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SAMPLING THEOREMS FOR LINEAR SHIFT-VARIANT SYSTEMS

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Abstract

Sampling theorem concepts are applied to certain classes of linear shift variant systems. Various sampling theorem characterizations arise from different bandlimiting assumptions on the system input and/or impulse response. These characterizations are also expressed in discrete form and in all cases considered, reduce to an identical computational form which can be evaluated with a generalized Z transform treatment. The Fourier duals of the sampling theorems, wherein the system is characterized by its frequency rather than impulse response, are also presented.

1. INTRODUCTION

Past application of sampling theorem concepts to linear shift variant systems has been to evaluate the system impulse response rather than to characterize the input-output relationship (1), although adequate sampling rates have been discussed briefly (2). In this paper, on the other hand, we present numerous conditions under which the superposition integral characterization of the input-output relation for linear shift variant systems can be expressed in sampling theorem type expansions. Certain ramifications of these treatments, such as digital characterization of the system process without loss of information and generalized Z transform treatment of discrete superposition relations, are also discussed.

In section 2, certain preliminary notational and computational conventions are given which are necessary in the development of the sampling theorems. Sections 3 through 5 present three different sampling theorems corresponding to various bandlimiting assumptions on the system input and/or impulse response. A direct consequence of these sampling theorems are infinite matrix characterizations of the system process which, except for sampling rates, are identical for all three sampling theorems. The use of the Z transform in treating these discrete characterizations is briefly discussed in section 7. In section 8 the Fourier duals of the sampling theorems are presented wherein sampling is performed in the frequency domain. Section 9 contains some concluding remarks.

The work of this paper was motivated by investigation of space-variant systems encountered in coherent optical processing (3). Substantial new material, however, is presented here. We note that causality is not a constraint in such spatial systems. For this reason, the sampling theorems here are presented for the more general case where causality is not a constraint. Although a causal signal (zero for negative time) can never be rigorously bandlimited (4), familiar time-bandwidth product approximations can be applied if approximate (5). The same is true for the system impulse response with respect to its first variable. There are many causal signals and systems, however, which can be considered "essentially" bandlimited. Even the concept of a bandlimited signal, at best, can only be considered as an approximation to real world phenomena (5).

For clarity of presentation, attention will be restricted to one dimension. Generalization to multidimensional systems is straightforward.

2. PRELIMINARIES

The response, $y(t)$, of a linear system to an input $u(t)$, can be expressed by the superposition integral:

$$\begin{aligned} y(t) &= S[u(t)] \\ &= \int_{-\infty}^{\infty} h(t-\tau;\tau)u(\tau)d\tau \end{aligned} \qquad (1)$$

where $S(\cdot)$ is the system operator and the system impulse response is formally defined as

$$h(t-\tau;\tau) = S[\delta(t-\tau)] \qquad (2)$$

where $\delta(t)$ denotes the Dirac delta. (For a causal system, $h(t-\tau;\tau)$ is zero for $t<\tau$.) This particular choice of impulse response notation has certain computational advantages (6). For example, we can directly express the output spectrum by

$$\begin{aligned} Y(f) &= F_t[y(t)] \\ &= F_\tau F_t[h(t;\tau)u(\tau)]\Big|_{v=f} \end{aligned} \qquad (3)$$

where the Fourier operators are defined by

$$F_t[s(t;\tau)] \triangleq \int_{-\infty}^{\infty} s(t;\tau)\exp(-j2\pi ft)dt \qquad (4)$$

and

$$F_\tau[s(t;\tau)] \triangleq \int_{-\infty}^{\infty} s(t;\tau)\exp(-j2\pi v\tau)d\tau \qquad (5)$$

Note that for the shift invariant case that $h(t;\tau) \rightarrow h(t)$. Equation 3 then takes on the familiar product form $Y(f) = F_t[h(t)]F_t[u(t)]$.

A transform of the impulse response which will be of interest is the variation spectrum defined as

$$H_\tau(t;v) = F_\tau[h(t;\tau)] \qquad (6)$$

The support of the variation spectrum is a measure of the manner in which the impulse response changes shape with respect to τ. We consider here the low-pass case for which $H_\tau(t;v)$ is identically zero outside the interval $|v| \leq W_v$. Such systems will be referred to as variation limited. The quantity $2W_v$ is appropriately termed the variation bandwidth.

Note that a shift invariant system has a variation bandwidth of zero and is thus truly invariant.

3. A SAMPLING THEOREM FOR VARIATION LIMITED SYSTEMS

We now will develop a sampling theorem applicable to variation limited systems with bandlimited inputs. For the bandlimited input, we again consider the low-pass case where $u(t)$ has bandwidth $2W_u$. Consider, then, the term $h(t;\tau)u(\tau)$ which is the argument of the Fourier operator in Eq. 3. Multiplication in the τ domain corresponds to convolution in the v domain. As such, if $u(\tau)$ has bandwidth $2W_u$ and $h(t;\tau)$ has a variation bandwidth of $2W_v$, then their product will have a bandwidth $2W_s$ equal to the sum of the component bandwidths:

$$2W_s = 2W_v + 2W_u \tag{7}$$

The product $h(t;\tau)u(\tau)$ can thus be expanded in a uniformaly converging (7) Whittaker-Shannon sampling theorem (8) in τ:

$$h(t;\tau)u(\tau) = \sum_n h(t;\tau_n)u(\tau_n) \cdot \text{sinc}\, 2W_s(\tau-\tau_n) \tag{8}$$

where $\tau_n = n/2W_s$ and $\text{sinc}\, x \triangleq \sin\pi x/\pi x$. Substituting into Eq. 3 and simplifying gives

$$Y(f) = \frac{1}{2W_s} \sum_n H_t(f;\tau_n)u(\tau_n) \times \exp(-j2\pi f\tau_n)G\left(\frac{f}{2W_s}\right) \tag{9}$$

where our transfer function is defined by

$$H_t(f;\tau) \triangleq F_t[h(t;\tau)] \tag{10}$$

and $G(f) = F_t[\text{sinc}\, t]$ is the gate function. Inverse Fourier transforming Eq. 9 gives

$$y(t) = \sum_n h(t-\tau_n;\tau_n)u(\tau_n) * \text{sinc}\, 2W_s t \tag{11}$$

where "*" denotes the convolution operation. We interpret Eq. 11 as follows: For bandlimited inputs, the output to a variation limited system can be computed by 1) sampling the input, 2) multiplying each input sample by its corresponding sample impulse response, 3) summing the results, 4) passing the sum through a suitable low-pass filter. As is shown in Figure 1, we can interpret this result as the representation of a variation limited system by a bank of of shift invariant systems each of which corresponds to a sample impulse response. The switching mechanism required to feed each filter its corresponding sample value is representative of the shift variance of the overall system. Note that Eq. 11 is not optimal in the sense of utilizing minimum sampling rates. That is, $u(\tau)$ only requires a sampling rate of $2W_u$ and $h(t;\tau)$ a sampling rate of $2W_v$ in τ. Both are here being sampled at a rate $2W_s$. The authors have shown however, that the sampling expansion utilizing the minimum allowable sampling rates is computationally less attractive (3).

4. AN ALTERNATE SAMPLING THEOREM

In the previous section, $h(t;\tau)$ was assumed to be bandlimited in τ. Note that this restriction does not necessarily assure that $h(t-\tau;\tau)$ is also bandlimited in τ. As such, we can derive an alternate sampling theorem for the case where $F_t[h(t-\tau;\tau)]$ is zero outside of the interval $|v| \leq W_h$. If our input has bandwidth $2W_u$, then the product $h(t-\tau;\tau) u(\tau)$ has bandwidth $2W_d = 2W_u + 2W_h$ in τ and can be expressed in the sampling theorem expansion:

$$h(t-\tau;\tau) = \sum_n h(t-\tau_n;\tau_n)u(\tau_n)\text{sinc}2W_d(\tau-\tau_n) \tag{12}$$

where, here, $\tau_n = n/2W_d$. Substituting into the superposition integral [Eq. 1] gives

$$y(t) = \sum_n h(t-\tau_n;\tau_n)u(\tau_n)\int_{-\infty}^{\infty} \text{sinc}2W_d(\tau-\tau_n)d\tau = \frac{1}{2W_d} \sum_n h(t-\tau_n;\tau_n)u(\tau_n) \tag{13}$$

Our expansion here is similar to that in Eq. 11 except for the sampling rate and the fact that no low-pass filtering is required. Note also, due to our bandlimiting constraints, the output in Eq. 13 is bandlimited with bandwidth $2W_d$.

5. A THIRD SAMPLING THEOREM

The sampling theorems thus far discussed require sampling in the τ or input domain. An alternate sampling theorem which utilizes output sampling occurs when $h(t;\tau)$ is bandlimited in t with (lowpass) bandwidth of, say, $2W_t$. (Note that this condition is equivalent to $h(t-\tau;\tau)$ being bandlimited in t.) Such a condition holds when the system response to an impulse input is bandlimited irrespective of the location of the input delta function. Under this bandlimited assumption, we can immediately express the impulse response in the sampling theorem expansion:

$$h(t;\tau) = \sum_n h(t_n;\tau)\,\mathrm{sinc}\,2W_t(t-t_n) \quad (14)$$

where $t_n = n/2W_t$. Substituting into Eq. 3 followed by simplification, leaves

$$Y(f) = \frac{1}{2W_t}\sum_n F_\tau[h(t_n;\tau)u(\tau)]\Big|_{v=f} \exp(-j2\pi f t_n)\,G\left(\frac{f}{2W_t}\right) \quad (15)$$

Inverse transforming yields:

$$y(t) = \sum_n h(t_n;t-t_n)u(t-t_n) * \mathrm{sinc}\,2W_t t \quad (16)$$

As before we have reduced the system characterization to a summation of convolutions. In this case, however, we do not have to place any bandlimiting constraints on our input.

We can interpret Eq. 16 as shown in Figure 2. Our input is fed into a tapped delay line which serves as the shift variance of the overall system. The outputs at various points along the delay line are then multiplied by the appropriate sample responses. All these products are then summed and passed through an appropriate low pass filter to give the corresponding system output.

6. DISCRETE CHARACTERIZATION

Inspection of the three sampling theorems thus far presented reveals that the corresponding system output is bandlimited and thus can also be expressed in a sampling theorem expansion. We now investigate direct computation of the required output sample values in terms of the sampled input and impulse response. The resulting computational forms, in the three cases considered, are identical.

(1) Consider first, the variation limited system with bandlimited input. From Eq. 11, we define the low-passed system impulse response as

$$\hat{h}(t-\tau_n;\tau_n) \triangleq 2W_s h(t-\tau_n;\tau_n) * \mathrm{sinc}\,2W_s t \quad (17)$$

Equation 11 can now be written

$$y(t) = \frac{1}{2W_s}\sum_n \hat{h}(t-\tau_n;\tau_n)u(\tau_n) \quad (18)$$

It follows immediately that

$$y(t_m) = \frac{1}{2W_s}\sum_n \hat{h}(t_m-\tau_n;\tau_n)u(\tau_n) \quad (19)$$

From Eq. 9, $y(t)$ has a bandwidth of $2W_s$. Thus, we require that $t_m = m/2W_s$. Note that Eq. 19 can be straightforwardly expressed in an infinite matrix form.

Suppose we now make the additional constraint that $h(t;\tau)$ is bandlimited in t with bandwidth $2W_t$. If $W_s > W_t$, then the low-passed impulse response in Eq. 17 is the same as our actual impulse response:

$$\hat{h}(t-\tau_n;\tau_n) = h(t-\tau_n;\tau_n) \;;\; W_s > W_t \quad (20)$$

Then, Eq. 19 becomes

$$y(t_m) = \frac{1}{2W_s} \sum_n h(t_m-\tau_n;\tau_n)u(\tau_n);\ W_s>W_t \quad (21)$$

where $t_n = \tau_n = n/2W_s$.

(2) Consider next the sampling theorem in section IV where $h(t-\tau;\tau)$ is bandlimited in τ. Since the output has bandwidth $2W_d$, it follows immediately from Eq. 13 that the desired output sample values are given by

$$y(t_m) = \frac{1}{2W_d} \sum_n h(t_m-\tau_n;\tau_n)u(\tau_n) \quad (22)$$

where, now $t_n = \tau_n = n/2W_d$.

(3) Lastly, consider the sampling theorem expansion in Eq. 16 where the output has bandwidth $2W_t$. The corresponding m^{th} output sample here is given by:

$$y(t_m) = \sum_n [h(t_n;t)u(t)*\text{sinc}\ 2W_t t]\Big|_{t=t_m-t_n} \quad (23)$$

where, now, $t_m = m/2W_t$.

A more computationally attractive form of Eq. 23 occurs when, in addition to being bandlimited in t, the system is variation limited and the input is bandlimited such that $2W_s < 2W_t$. In this case Eq. 15 becomes

$$\begin{aligned} Y(f) &= \frac{1}{2W_t} \sum_n F_t[h(t_n;t)u(t)] \\ &\quad \times \exp(-j2\pi f t_n) G\left(\frac{f}{2W_t}\right) G\left(\frac{f}{2W_s}\right) \\ &= \frac{1}{2W_t} \sum_n F_t[h(t_n;t)u(t)] \\ &\quad \exp(-j2\pi f t_n);\ W_s < W_t \end{aligned} \quad (24)$$

Inverse transforming and evaluating at $t = t_m$ gives

$$y(t_m) = \frac{1}{2W_t} \sum_n h(t_m-\tau_n;\tau_n)u(\tau_n);\ W_s < W_t \quad (25)$$

where $t_n = \tau_n = n/2W_t$.

Inspection of the results of the three discrete characterizations above [Eqs. 21, 22 and 25] reveals computationally identical forms. Our assumptions is all cases are (1) $u(\tau)$ is bandlimited, (2) either $h(t;\tau)$ or $h(t-\tau;\tau)$ is bandlimited in τ and, for the first [Eq. 21] and third [Eq. 25] cases, that (3) $h(t;\tau)$ [and thus $h(t-\tau;\tau)$] is bandlimited in t. We can combine the three discrete matrix type relationships into a single expression:

$$y(t_m) = \frac{1}{2W} \sum_n h(t_m-\tau_n;\tau_n)u(\tau_n) \quad (26)$$

where

$$W = \min[W_d,\ \max(W_t,\ W_s)] \quad (27)$$

and $t_n = \tau_n = n/2W$. We again stress, that in all cases considered, sampling of both the input and impulse response is performed at a rate above the required minimum allowable sampling rate.

7. Z TRANSFORM TREATMENT

For shift invariant systems, Eq. 26 takes on the form of a discrete convolution which is traditionally treated with the Z transform (8). We will now show that due to our choice of impulse response notation, such treatment can be generalized to the shift variant case.

We define two Z transforms of a two variable discrete sequence $s(m,n)$ by

$$Z_n[s(m,n)] = \sum_n z^{-n}\, s(m,n) \quad (28)$$

and

$$Z_m[s(m,n)] = \sum_m z^{-m}\, s(m,n) \quad (29)$$

Note the similarity of the spirit of these definitions to the Fourier transform operations in Eqs. 4 and 5.

Denote the Z transform of $y(\frac{m}{2W})$ by $\hat{Y}(z)$. From Eq. 26, it follows that

$$\hat{Y}(z) = Z_m[y(t_m)]$$

$$= \frac{1}{2W}\sum_n [\sum_m z^{-m}h(t_m-\tau_n;\tau_n)]u(\tau_n)$$

$$= \frac{1}{2W}\sum_n Z_m[h(t_m;\tau_n)]z^{-n}u(\tau_n) \qquad (30)$$

$$= \frac{1}{2W} Z_n Z_m[h(t_m;\tau_n)u(\tau_n)]$$

This is the generalized Z transform treatment of a discrete shift variant process. Note, as was in the case with Eq. 3 the result reduces to the more familiar product form for the shift invariant case.

8. FOURIER DUAL SAMPLING THEOREMS

The sampling theorems thus far presented can also be applied in a Fourier dual sense to the frequency domain. The corresponding constraints here, take on a physically different meaning and thus widen the class of systems which can be characterized in sampling theorem type expansions.

To change the computational form of the superposition integral, we apply Parseval's theorem to Eq. 1:

$$y(t) = \int_{-\infty}^{\infty} k(t-cv;v)U(v)dv \qquad (31)$$

where $U(v) = F_\tau[u(\tau)]$ and

$$k(t-cv;v) = \int_{-\infty}^{\infty} h(t-\tau;\tau)\exp(j2\pi v\tau)d\tau \qquad (32)$$

The kernel, $k(\cdot,\cdot)$ is recognized as the system frequency response:

$$k(t-cv;v) = S[\exp(j2\pi v\tau)] \qquad (33)$$

The constant c is included simply to maintain dimensional consistancy between the time variable t and frequency variable v.

(1) Consider first the Fourier dual of the sampling theorem for variation limited systems. Here, we require that $F_\tau^{-1}[k(t;v)]$ be identically zero outside the interval $|\tau| \leq T_v$. Also U(v) must be "bandlimited." That is, our input, $u(\tau)$, must be nonzero only over the interval $|\tau| \leq T_u$. The resulting sampling theorem, then, is simply the Fourier dual of Eq. 11:

$$y(t) = \frac{1}{c}\sum_n k(t-cv_n;v_n)U(v_n)*\text{sinc}(2T_s t/c) \qquad (34)$$

where $2T_s = 2T_u + 2T_v$ and $v_n = n/2T_s$.

(2) Consider next the Fourier dual of the sampling theorem in section 4. Here, we require $F_\tau^{-1}[k(t-cv;v)]$ is zero for $|\tau| > T_h$ and, again, that $u(\tau)$ is zero for $|\tau| > T_u$. The Fourier dual of the sampling theorem in Eq. 13 follows immediately as

$$y(t) = \frac{1}{2T_d}\sum_n k(t-cv_n;v_n)U(v_n) \qquad (35)$$

where $2T_d = 2T_h + 2T_u$ and $v_n = n/2T_d$.

(3) Lastly, we inspect the Fourier dual of the sampling theorem presented in section 5. Our constraint in this case is that k(t,v) is bandlimited in t with bandwidth $2W_t$. Note that this constraint is the same as requiring $h(t;\tau)$ to be bandlimited in t. The resulting sampling theorem expansion corresponding to Eq. 16 is

$$y(t) = \frac{1}{c}\sum_n k[t_n;\frac{t-t_n}{c}]\,U(\frac{t-t_n}{c})*\text{sinc}2W_t t \qquad (36)$$

where $t_n = n/2W_t$.

The three sampling theorems presented here can obviously be placed in discrete form as was done in section 6. For brevity, these discrete cases will not be presented but can be straightforwardly derived utilizing previous notions.

9. CONCLUSIONS

We have presented several sampling theorems applicable to various classes of shift variant systems envolving certain bandlimiting constraints on the system impulse response and/or input. The system output, in certain instances, is bandlimited and the computational form required to evaluate the values for its sampling theorem expansion was shown to result in an infinite matrix relation. The computational forms in each of the three cases considered are identical differing only in sampling rate. The matrix type relationship was shown to be able to be evaluated in a generalized Z transform treatment. Fourier duals of the sampling theorem, where sampling is largely performed in the frequency domain, were also presented. Possible areas of application of the sampling theorems include signal and image processing as well as shift variant system synthesis with a number of shift invariant systems and/or tapped delay lines. Investigation into implementation of the sampling theorems with coherent optical processors is also presently under way (9).

10. REFERENCES

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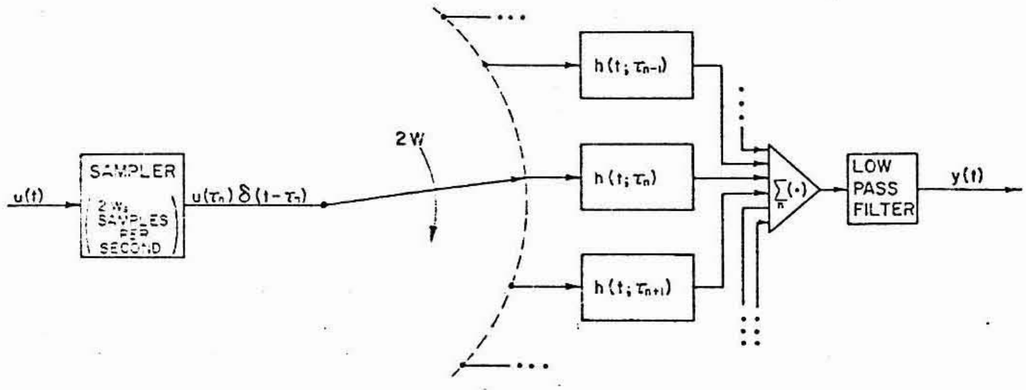

Figure 1: An implementation of the sampling theorem presented in section 3. Sample values of the input are fed into a bank of shift invariant filters each of which corresponds to a sample of the parent shift variant impulse response.

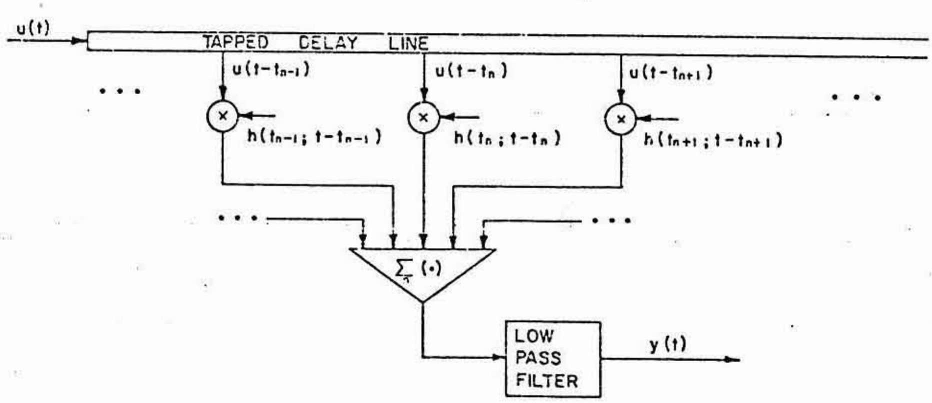

Figure 2: An implementation of the sampling theorem presented in section 5. Delayed versions of the input are multiplied by corresponding sample impulse responses, summed, and passed through a low pass filter to give the output of the parent shift variant system.