
$$\omega = -\frac{1}{2} + \sqrt{\frac{1}{4} + \omega_0^2}$$

$BW = \omega_2 - \omega_1 = 1$

$\omega_1 \omega_2 = \left(\sqrt{\tfrac{1}{4}+1} - \tfrac{1}{2}\right)\left(\sqrt{} + \tfrac{1}{2}\right) = \tfrac{1}{4} + 1 - \tfrac{1}{4} = 1 \Rightarrow \underline{\underline{\omega_0^2}}$

Design Procedure for BP Filter (with geometric symmetry)

1. Determine $BW = \omega_2 - \omega_1$
   $\omega_0 = \sqrt{\omega_1 \omega_2}$



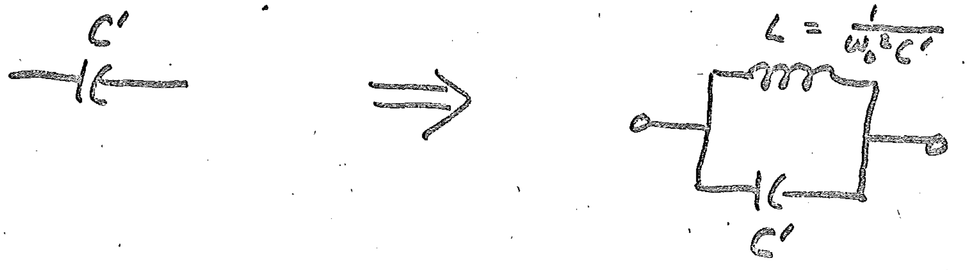
2. BP spec. $\Longrightarrow$ LP spec. with BW normalized to 1 $\left[S = \dfrac{s^2 + \omega_0^2}{(BW)\, s} \right]$

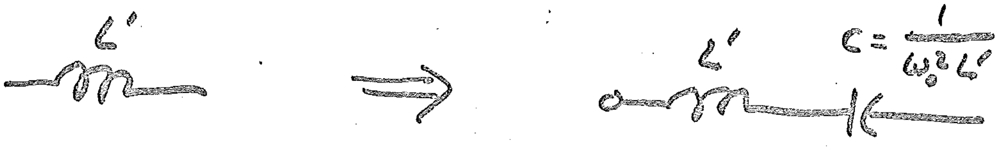
3. Realize the LP prototype filter

4. Denormalize LP components with respect to BW and impedance level.

$$L' = \frac{1}{BW} L \; K_z$$
$$C' = \frac{1}{BW} C \; \frac{1}{K_z}$$

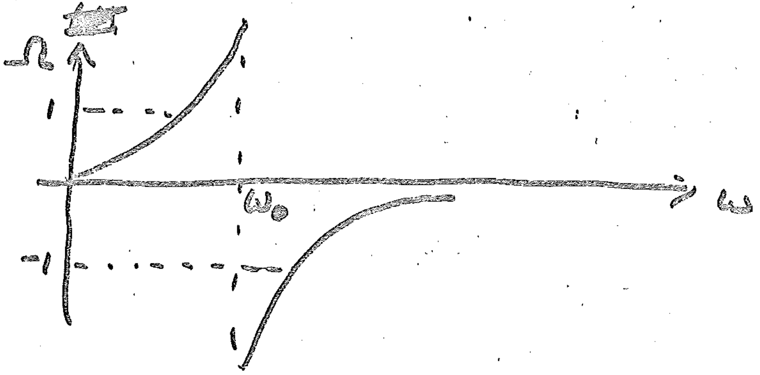
5. Convert to BP filter with center frequency ω_0





LP → BE

$$\mathcal{S} = \frac{s}{s^2+\omega_0^2} \qquad \Omega = \frac{\omega}{\omega_0^2-\omega^2}$$

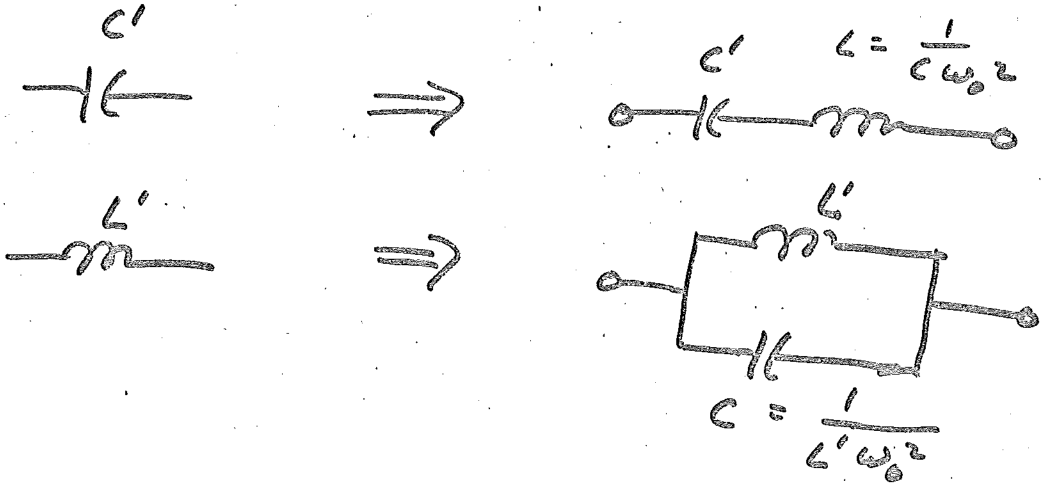


Including BW
$$\mathcal{S} = \frac{BW\ s}{s^2+\omega_0^2}$$

$$\Omega = \frac{(BW)\ \omega}{\omega_0^2-\omega^2}$$

After obtaining LP prototype filter, Denormalize BW & impedance level.

Then convert to BE filter



Note: These transformations do not preserve linear phase characteristics. — $\frac{d\beta}{d\omega} = \frac{d\beta}{d\Omega}\ \frac{d\Omega}{d\omega}$

BOB MARKS

IT IS DESIRED TO REALIZE A HIGH PASS FILTER WITH NORMALIZED TRANSFER FUNCTION:

$$G_{12}(s) = \frac{h s^2}{s^2 + ds + 1}$$

THE DESIGN PROCEDURE FOLLOWS FROM "A PRACTICAL METHOD OF DESIGNING RC ACTIVE FILTERS" BY SALLEN AND KEY IN *ACTIVE INDUCTORLESS FILTERS* ED. MITRA. FOR $h = 1$ THE TRANSFER FUNCTION $|G(j\omega)|$ IS SHOWN IN FIG. 5b ON Pg. 60. THE NORMALIZED TRANSFER AT $\omega_0 = 1$ IS $1/d$, THE PARAMETED d IS EFFECTIVELY A MEASURE OF THE CIRCUITS Q, AS CONCLUDED FROM SUCH EQUATIONS AS Eq. 3 FROM "VERY HIGH Q INSENSITIVE ACTIVE CIRCUITS" BY TARMY AND GHAUSI, Pg 84; AND Eqs. 23 & 24 FROM "R.C. AMPLIFIERS..." BY GEFFE ON Pg. 74. THUS:

$$Q = 1/d$$

IS A MEASURE OF A "HIGH PASS Q"

LET THE FILTER SPECIFICATIONS BE FOR A HIGH PASS SECOND ORDER BUTTERWORTH TYPE. AGAIN:

$$G_{12}(s) = \frac{hs^2}{s^2 + ds + 1}$$

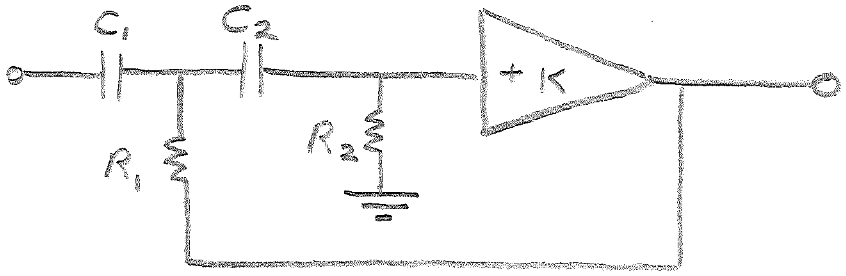
THUS: $$G_{12}(j\omega) = \frac{-h\omega^2}{-\omega^2 + jd\omega + 1}$$

$$|G_{12}(j\omega)|^2 = \frac{h^2\omega^4}{(\omega^2-1)^2 + d^2\omega^2}$$

IF $d = \sqrt{2}$:

$$|G_{12}(j\omega)|^2 = \frac{h^2\omega^4}{\omega^4 + 1}$$

BELOW IS THE CHOSEN HIGH PASS CONFIGURATION, AS SHOWN ON Pg 55 IN FIGURE 3;



DEFINE:

$T_1 = R_1C_1$; $T_2 = R_2C_2$; $\gamma = \frac{C_2}{C_1}$

RELATIONSHIPS:

$T_1T_2 = 1$; $h = K$

FOR COMPUTATIONS CONCERNING γ AND T_1, INTEREST IS CENTERED ON GROUP I ON Pg. 56. ASSIGNING A VALUE OF $\frac{3}{2}$ TO γ, EQUATION d OF GROUP I DICTATES

$$K_{MIN} = \frac{4(1+\gamma)-d^2}{4(1+\gamma)}$$

$$= \frac{(4)(2.5)-2}{4(2.5)}$$

$$= 0.8$$

SO LET THE VALUE OF K HAVE A CONVENIENT MAGNITUDE OF UNITY. THE CORRESPONDING VALUE OF T_1 IS GIVEN BY EQUATION b:

$$T_1 = \frac{d}{2(1+\gamma)}\left[1 \pm \sqrt{1 - \frac{4(1+\gamma)(1-K)}{d^2}}\right]$$

$$= \frac{d}{(1+\gamma)}$$

$$= \frac{1.41}{(2.5)}$$

$$= 0.564 \text{ SEC}$$

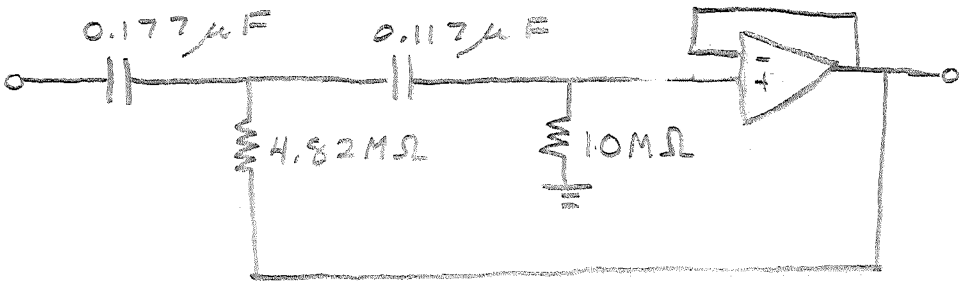
THEN:

$T_2 = {}^{1}/T_1 = 1.77$ SEC

AN ARBITRARY VALUE OF 10^7 IS ASIGNED R_2

$$\Rightarrow C_2 = \frac{T_2}{R_2}$$
$$= 0.177\ \mu F$$
$$C_1 = \frac{C_2}{\gamma} = \left(\frac{2}{3}\right)(0.177)$$
$$= 0.117\ \mu F$$
$$R_1 = \frac{T_1}{C_1} = \frac{0.564}{0.117}$$
$$= 4.82 \times 10^6$$

YIELDING THE CIRCUIT BELOW



TO SHIFT TO $f_0 = 10^2$ HZ, THE RESISTIVE ELEMENTS ARE DIVIDED BY $\omega_0 = 2\pi \times 10^2$

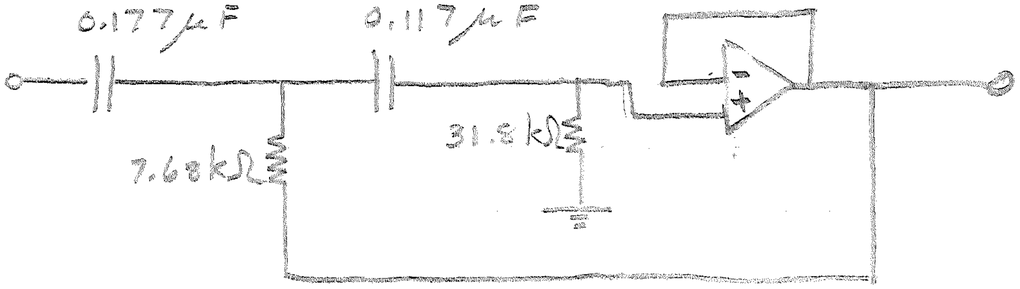
($f_0 \triangleq$ FREQUENCY OF GREATEST TRANSFER)

$$\Rightarrow R_2 = \frac{10^7}{2\pi \times 10^2} = 3.18 \times 10^4 \, \Omega$$

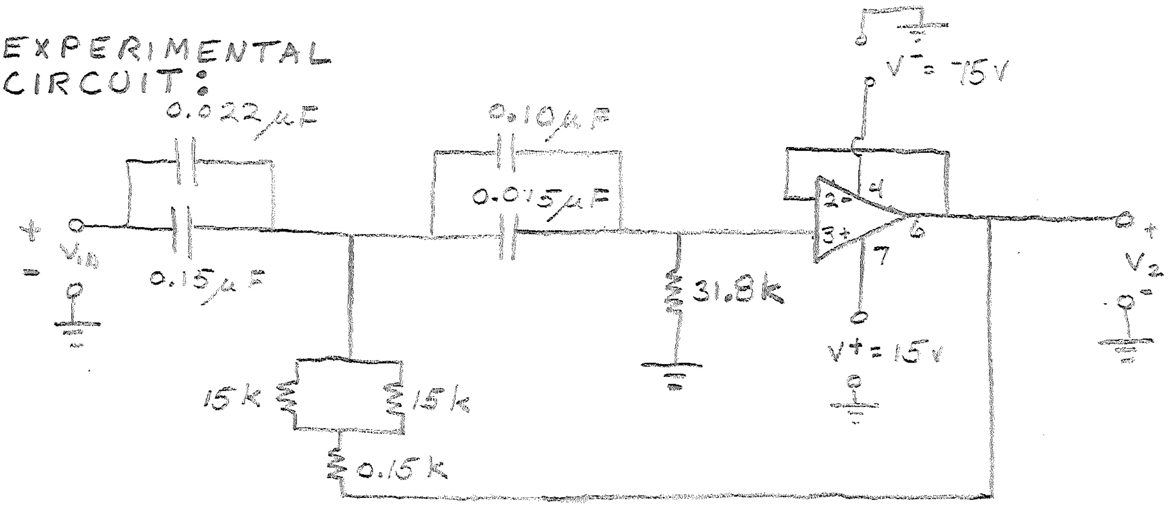
$$\doteq 31.8 \text{ k}\Omega$$

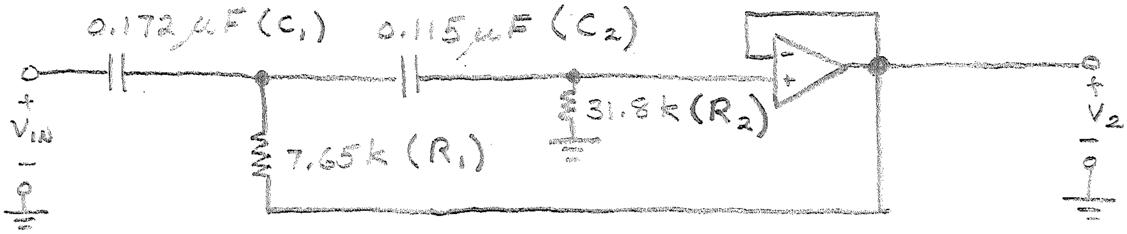
$$R_1 = \frac{4.82 \times 10^6}{2\pi \times 10^2} \doteq .768 \times 10^4$$

$$= 7.68 \text{ k}\Omega$$



EXPERIMENTAL CIRCUIT:





(FIG C)

741 FAIRCHILD I.C. OP AMP

DATA:

f (HZ)	V (VOLTS)	db	
100	8.4	0.75	(PEAK)
75	7.75	0	(0 db)
55	5.45	-3	(-3db POINT)
1000	7.70	-0.2	(ASYM. VALUE)

FOR THIS AND OTHER DATA:

INPUT: $V_0 = 7.75$ VOLTS (RMS) $\triangleq$ 0db

MEASUREMENT PERCISIONS:

f ⇒ ±2 HZ

V ⇒ ±0.05V

db ⇒ ±0.1db

SENSITIVITY CONSIDERATIONS

IT IS NOTED UNDER METHOD 2 OF POLE DESENSITATION (Pg 30) OF "THE OPERATIONAL AMPLIFIER IN LINEAR ACTIVE NETWORKS" BY MOSCHYTZ THAT THE GAIN SENSITIVITY FOR UNIT FORWARD GAIN FEEDBACK CONFIGURATIONS IS NEGLIBLE, AND WILL THUS NOT BE CONSIDERED.

THE SENSITIVITY OF A FUNCTION ψ WITH RESPECT TO AN ELEMENT x IS DEFINED AS;

$$S_x^{\psi} = \frac{\delta \psi}{\delta x} \frac{x}{\psi}$$

THE QUESTION ARISES AS TO THE CHOOSING OF ψ AND x DESCRIPTIVE OF THE NETWORK. IT IS NOTED THAT THE PARAMETER d IS COMPLETELY DESCRIPTIVE OF THE MAGNITUDE OF THE HIGH PASS FILTER FOR A GIVEN $h=k$. (SEE FIG b, Pg. 60). SO FOLLOWING IS AN ANALYSIS OF THE SENSITIVITY OF d WITH RESPECT TO THE PASSIVE ELEMENTS.

FROM EQUATION (a), GROUP I:

$$d = \frac{1-K}{T_1} + T_1(1+\gamma)$$
$$= T_1(\gamma + 1) \quad ; \quad K = 1$$
$$= R_1C_1\left(\frac{C_2}{C_1} + 1\right)$$
$$= R_1C_2 + R_1C_1$$

— This assumes that $w_0 = 1$ and (EQ B) are include additional constraint

d's SENSITIVITY TO THE PASSIVE ELEMENTS FOLLOWS;

a) $S^{d}_{C_2}$

$$\frac{\delta d}{\delta C_2} = R_1 \Rightarrow S^{d}_{C_2} = \frac{R_1 C_2}{d}$$
$$= \frac{(7.64 \times 10^{3})(1.17 \times 10^{-7})}{\sqrt{2}}$$
$$= 6.3 \times 10^{-4}$$

b) $S^{d}_{R_1}$

$$\frac{\delta d}{\delta R_1} = C_1 + C_2 \Rightarrow S^{d}_{R_1} = \frac{R_1(C_1 + C_2)}{d}$$
$$= \frac{(7.68 \times 10^{3})(2.94 \times 10^{-7})}{\sqrt{2}}$$
$$= 1.59 \times 10^{-3}$$

c) $S^{d}_{C_1}$

$$\frac{\delta d}{\delta C_1} = R_1 \Rightarrow S^{d_1}_{C_1} = \frac{R_1 C_1}{d}$$

$$= \frac{(7.65 \times 10^{3})(1.77 \times 10^{-7})}{\sqrt{2}}$$

$$= 9.63 \times 10^{-4}$$

d) $S^{d}_{R_2}$

FROM GROUP III, EQ. a :

$$d = \frac{1+\gamma}{T_2} = \frac{1+\gamma}{R_2 C_2} \quad ; \; K = 1$$

THIS FOLLOWS ALSO FROM $T_1 T_2 = 1$ AND EQATION Ⓑ ON PREVIOUS PAGE

$$\frac{\delta d}{\delta R_2} = \frac{-C_2(\gamma+1)}{(R_2 C_2)^2} = \frac{-(\gamma+1)}{R_2^2 C_2}$$

$$S^{d}_{R_2} = \frac{-(\gamma+1)}{R_2^2 C_2} \frac{R_2}{d}$$

$$= \frac{-(\gamma+1)}{d R_2 C_2}$$

$$= \frac{-2.5}{\sqrt{2}(3.18 \times 10^{4})(1.77 \times 10^{-7})}$$

$$= -3.16 \times 10^{2}$$

THE FOLLOWING DATA STEMS FROM VARIATIONS OF A SINGLE PASSIVE ELEMENT FROM FIG. C

(9) SENSITIVITY DATA $C_2 = 0.235\ \mu F$

f	V	db
68	8.6	1
187	7.25	0
40	5.45	-3
10^3	7.6	-0.3

$C_2 = 0.066$

f	V	db
150	7.8	$\sim +0.05$
137	7.25	0
83	5.45	-3
10^3	7.6	-0.02

(b) SENSITIVITY : $R_1 = 11.15$ k

f (± 2 HZ)	V (± 0.05 v)	db (± 0.1 db)	
100	7.75	0	(PEAK)
52	5.45	-3	
1000	7.20	-0.3 db	

SENSITIVITY: $R_1 = 5.15$ k

100	9.7	1.8
76	7.25	0
63	5.45	-3
1000	7.2	-0.25

(c) SENSITIVITY DATA: $C_1 = 0.242\ \mu F$

f	v	db
86	8.20	+0.5
66	7.25	0
50	5.45	-3db
1000	7.70	-0.2db

$C_1 = 0.90\ \mu F$

f		
142	8.5	0.8
102	7.25	0
84	5.45	-3
1000	7.70	-0.2

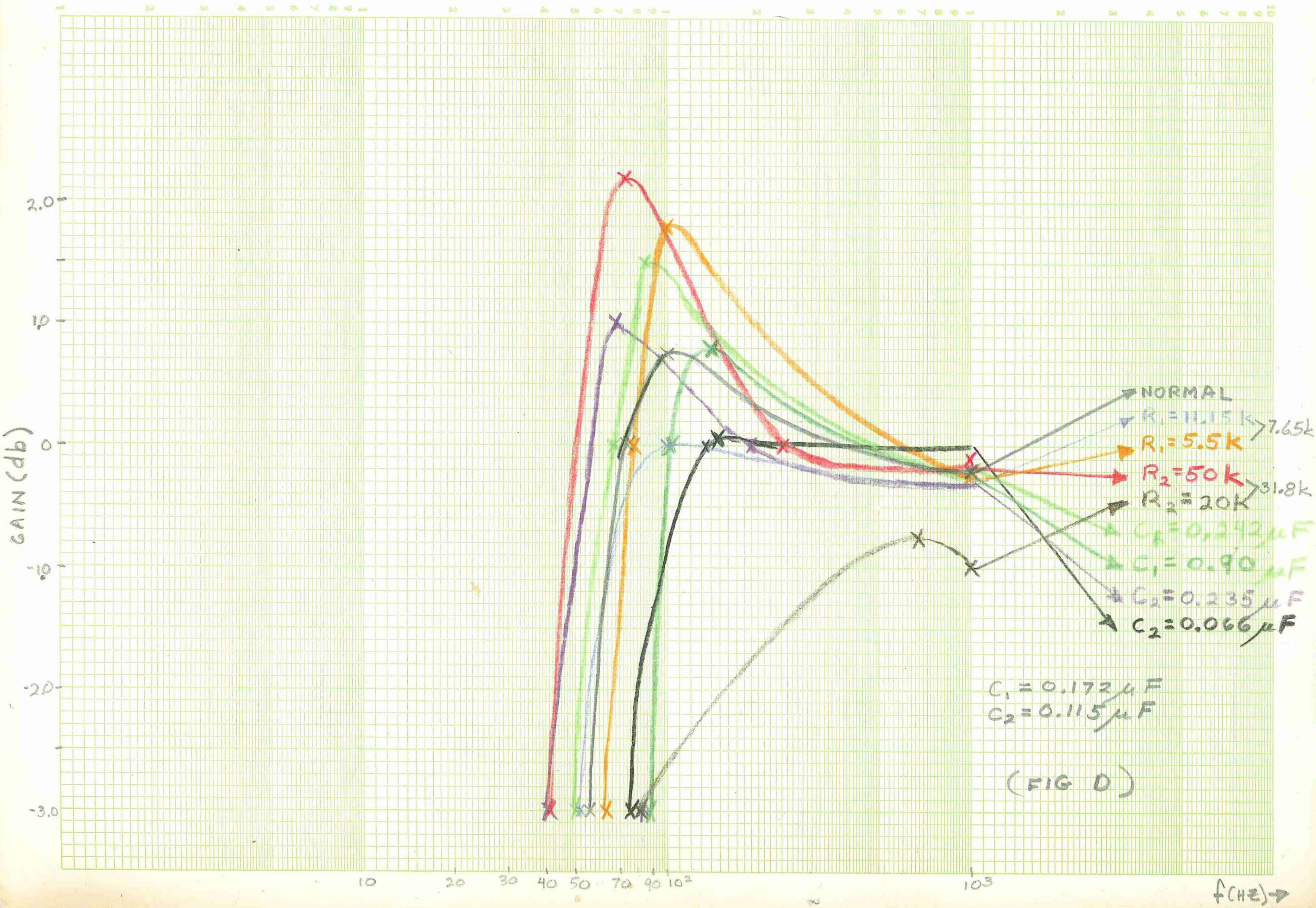
(d) SENSITIVITY DATA: $R_2 = 50.0\ k$

f	v	db
72 (PEAK)	10	2.2
240	7.75	0
42	5.45	-3
1000	7.375	-0.1

$R_2 = 20\ k$

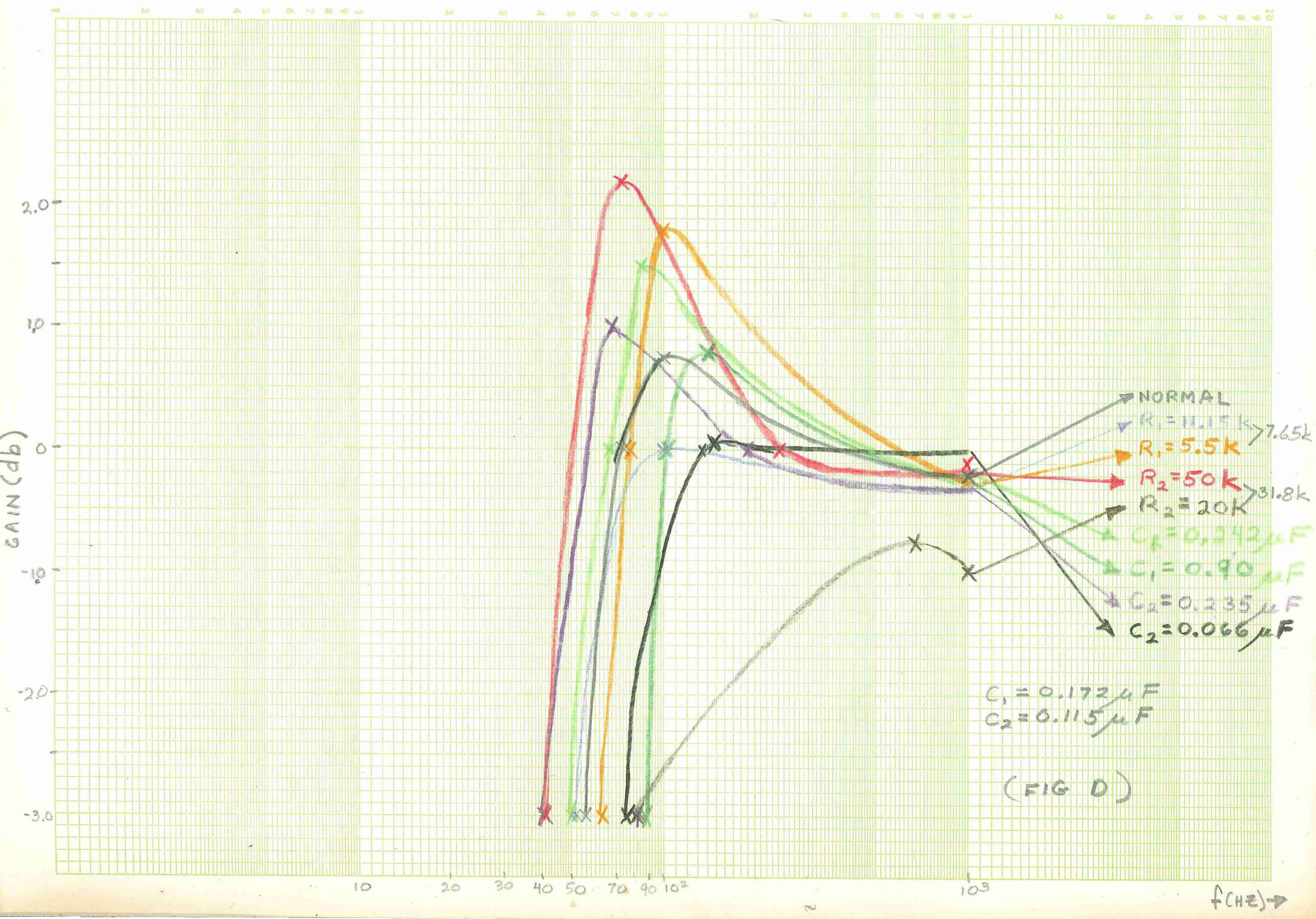
670	7.6	-0.3
82	5.45	-3
1000	7.0	-0.5

FOLLOWING ARE ROUGH ATTENUATION SKETCHES FOR THE ABOVE DATA.



$C_1 = 0.172\,\mu F$
$C_2 = 0.115\,\mu F$

(FIG D)



$C_1 = 0.172 \mu F$
$C_2 = 0.115 \mu F$

(FIG D)

THE MOST NOTABLE DEVIATIONS OCCUR WITH CHANGES IN THE R_2 VALUE (~30% DOWN, ~70% UP) CORRESPONDING TO THE RED AND BROWN SKETCHES OF FIGURE D. AS WAS COMPUTED EARLIER, d HAS HIGH SENSITIVITY WITH RESPECT TO R_2, THUS THE MORE DRAMATIC CURVE ALTERATIONS, EVEN SO $S^{d}_{R_2}$ AS COMPUTED SEEMS NOT TO BE AS LARGE IN EXPERIMENT.

THE LOWER SENSITIVITIES WITH RESPECT TO R_1, C_1, AND C_2 ARE ILLUSTRATED UPON NOTING VALUE DEVIATIONS FROM THOSE PERSCRIBED RANGE UP TO 100% FOR SIGNIFICANT TRANSFER CURVE ALTERATIONS, THIS IDEA WAS ALSO <u>BORE</u> OUT FROM NO NOTABLE TRANSFER ALTERATIONS FROM PASSIVE ELEMENT DEVIATIONS ON THE ORDER OF 5%. (DATA NOT INCLUDED.)

A FINAL NOTE CONCERNS THE NON-MAXIMALLY FLAT TRANSFER FUNCTION, WHICH IS SUPPOSEDLY A SECOND ORDER FREQUENCY TRANSFORMED BUTTERWORTH.

ie) $|G_{12}(j\omega)|^2_{LP} = \frac{1}{\omega^4+1}$

FOR THE LOW PASS CASE. TRANSFORING TO A HIGH PASS IS ACHIEVED BY SUBSTITUTION OF $\frac{1}{\omega}$ FOR ω:

$$|G_{12}(j\omega)|^2_{HP} = \frac{1}{\frac{1}{\omega^4}+1}$$

$$= \frac{\omega^4}{\omega^4+1},$$

THE PERSCRIBED TRANSFER FUNCTION. FOR $d \neq \sqrt{2}$,

$$|G_{12}(j\omega)|^2_{HP} = \frac{\omega^4}{\omega^4+(d^2-2)\omega^2+1}$$

must be an error somewhere!

POSSIBLY PURE ELIMINATION OF THE DENOMINATOR'S SECOND ORDER COEFFICIENT DOES NOT TAKE PLACE IN EXPERIMENT, ACCOUNTING FOR THE PEAK. EXAMINATION OF FIG. (d) SUGGESTS USE OF $C_2 = 0.066 \mu F$ FOR FLATTER

RESPONSE IN THE PASS BAND,

February 24, 1973

EE-533 NETWORK SYNTHESIS II

Take-Home Final Exam Due 12:00 Noon, March 1, 1973

98

1. Work all 4 problems.
2. Annotate your solutions so I can follow your procedure. (Refer to the equation number if you are taking the equation from a textbook.)
3. Sign the statement below.

39/40

I certify that I have neither given nor received aid from another person on this examination.

Robert J. Marks II

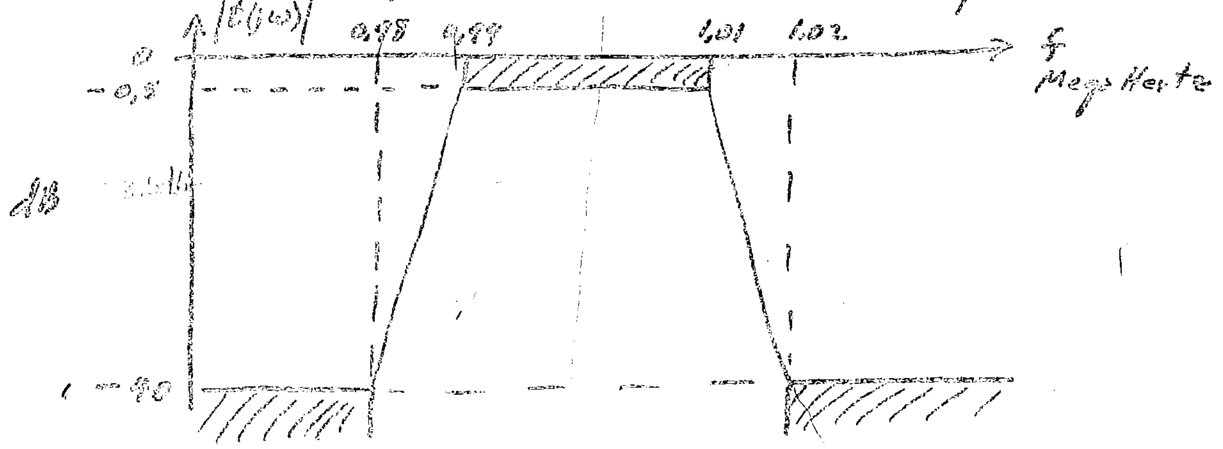
Signature

1. Use a band-pass to low-pass transformation to convert the following specification and thus determine the transmission function of the low-pass prototype filter if the filter is required to be

a) a Butterworth type, $|t(j\omega)|^2 =$ ______

b) a Chebyshev type, $|t(j\omega)| =$ ______

(Do *not* realize the filter. It is *not* necessary that the transformation be the same for both types.)

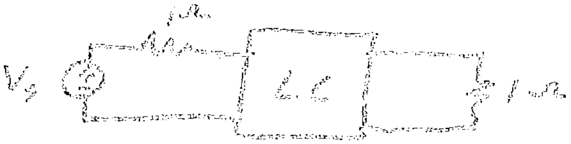


Attenuation ≤ 0.5 dB for $0.99 \leq f_{MHz} \leq 1.01$

Attenuation ≥ 40 dB for $\begin{cases} f_{MHz} \leq 0.98 \\ f_{MHz} \geq 1.02 \end{cases}$

use $f_0 = \sqrt{.99 \times 1.01} \cong 1.0$ MegaHertz

1. Realize a 3rd order Chebyshev lowpass filter with a passband ripple of 1.0 dB which is terminated in 1 ohm at both ends.

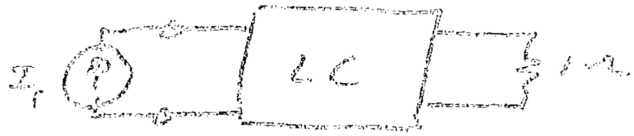


$$|t(j\omega)|^2 = \frac{1}{1 + 0.2589(4\omega^3 - 3\omega)^2}$$

Network configuration and values = ?

(Note: Equations 13, 28 may be of some help.)

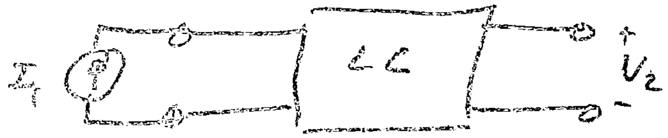
3) Repeat the specifications of problem 2 but realize a filter terminated in 1 ohm at the output.



$$|z_{12}(j\omega)|^2 = \frac{1}{1 + 0.2589\,(4\omega^3 - 3\omega)^2}$$

Network configuration and element values = ?

4) Repeat the specifications of problem 2 but realize a filter operating into an open circuit.



z_{12} = ?

z_{22} = ?

$$|z_{12}(j\omega)|^2 = \frac{1}{1 + 0.2589(4\omega^3 - 3\omega)^2}$$

Network configuration and element values = ?

1) FROM "INTRODUCTION TO FILTERS" BY ZVEREZ, Pg 78

$$f_b = \sqrt{f_c\, f_{-c}} = \sqrt{(0.99)(1.01)} = 1 \quad (MHZ)$$

$$\gamma_c = \frac{f_c}{f_b} = 1.01$$

$$\gamma_{-c} = \frac{1}{\gamma_c} = \frac{1}{1.01} = 0.99$$

$$\Omega_c = 1 = a(\gamma_c - \gamma_{-c})$$

$$= a(0.02) \Rightarrow a = 50$$

$$\Omega_s = a(\gamma_s - \gamma_{-s})$$

$$\gamma_s = \frac{f_c}{f_b} = 1.02$$

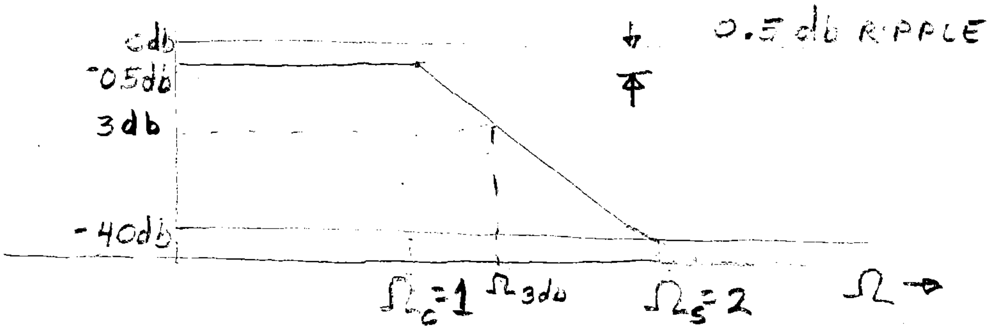
9

$$\gamma_{-s} = \frac{1}{\gamma_s} = 0.98$$

$$\Rightarrow \Omega_s = 50(0.04)$$

✓

$$= 2.00$$



(FIG 1A)

AN EQUIVALENT TRANSFORMATION TO THE NORMALIZED LOW PASS FILTER $t_{LP}(f)$ FROM THE ABOVE BAND PASS SPECIFICATIONS FOR THE GIVEN BAND-PASS TRANSFER FUNCTION $t_{BP}(f)$ IS: $t_{LP}(f) = t_{BP}\left[\frac{f - 10^6}{10^4}\right]$

FROM Pg 88 OF GRAPH HANDOUT BY EVEREZ, THE CHEBYSHEV FILTER MUST ATTENUATE 40 db @ $\Omega = 2 \Rightarrow n > 4$, SO LET $n = 5$.

FOR 0.5 db RIPPLE = 0.056

$$1 - \frac{1}{\sqrt{1+\epsilon^2}} = 0.056$$ ⇒ Eq. 13.18 IN VAN VALKENBURG

$$-\frac{1}{\sqrt{1+\epsilon^2}} = 0.944$$

$$-\sqrt{1-\epsilon^2} = 1.06$$

$$1+\epsilon^2 = 1.124$$ ✓

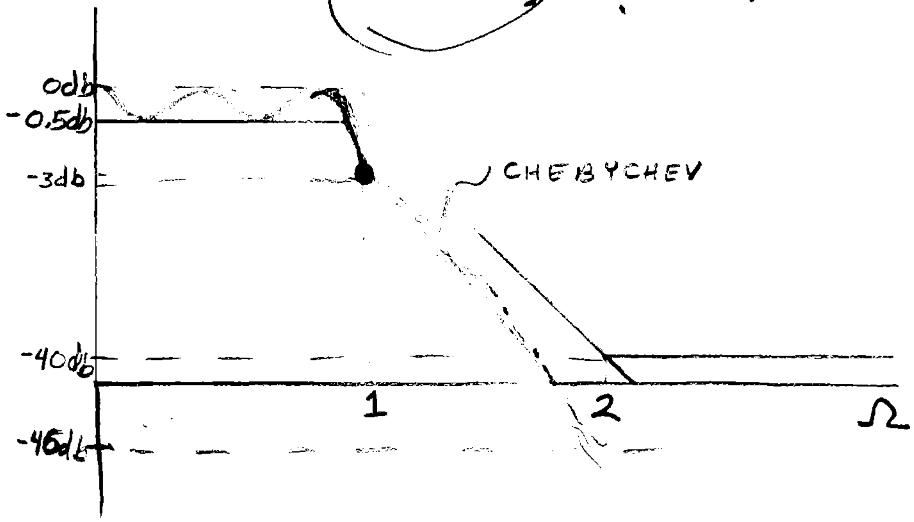
$$\epsilon^2 = 0.124$$

THUS:

$$|G_{12}(j\omega)|^2 = \frac{1}{1 + 0.124\, C_5^2(\omega)}$$

$$= \frac{1}{1 + 0.124(16\omega^5 - 20\omega^4 + 5\omega)^2}$$

THIS FUNCTION ALSO YIELDS A ROLL OFF OF ABOUT 46 db/OCTAVE, GREATER THAN THE DESIRED 39.5 db/OCTAVE. ATTENUATION @ $\Omega = 1$ IS 3db, ? 0.5 dB



SO THE NORMALIZED CHEBYCHEV L.P. B.W. IS UNITY AS DESIRED. FOR $n=4$, ATTENUATION AT $\Omega=2$ IS ~35db, SO THE ROLL OFF IS NOT ADEQUATELY STEEP. IT MUST THUS BE CONCLUDED THAT MEETING THE $\Omega=2$, ATTEN = 40.0 db SPECIFICATION IN THE CHEBYCHEV REALIZATION MAY ONLY BE ACHIEVED BY AN ALTERNATE RIPPLE SPECIFICATION OR POSSIBLY BY A DIFFERENT FREQUENCY TRANSFORMATION. HOWEVER, IN THAT OVER-ATTENUATION IN A FILTER'S ROLL OFF IS SELDOM CONCERN FOR ALARM, THE PRECEEDING CHEBYCHEV DESIGN STANDS AS A REASONABLE SOLUTION.

AN ALTERNATE FREQUENCY TRANSFORMATION IS NORMALIZATION OF THE 3db DOWN POINT TO UNITY, PRODUCING ON ONE HAND, A FLATTER RESPONSE IN THE PASS BAND, BUT ON THE OTHER, A FASTER RATE OF ATTENUATION, AND THUS IN GENERAL A LARGER n. THE 3db DOWN POINT FOR THE NORMALIZED LOW-PASS IS FOUND AS FOLLOWS: (FROM FIG. 1)

$$\frac{1-\Omega_{3db}}{2.5} = \frac{1}{39.5}$$

$$\Omega_{3db} = 1 + \frac{2.5}{39.5}$$
$$= 1.063$$

THE 40db SPECIFICATION THUS OCCURS @ $\Omega = \frac{2}{1.063} = 1.88$

CAREFULL ANALYSIS OF THE CHEBYSHEV 0.5 db RIPPLE CURVES STILL YIELDS $n = 5$, AND THE PREVIOUS TRANSFER FUNCTION REMAINS UNCHANGED, YET FREQUENCY SCALING TO THE PASSBAND IS ✓ ALTERED SLIGHTLY

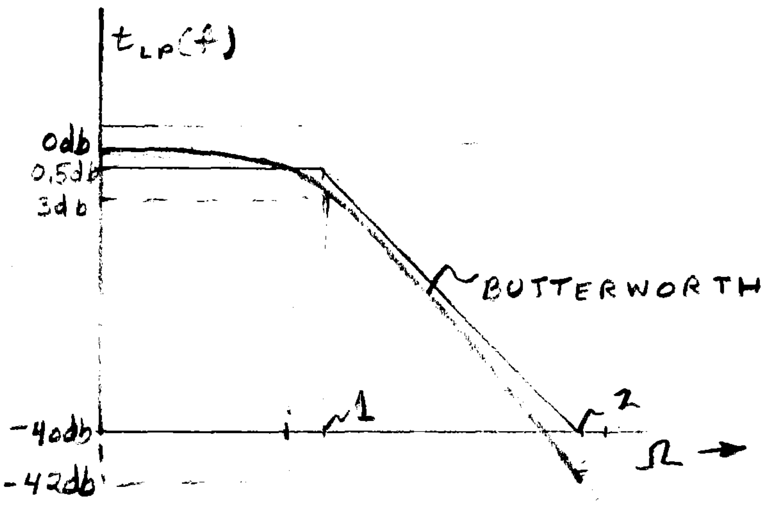
FROM PG 82 OF SAME HANDOUT, THE BUTTERWORTH MUST ATTENUATE 40db @ $\Omega = 2 \Rightarrow n > 6$, SO LET $n = 7$

THUS:

$$|G_{12}(j\omega)|^2 = \frac{1}{1 + \omega^{2n}}$$

$$= \frac{1}{1 + \omega^{14}}$$

AGAIN, THERE IS OVER ATTENUATION AT $\Omega = 2$ ~(42db). THE NORMALIZED B.W. IS OF COURSE UNITY



IF THE $\Omega_{3db} = 1$ NORMALIZATION IS EMPLOYED, IT IS NOTED THAT THE 40db ATTENUATION CRITERIA IS NOT QUITE MET @ $\Omega = 1.88$ FOR $n = 7$. THEN FOR FLATTER RESPONSE IN THE PASSBAND AND A QUICKER RATE OF ATTENUATION, LET $n = 8$

THUS:

$$|G_{12}(j\omega)|^2 = \frac{1}{1 + \omega^{16}}$$

IT IS TO BE NOTED THAT BOTH ?
THE BUTTERWORTH AND CHEBYSHEV '
CURVES ZVEREZ A NORMALIZED TO -3db @ $\Omega = 1$, SO THAT THE -3db FREQUENCY NORMALIZATION FOR THE LOW-PASS PROTOTYPE IS IN FACT THE CORRECT PROCEDURE.

2) $|t(j\omega)|^2 = \frac{1}{1 + 0.259(4\omega^3 - 3\omega)^2}$

$= \frac{1}{1 + \epsilon^2 C_n^2(\omega)} \Rightarrow n = 3$

$\epsilon^2 = 0.259 \Rightarrow \epsilon = 0.509$

POLE LOCATIONS FOR n^{TH} ORDER CHEBYSHEV

$\sigma_k = \pm \sinh a \sin \frac{2k-1}{r} \frac{\pi}{2}$; $k = 1, 2, \ldots, 2r$

$\omega_k = \cosh a \cos \frac{2k-1}{r} \frac{\pi}{2}$

$\Rightarrow a = \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon}$

$= \frac{1}{3} \sinh^{-1}(1.97)$

$= (\frac{1}{3})(1.43) = 0.477$

$\sinh a = 0.499$; $\cosh a = 1.117$

$\sigma_1 = 0.499 \sin \frac{\pi}{6}$ $\qquad$ $\omega_1 = 1.117 \cos \frac{\pi}{6}$

$= 0.250$ $\qquad$ $= 0.969$

$\sigma_2 = 0.499 \sin \frac{\pi}{2}$ $\qquad$ $\omega_2 = 1.117 \cos \frac{\pi}{2}$

$= 0.499$ $\qquad$ $= 0$

$\sigma_3 = 0.499 \sin \frac{5\pi}{6}$ $\qquad$ $\omega_3 = 1.117 \cos \frac{5\pi}{6}$

$\overset{+}{=} 0.250$ $\qquad$ $= -0.969$

SO DUE TO QUADRA$\overset{?}{\underline{D}}$AL SYM. OF CHEBYCHEV POLYNOMIAL SINGULARITIES:

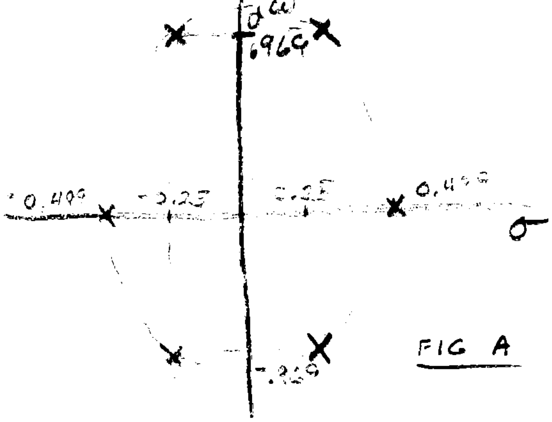


FIG A

10

PROCEDURE FROM
TABLE 15-1
Pg. 435 of VAN VALKENBURG

$$|\rho(j\omega)|^2 = 1 - |t(j\omega)|^2$$

$$= \frac{0.2589(4\omega^3-3\omega)^2}{1+0.2589(4\omega^3-3\omega)^2}$$

$$= \frac{\epsilon^2(16\omega^6-24\omega^4+9\omega^2)}{1+\epsilon^2(16\omega^6-24\omega^4+9\omega^2)}$$

$$\rho_1(s)\rho_1(-s) = \frac{-\epsilon^2(16s^6+24s^4+9s^2)}{1-\epsilon^2(16s^6+24s^4+9s^2)} = \frac{C(-s^2)}{B(-s^2)}$$

$$\frac{1}{B(-s^2)} = \frac{1}{1-\epsilon^2(16s^6+24s^4+9s^2)} \qquad (\text{EQ B})$$

$$= |t(j\omega)|^2_{s=j\omega}$$

THUS $B(-s^2)$ HAS ZEROS FROM FIGURE A AT:

$$(s+0.499)(s-0.499)(s+0.25+j0.969)(s+0.25-j0.969)$$
$$(s-0.25+j0.969)(s-0.25-j0.969)$$

UPON EXPANSION, THE COEFFICIENT OF THE SIXTH ORDER TERM FROM EQ. B MUST BE $-\epsilon^2 16$, THUS

$$B(-s^2) = -16\epsilon^2(s+0.499)(s-0.499)$$
$$(s+0.25+j0.969)(s+0.25-j0.969)$$
$$(s-0.25+j0.969)(s-0.25-j0.969)$$

$$C(-s^2) = -\epsilon^2(16s^6 + 24s^4 + 9s^2)$$

$$= -\epsilon^2(4s^3 + 3s)^2$$

$$= -\epsilon^2 s^2 (4s^2 + 3)^2$$

$$\Rightarrow \rho_1(s)\rho_1(-s) = \frac{C(-s^2)}{B(-s^2)}$$

$$= \frac{+s^2(4s^2+3)^2}{+16(s+0.499)(s-0.499)(s+0.25+j0.969)(s+0.25-j0.969)}$$

$$\times (s-0.25+j0.969)(s-0.25-0.969)$$

$$= \frac{s(4s^2+3)}{4(s+0.499)} \times \left[\frac{1}{(s+0.25+j0.969)(s+0.25-j0.969)}\right]$$

$$\times \frac{(-s)\{4(-s)^2+3\}}{4\{(-s)+0.499\}}\left[\frac{1}{\{(-s)+0.25+0.969)\{(-s)+0.225-j0.969\}}\right]$$

EMPLOYING LEFT HALF PLANE SINGULARITIES AND CONJUGATE j-AXIS ZEROS:

$$\rho_1(s) = \frac{s(4s^2+3)}{4(s+0.499)}\left[\frac{1}{(s+0.25+j0.969)(s+0.25-j0.969)}\right]$$

AND

$$\rho_1(s) = s(4s^2+3)$$

$$= 4s^3 + 3s$$

$$q_1(s) = 4(s+0.499)(s+0.25+j0.969)(s+0.25-j0.969)$$

$$= (4s+2.0)[(s+0.25)^2 + (0.969)^2]$$

$$= [4s+2.0][s^2+0.50s+1.01]$$

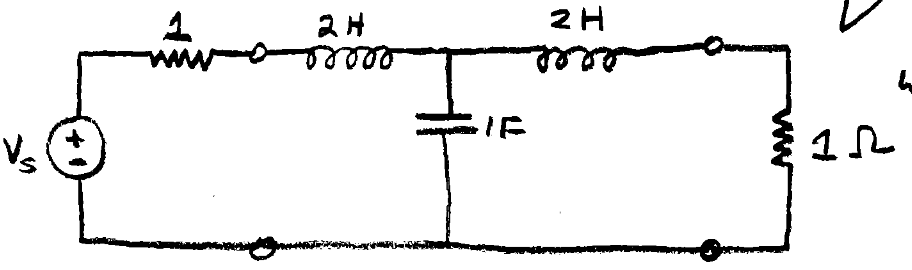
$$= 4s^3 + 4s^2 + 5.04s + 2.02$$

$$\frac{Z_{11}}{R_1} = \frac{q_1 + p_1}{q_1 - p_1} \quad ; \quad R_1 = \cancel{2}\,1$$

$$\Rightarrow Z_{11} = \frac{8s^3 + 4s^2 + 8.04s + 2.02}{4s^2 + 2.04s + 2.02}$$

ALL ZERO'S OF $t(s)t(-s)$ ARE AT ∞, SO Z_{11} MAY BE DEVELOPED AS AN LADDER. TERMS SHOULD CANCEL TWO AT A TIME, THUS THE SYNTHETIC DIVISION "FUDGING" BELOW:

Y			Z
s	$4s^2 + 2.04s + 2.02$	$8s^3 + 4s^2 + 8s + 2.02$	$2s$
	$4s^2 + 2s$	$8s^3 + 4s^2 + 4s$	
1	2	$4s + 2$	$2s$
	2	$4s$	
	0	2	



within ±3%

$$3)\ |Z_{12}(j\omega)|^2 = \frac{1}{1+0.2589(4\omega^2-3\omega)^2}$$

$$Z_{12}(s)\,Z_{12}(-s) = \frac{1}{B(-s^2)}$$ FROM PROBLEM 2

$$= [16\epsilon^2(s+0.499)(s-0.499)(s+0.25+j0.969)(s+0.25-j0.969)(s-0.25+0.969)(s-0.25-j0.969)]$$

$$= \left[\frac{1}{4\epsilon(s+0.499)(s+0.25+j0.969)(s+0.25-j0.969)}\right]$$

$$\times\left[\frac{1}{4\epsilon\{(-s)+0.499\}\{(-s)+0.25+j0.969\}\{(-s)+0.25-j0.969\}}\right]$$

$$\Rightarrow Z_{12}(s) = \frac{1}{4\epsilon(s+0.499)(s+0.25+j0.969)(s+0.25-j0.969)}$$

$$= \frac{1/\epsilon}{4s^3+4s^2+5.04s+2.02} \quad \text{Ⓒ}$$

$$= \frac{1/\epsilon \quad \frac{1}{4s^3+5.04s}}{1+\frac{4s^2+2.02}{4s^3+5.04s}} = \frac{Z_{12}}{Z_{22}+1}$$

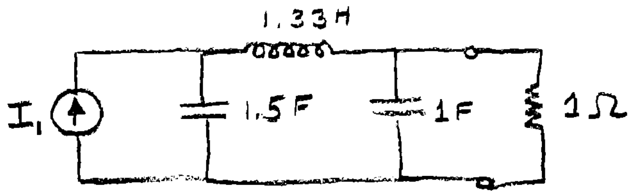
↑ Pg 46, TABLE 2-3

$$\Rightarrow Z_{12} = \frac{1}{\epsilon}\ \frac{1}{4s^3+5.04s}$$

$$Z_{22} = \frac{4s^2+2.02}{4s^3+5.04s}$$

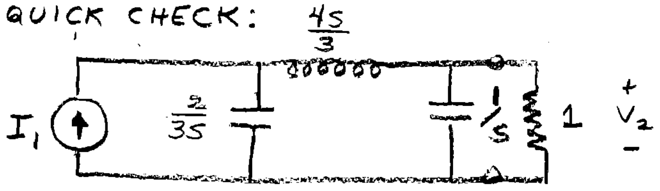
10

$$
\begin{array}{r|r|r|l}
 & & & Y \\
1.33S & 4S^2 + 2.02 & 4S^3 + 5.04S & S \\
 & 4S^2 & 4S^3 + 2.02S & \\
\hline
 & 2.02 & 3.02S & 1.5S \\
 & & 3.02S & \\
\end{array}
$$

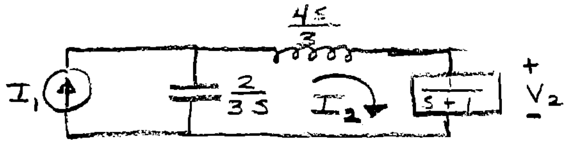


✓

A QUICK CHECK:



$Z_{12}(s) = \frac{V_2}{I_1}$



$$V_2 = \frac{1}{s+1} I_2$$

$$\left(\frac{4s}{3} + \frac{2}{3s} + \frac{1}{s+1} \right) I_2 = \frac{2}{3s} I_1$$

$$\Rightarrow \frac{V_2}{I_1} = \frac{1}{s+1} \; \frac{\frac{2}{3s}}{\frac{4s}{3} + \frac{2}{3s} + \frac{1}{s+1}}$$

$$= \frac{\frac{2}{3}(s+1)}{(s+1)\left(\frac{4s^2}{3}(s+1) + \frac{2}{3}(s+1) + s \right)}$$

$$= \frac{\frac{2}{3}(s+1)}{(s+1)\left[\frac{4s^3}{3} + \frac{4s^2}{3} + \frac{2}{3}s + \frac{2}{3} + s \right]}$$

$$= \frac{2\cancel{(s+1)}}{\cancel{(s+1)}(4s^3 + 4s^2 + 2s + 2 + 3s)}$$

$$= \frac{2}{4s^3 + 4s^2 + 5s + 2}$$

✓

EQUIVALENT TO ©; NOTE $\frac{1}{\epsilon} = \frac{1}{0.509} \simeq 2$

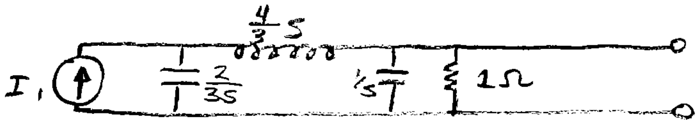
4) $|Z_{12}(j\omega)|^2 = \dfrac{1}{1 + 0.2589(4\omega^3 - 2\omega)^2}$

FROM PROBLEM ③

$$Z_{12}(s) = \frac{1/\epsilon}{4s^3 + 4s^2 + 5.04s + 2.02}$$

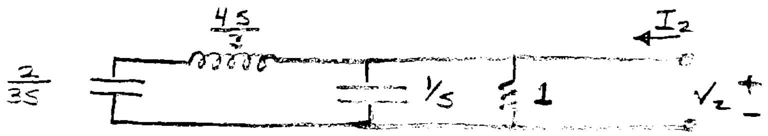
$$= \frac{1}{4\epsilon(s+0.5)(s+0.25+j0.969)(s+0.25-j0.969)}$$

NOT THE TRANSFER FUNCTION FOR A PRACTICAL LC NETWORK. IT IS TO BE NOTED THE SOLUTION TO PROBLEM 3 HAS AN IDENTICAL TRANSFER CHARACTERISTIC, WITH THE 1 Ω LOAD RESISTER INCLUDED IN THE NETWORK:

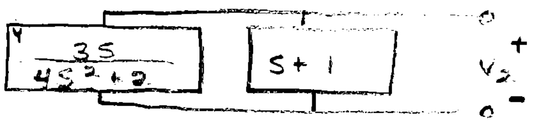


ANALYSIS OF $Z_{12}(s)$ HAS ALREADY BEEN SHOWN TO MEET THE GIVEN $Z_{12}(s)$ FUNCTION. FOLLOWING IS ANALYISIS OF $Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$

10



$$Z = \frac{4s}{3} + \frac{2}{3s} = \frac{4s^2+2}{3s}$$



$$\frac{3s}{4s^2+2} + s + 1 = \frac{3s + (4s^2+2)(s+1)}{4s^2+1}$$

$$= \frac{3s + (4s^3+4s^2+2s+2)}{4s^2+2}$$

$$\Rightarrow Z_{22} = \frac{4s^2+2}{4s^3+4s^2+5s+2}$$

✓

NOTE EQUIVALENCE OF DENOMINATOR TO Z_{12}