A sampling theorem for space-variant systems

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A sampling theorem applicable to that class of linear systems characterized by sufficiently slowly varying line-spread functions is developed. For band-limited inputs such systems can be exactly characterized with knowledge of the sampled system line-spread function and the corresponding sampled input. The desired sampling rate is shown to be determined by both the system and the input. The corresponding output is shown to be band limited. A discrete matrix representation of the specific system class is also presented. Applications to digital processing and coherent space-variant system representation are suggested.

INTRODUCTION

This paper presents a sampling theorem applicable to that class of linear space-variant systems characterized by sufficiently slowly varying line-spread functions. For band-limited inputs, such systems can be exactly characterized with knowledge of the sampled system line-spread function and the corresponding sampled input. The resulting sampling theorem expression for the (band-limited) system output is simply a summation of convolutions. A discrete matrix representation of the specific system class is also presented.

Areas of possible application for the result include digital signal processing and the representation of coherent space-variant systems. Application limitations are also briefly discussed.

Previous work in this area has been limited to sampling theorem expansion of the system line-spread function[1,2] without regard to the input. Huang[3] has discussed the minimum required sampling rates taking the input into account.

For clarity of presentation, and without loss of generality, attention will here be restricted to one dimension. Generalization to two or more dimensions may be accomplished by straightforward extension.

SAMPLING THEOREM

The output $g(x)$ of a linear system for a corresponding input $f(x)$ may be expressed via the superposition integral

$$g(x) = \int_{-\infty}^{\infty} f(\xi)\, h(x-\xi;\xi)\, d\xi \ , \tag{1}$$

where $h(x-\xi;\xi)$, the system line-spread function, is the system response to an input Dirac delta located at the point $x=\xi$ (after the notation of Lohmann and Paris[4]).

When the line-spread function is no longer a function of its second argument, the system is isoplanatic (space invariant), and Eq. (1) becomes the convolution integral

$$g(x) = \int_{-\infty}^{\infty} f(\xi)\, h(x-\xi)\, d\xi = f(x) * h(x) \ . \tag{2}$$

The direct statement of a space-variant system's output spectrum may be found through application of Fourier transform operators to the superposition integral [Eq. (1)]:

$$G(f_x) = \mathfrak{F}_x[g(x)]$$

$$= \int_{-\infty}^{\infty} f(\xi)\, \mathfrak{F}_x[h(x;\xi)] \exp(-j2\pi f_x \xi)\, d\xi$$

$$= \mathfrak{F}_\xi\, \mathfrak{F}_x [f(\xi)\, h(x;\xi)]\big|_{v=f_x} \ , \tag{3}$$

where v and f_x are the frequency variables associated, respectively, with ξ and x, and where, for a given two-dimensional function $p(x;\xi)$, the Fourier transform operators are defined as

$$\mathfrak{F}_x[p(x;\xi)] \triangleq \int_{-\infty}^{\infty} p(x;\xi) \exp(-j2\pi f_x x)\, dx \tag{4}$$

and

$$\mathfrak{F}_\xi[p(x;\xi)] \triangleq \int_{-\infty}^{\infty} p(x;\xi) \exp(-j2\pi v \xi)\, d\xi \ . \tag{5}$$

Roughly, $\mathfrak{F}_\xi(\cdot)$ operates on the *input* variable ξ, while $\mathfrak{F}_x(\cdot)$ operates on the *output* variable x.

We now define the system's spatial transfer function as

$$H_x(f_x;\xi) \triangleq \mathfrak{F}_x[h(x;\xi)] \ . \tag{6}$$

Equation (3) may now be written

$$G(f_x) = \mathfrak{F}_\xi[f(\xi)\, H_x(f_x;\xi)]\big|_{v=f_x} \ . \tag{7}$$

In a similar fashion, we define the system's *variation spectrum* as

$$H_\xi(x;v) \triangleq \mathfrak{F}_\xi[h(x;\xi)] \ . \tag{8}$$

The variation spectrum is a measure of how the line-spread function varies with changing ξ. We say the line-spread function varies sufficiently slowly if the variation spectrum is band limited[5] in v for all x:

$$H_\xi(x;v) = 0 \text{ for } |v| > W_v \text{ for all } x \ . \tag{9}$$

The bandwidth $2W_v$ is appropriately termed the *variation bandwidth*. Note than an isoplanatic system has a variation bandwidth of zero, and is thus truly "invariant."

Consider, then, the following form of the superposition integral's integrand contained in Eq. (3):

$$f(\xi)\, h(x;\xi) \ . \tag{10}$$

Multiplication in the space (ξ) domain corresponds to convolution in the frequency (v) domain. As such, if $f(\xi)$ and $h(x;\xi)$ have respective bandwidths in v of $2W_f$ and $2W_v$, then their product will have a bandwidth $2W$ equal to the sum of the component bandwidths:

$$2W = 2W_f + 2W_v \ . \tag{11}$$

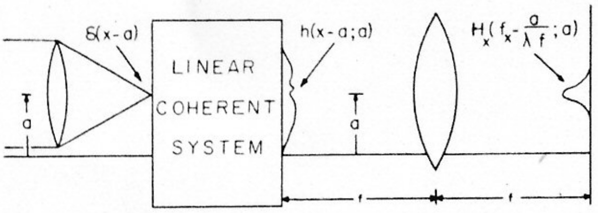

FIG. 1. Generation of a sample line-spread function and corresponding sample-transfer function for an arbitrary coherent space-variant system.

One may then apply the Whittaker-Shannon sampling theorem[6] to the product integrand to give

$$f(\xi)\,h(x;\xi)=\sum_{n=-\infty}^{\infty} f(\xi_n)\,h(x;\xi_n)\,\mathrm{sinc}2W(\xi-\xi_n)\ , \tag{12}$$

where $\xi_n = n/2W$ and $\mathrm{sinc}y = \mathfrak{F}[\mathrm{rect}x]$ where

$$\mathrm{rect}(x) \triangleq \begin{cases} 1, & |x| \le \frac{1}{2}\ , \\ 0, & |x| > \frac{1}{2}\ . \end{cases} \tag{13}$$

Substituting Eq. (12) into Eq. (3) followed by simplification leaves

$$G(f_x)=\frac{1}{2W}\sum_n f(\xi_n)\,H_x(f_x;\xi_n) \times \exp(-j2\pi f_x\,\xi_n)\,\mathrm{rect}\left(\frac{f_x}{2W}\right)\ , \tag{14}$$

or equivalently, in the space domain,

$$g(x)=\sum_n f(\xi_n)\,h(x-\xi_n;\,\xi_n) * \mathrm{sinc}(2Wx)\ . \tag{15}$$

Thus, providing that $h(x;\xi)$ and $f(\xi)$ are band limited in ξ, the output to a linear space-variant system can be computed by (i) sampling the input, (ii) multiplying each input sample by its corresponding line-spread function, (iii) summing these products, and (iv) passing the sum through a suitable low-pass filter.

APPLICATION

It has previously been suggested that multielement coherent space-variant systems may be represented by a number of sample transfer functions.[7-10] The vast storage capacity of the volume hologram[11] may be utilized for sequential angle-multiplexed recording of these sample functions. The resulting volume hologram should exhibit the input–output relationship of the original system to a good approximation. Such a system representation provides for increased orientation stability, reduced weight, and real-space condensation.

For coherent optical systems, a sample transfer function can easily be realized as in Fig. 1.[7,8,10] The impulse input to the system is generated by focusing an incident plane wave to a line source at the input coordinate $\xi = a$. The corresponding line-spread function $h(x-a;a)$ is Fourier transformed by a displaced thin lens to yield in the back focal plane an amplitude distribution proportional to a scaled and shifted version of the sample spatial transfer function $H_x(f_x - a/\lambda f;a)$. The scaled spatial frequency is given by

$$f_x = x/\lambda f\ , \tag{16}$$

where λ is the wavelength of the spatially coherent illumination and f is the focal length of the Fourier transforming lens. The amplitude and phase of a number of such sample transfer functions may then be angle multiplexed within a single volume hologram. The hologram, in principle, may then be utilized as a space-variant equivalent to a Vander Lugt filter.[6] Such schemes have been proposed by Deen, Walkup, and Hagler,[7] and by Marks.[10] The method of Deen *et al.* falls short of direct implementation of the sampling theorem [Eq. (14)] only by not including the required low-pass filter of bandwidth $2W$.

In practice, one is of course limited to recording only a finite number of holograms, short of the countably infinite number required by the sampling theorem. We overcome this problem by application of the familiar space-bandwidth product estimate of the number of samples required for a good approximation. If the system input $f(\xi)$ is essentially zero outside of the interval $|\xi| \le a$, and the spectrum (in v) of the integrand $f(\xi)h(x;\xi)$ is essentially zero outside of the interval $|v| \le W$, then the required number of samples for a good approximation is[12]

$$S = 4Wa\ . \tag{17}$$

Truncation will of course result in a degree of error.[13]

EXAMPLE

As an example application of the space-variant system sampling theorem, consider the ideal band-limited coherent imaging system with magnification $M \ne 1$. While a simple *mathematical* coordinate transformation reduces the imaging system to an isoplanatic system,[6] the ideal imaging system having nonunity magnification must rigorously be classified as space variant.[14] That is, in the *physical* sense, one may not use a single planar holographic filter in the Fourier plane to represent a simple magnifier.

The line-spread function of the imaging system to be considered is

$$\begin{aligned} h(x-\xi;\xi) &= 2f_0\,\mathrm{sinc}2f_0[x - M\xi] \\ &= 2f_0\,\mathrm{sinc}2f_0[(x-\xi)-(M-1)\xi]\ , \end{aligned} \tag{18}$$

where f_0 is the cutoff frequency of the system. Note that for an arbitraily large value of f_0, Eq. (18) approaches the displaced Dirac delta function characterizing an ideal imaging system. From Eq. (18) we may write

$$h(x;\xi) = 2f_0\,\mathrm{sinc}2f_0[x-(M-1)\xi]\ . \tag{19}$$

The spatial transfer function [Eq. (6)] is then given by

$$H_x(f_x;\xi) = \exp[-j2\pi(M-1)f_x\xi]\,\mathrm{rect}(f_x/2f_0)\ . \tag{20}$$

Substituting into Eq. (7) yields the system output spectrum

$$G(f_x) = F(Mf_x)\,\mathrm{rect}(f_x/2f_0)\ , \tag{21}$$

where

$$F(f_x) = \mathfrak{F}_x[f(x)]\ . \tag{22}$$

Contained in Eq. (21) is the inherent low-pass nature of the imaging system.

To apply the sampling theorem, we need first look at the system's variation spectrum. Appropriately transforming Eq. (19) gives

$$H_t(x;v)=\frac{1}{M-1}\exp\left[-j2\pi v\left(\frac{x}{M-1}\right)\right]\mathrm{rect}\left(\frac{v}{2f_0(M-1)}\right) . \tag{23}$$

The finite system variation bandwidth is thus

$$2W_v=2f_0\,|M-1| \ . \tag{24}$$

Note that for $M=1$, the line-spread function of Eq. (18) is isoplanatic and the corresponding variation bandwidth is zero.

Since the variation bandwidth of Eq. (23) is finite, the sampling theorem is directly applicable. Suppose an input of bandwidth $2W_f$ is sampled at the rate $2W=2W_f+2W_v$. The corresponding sampled expansion for the output spectrum [Eq. (14)] becomes

$$G(f_x)=\frac{1}{2W}\sum_n f(\xi_n)\exp(-j2\pi f_x M\xi_n)\,\mathrm{rect}\left(\frac{f_x}{2W}\right)\mathrm{rect}\left(\frac{f_x}{2f_0}\right) . \tag{25}$$

This relationship is recognized as a Fourier series expansion of the output spectrum of Eq. (21) with period $2W$. The $\mathrm{rect}(x/2W)$ term merely retains the desired zero-order term.

MATRIX REPRESENTATION

The space-variant sampling theorem results can be utilized to express the system input-output relationship in exact matrix form. Such a relationship would find use in digital applications.

Consider first the output spectrum expansion of Eq. (14). The $\mathrm{rect}(f_x/2W)$ term dictates that a band-limited input to a space-variant system with finite variation bandwidth must result in an output with bandwidth not exceeding $2W$. The output spectrum may thus be expressed by the Whittaker-Shannon sampling theorem as

$$G(f_x)=\frac{1}{2W}\sum_n g(x_n)\exp(-j2\pi f_x x_n)\,\mathrm{rect}\left(\frac{f_x}{2W}\right) , \tag{26}$$

where

$$x_n=n/2W \ . \tag{27}$$

From this expansion we will obtain the desired output sample values given by $g(x_n)$. Equating Eq. (26) with Eq. (14) and multiplying both sides by $\exp(j2\pi f_x\,\xi_m)$ gives

$$\sum_n g(x_n)\exp\left(-\frac{j2\pi f_x(n-m)}{2W}\right)\mathrm{rect}\left(\frac{f_x}{2W}\right)$$
$$=\sum_n f(\xi_n)H_x(f_x;\xi_n)\exp\left(\frac{j2\pi f_x(m-n)}{2W}\right)\mathrm{rect}\left(\frac{f_x}{2W}\right) . \tag{28}$$

We define the low-pass filtered sample transfer function as

$$\hat{H}_x(f_x;\xi_n)=H_x(f_x;\xi_n)\,\mathrm{rect}(f_x/2W) , \tag{29}$$

and recognize that

$$\int_{-\infty}^{\infty}\exp\left(-\frac{j2\pi f_x(n-m)}{2W}\right)\mathrm{rect}\left(\frac{f_x}{2W}\right)df_x$$
$$=2W\,\mathrm{sinc}(n-m)=2W\delta_{nm} , \tag{30}$$

where δ_{nm} is the Kronecker delta. Thus, integration of Eq. (28) over all f_x gives

$$g(x_m)=\frac{1}{2W}\sum_n f(\xi_n)\,\hat{h}(x_m-\xi_n;\xi_n) , \tag{31}$$

where $\hat{h}(x;\xi_n)$ and $\hat{H}_x(f_x;\xi_n)$ are Fourier transform pairs. This relationship can be viewed as an infinite matrix representation of the superposition integral [Eq. (1)]. Coupled with the space-bandwidth product [Eq. (17)] as a measure of the number of required samples, such a relationship would appear to have interesting applications in digital signal processing.

CONCLUSION

Linear space-variant systems with line-spread functions of finite variation bandwidth may be represented exactly in sampled form for band-limited inputs. Employing a sampling rate equal to the sum of the input and variation bandwidths yields a relationship in which each sampled input point is assigned a corresponding line-spread function. This result gives further credibility to the concept of holographic representation of linear space-variant systems with volume holograms. The corresponding exact matrix characterization of the space-variant system input–output relationship has analogous applications in digital signal processing.

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APPENDIX

The sampling theorem expressions in Eqs. (14) and (15) are not optimum in the sense of utilizing maximum allowable sampling intervals. That is, we are sampling both the input and line-spread function at a rate of $2W$, while the minimum required sampling rates are $2W_f$ and $2W_v$, respectively. As will be shown, however, the resulting expression employing these minimum sampling rates is rather unattractive for computation and implementation purposes.

Consider, then, the following sampling theorem expansion of a space-variant system's line-spread function[1]:

$$h(x;\xi)=\sum_p h(x;\xi_p)\,\mathrm{sinc}2W_v(\xi-\xi_p) , \tag{A1}$$

where $2W_v$ is the variation bandwidth and $\xi_p=p/2W_v$. One may similarly apply the sampling theorem to the system input to give

$$f(\xi)=\sum_k f(\xi_k)\,\mathrm{sinc}2W_f(\xi-\xi_k) , \tag{A2}$$

where $\xi_k=k/2W_f$ and $2W_f$ is the input's bandwidth. Substituting Eqs. (A1) and (A2) into Eq. (3) gives

$$G(f_x) = \frac{1}{4W_f W_v} \sum_k \left\{ f(\xi_k) \sum_p H_x(f_x;\xi_p) \left[\operatorname{rect}\left(\frac{f_x}{2W_v}\right) \times \exp(-j2\pi f_x \xi_p) \right] * \left[\operatorname{rect}\left(\frac{f_x}{2W_f}\right) \exp(-j2\pi f_x \xi_k) \right] \right\} . \tag{A3}$$

Equation (A3) is identical to Eq. (14) yet employs larger sampling intervals. The above relationship, however, has the disadvantage of not assigning each sample input value to a single corresponding sampled line-spread function.

Lastly, note that the two convolving rect's in Eq. (A3) give an upper bound on the output bandwidth of $2W = 2W_f + 2W_v$. This constraint is the same as contained in Eq. (14).

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