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SOME PRELIMINARY RESULTS ON DETECTION IN LAPLACE NOISE

by

Robert J. Marks II,* Gary L. Wise,** Douglas G. Haldeman,† and John L. Whited††

Abstract

The problem of the discrete time detection of a known constant signal in additive white stationary Laplace noise is considered. Both the Neyman-Pearson optimal detector and the linear detector are treated. Convenient expressions for the distribution functions of the test statistics are given. These expressions allow one to determine the performance of the detectors in a computationally convenient manner.

Introduction

Recently, there has been considerable interest in the detection of signals in non-Gaussian noise. Although the assumption of Gaussian noise is frequently justified, such as in UHF, in other cases, such as ELF, the assumption is definitely unjustified. One form of frequently encountered non-Gaussian noise is that known as impulsive noise. Impulsive noise is typically characterized as noise whose distribution has an associated "heavy tail" behavior. That is, the probability density function (pdf) approaches zero more slowly than a Gaussian pdf. The references in [1] and [2] give a summary of some forms of impulsive noise and situations where it arises. In this paper we consider the discrete time detection of a known constant signal in additive white Laplace noise. That is, the pdf of the noise is given by

$$f(n) = \frac{\gamma}{2} e^{-\gamma|n|} \quad . \qquad (1)$$

Notice that Laplace noise has the "heavy tail" behavior associated with impulsive noise.

The Laplace distribution is popular in statistics and many of its properties have been studied [3]. Furthermore, it is used as a noise model in engineering studies. For example, Miller and Thomas [1] used Laplace noise in a numerical study of relative efficiency. Bernstein, et.al. [4] comment on the non-Gaussian nature of ELF atmospheric noise, and they give a plot of a typical experimentally determined pdf associated with such noise [4, Figure 10]. This experimentally determined pdf is similar to a Laplace pdf, and on a linear graph the difference is barely distinguishable. Mertz [5] proposed the following pdf for the amplitude distribution of impulsive noise:

$$\tilde{f}(x) = \nu h^{\nu}(x + h)^{-(\nu+1)} \quad , \quad x \geq 0 \quad .$$

Notice that if we let

$$\nu = \frac{h}{\gamma} - 1 \quad ,$$

then

$$\lim_{h\to\infty} \tilde{f}(x) = \gamma e^{-\gamma x} \quad , \quad x \geq 0 \quad .$$

Thus the limiting case of the Mertz model for the amplitude distribution of impulsive noise is identical to the distribution of the amplitude of Laplace noise. Kanefsky and Thomas [6] considered a class of generalized Gaussian noises, obtained by generalizing the Gaussian density to obtain a variable rate of exponential decay. The Laplace distribution is within this class of generalized Gaussian distributions. Also, Duttweiler and Messerschmitt [7] refer to the Laplace distribution as a model for the distribution of speech.

In the next section we present a brief summary of the problem. Then in the following section we develop convenient expressions for the distribution functions of the test statistic under both hypotheses. Finally, some comments on the linear detector are given and several examples are given to illustrate the usefulness of the results.

Preliminaries

We consider testing for the presence or absence of a positive, constant signal s, in additive Laplace noise. We assume that the noise samples are statistically independent. (A restricted receiver bandwidth might cause this assumption to be violated.) The problem is modeled as the following hypothesis testing problem:

$$H_0: \; x_i = n_i \qquad\qquad i = 1,2,\ldots,N$$
$$H_1: \; x_i = s + n_i \quad . \qquad s > 0$$

Based on the observations $\{x_i, \; i = 1,2,\ldots,N\}$, we are to decide to announce that the signal is absent or present. The quantity α will denote the probability of false alarm; that is, α is the probability of incorrectly announcing H_1. Similarly, β, the detection probability, is the probability of correctly announcing H_1.

The Neyman-Pearson optimal detector is a detector which, for a fixed α, will maximize β. The optimal detector for our problem is well known [8], and is illustrated in Figure 1. The

*Department of Electrical Engineering, Texas Tech University, Lubbock, Texas 79409.

**Department of Electrical Engineering, The University of Texas at Austin, Austin, Texas 78712.

†TRW Defense and Space Systems, Redondo Beach, California 90278.

††Rockwell International, Collins Government Telecommunications Division, 1200 North Alma Road, Richardson, Texas 75207.

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observations are passed through a zero memory nonlinearity $g(\cdot)$, and then summed. The result is then compared to a threshold. The threshold T is chosen to give the desired false alarm probability. The nonlinearity $g(\cdot)$, illustrated in Figure 2, is given by the following expression:

$$g(x) = \begin{cases} \gamma s, & x > s \\ 2\gamma x - \gamma s, & 0 \le x \le s \\ -\gamma s, & x < 0 \end{cases} \quad . \qquad (2)$$

For the optimal detector, the test statistic t is given by the following sum of independent, identically distributed random variables:

$$t = \sum_{i=1}^{N} g(x_i) \quad .$$

If the distribution of this sum were known, then the detection and false alarm probabilities could be found, and the performance of the detector would be known. However, past attempts at obtaining a simple expression for this distribution have not been successful. A lengthy and complex recursion scheme for obtaining this distribution has been considered by Miller and Thomas [1, 9]. If N were sufficiently large, the Central Limit Theorem would apply, and the distribution of t would be approximately normal. However, the small sample performance of the detector would still be unknown (see, for example, [1, 10]). Alternatively, one could establish bounds on the detection and false alarm probabilities, and thus establish a bound on detector performance; or Monte Carlo simulation may be employed. In general, however, it would be desirable to have a convenient expression for the probability distribution of the test statistic t.

The Neyman-Pearson Optimal Detector

In this section we derive an expression for the distribution of the test statistic for the Neyman-Pearson optimal detector. The test statistic is obtained by passing each of the observations through the nonlinearity $g(\cdot)$, given by Eq. (2), and summing the outputs.

We first consider the case of no signal, i.e. H_0. If X_i has a Laplace distribution given by Eq. (1), then $g(X_i)$ will have the following distribution function:

$$F(x) = \frac{1}{2}\,u(x+\gamma s) + \frac{1}{4}\int_{-\infty}^{x} \exp[-\tfrac{1}{2}(\gamma s+v)]\,G\!\left(\frac{v}{2\gamma s}\right)dv + \frac{1}{2}\,e^{-\gamma s}\,u(x-\gamma s) \quad , \qquad (3)$$

where $u(\cdot)$ denotes the unit step function given by

$$u(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

and $G(\cdot)$ denotes the gate function given by

$$G(x) = \begin{cases} 1, & |x| \le \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases} \quad .$$

The distribution function $F_N(\cdot)$ of the test statistic t is given by

$$F_N(x) = \int_{-\infty}^{\infty} F_{N-1}(x-v)\,dF(v) \quad ,$$

where $F_1(x) = F(x)$.

The Fourier-Stieltjes transform of Eq. (3) is given by

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x}\,dF(x) \quad ,$$

where j denotes the imaginary unit. A straightforward calculation yields that

$$\hat{F}(\omega) = \exp\left(-\frac{\gamma s}{2}\right)\left(\cosh\left[\left(\frac{1}{2}+j\omega\right)\gamma s\right] + \frac{\sinh\left[\left(\frac{1}{2}+j\omega\right)\gamma s\right]}{1+2j\omega}\right) \quad . \qquad (4)$$

Letting $\hat{F}_N(\omega)$ denote the Fourier-Stieltjes transform of $F_N(x)$, we get that

$$\hat{F}_N(\omega) = \left[\hat{F}(\omega)\right]^N \quad . \qquad (5)$$

Using the binomial expansion, it follows from Eqs. (4) and (5) that

$$\hat{F}_N(\omega) = \exp\left(-\frac{N\gamma s}{2}\right)\sum_{k=0}^{N}\binom{N}{k}\frac{\left\{\sinh\left[\left(\frac{1}{2}+j\omega\right)\gamma s\right]\right\}^k}{2^k\left(\frac{1}{2}+j\omega\right)^k} \cdot \left\{\cosh\left[\left(\frac{1}{2}+j\omega\right)\gamma s\right]\right\}^{N-k} \quad .$$

Expressing the hyperbolic sine and hyperbolic cosine in terms of complex exponentials results in

$$\hat{F}_N(\omega) = \exp\left(-\frac{N\gamma s}{2}\right)\sum_{k=0}^{N}\binom{N}{k}\frac{1}{2^k\left(\frac{1}{2}+j\omega\right)^k} \cdot \left[\frac{\exp\left[\left(\frac{1}{2}+j\omega\right)\gamma s\right]-\exp\left[-\left(\frac{1}{2}+j\omega\right)\gamma s\right]}{2}\right]^k \cdot \left[\frac{\exp\left[\left(\frac{1}{2}+j\omega\right)\gamma s\right]+\exp\left[-\left(\frac{1}{2}+j\omega\right)\gamma s\right]}{2}\right]^{N-k} \quad .$$

Using the binomial expansion again yields

$$\hat{F}_N(\omega) = \exp\left(-\frac{N\gamma s}{2}\right)\sum_{k=0}^{N}\binom{N}{k}\frac{2^{-k}\,2^{k-N}}{2^k\left(\frac{1}{2}+j\omega\right)^k}$$

$$\cdot\sum_{p=0}^{k}\binom{k}{p}(-1)^p\exp\left[-\left(\frac{1}{2}+j\omega\right)\gamma sp\right]\exp\left[\left(\frac{1}{2}+j\omega\right)\gamma s(k-p)\right]$$

$$\cdot\sum_{q=0}^{N-k}\binom{N-k}{q}\exp\left[-\left(\frac{1}{2}+j\omega\right)\gamma sq\right]\exp\left[\left(\frac{1}{2}+j\omega\right)\gamma s(N-k-q)\right] .$$

A straightforward simplification results in the following expression:

$$\hat{F}_N(\omega) = \sum_{k=1}^{N}\binom{N}{k}2^{-(N+k)}\sum_{p=0}^{k}\binom{k}{p}(-1)^p$$

$$\cdot\sum_{q=0}^{N-k}\binom{N-k}{q}\exp[-(p+q)\gamma s]\frac{\exp[j\omega(N-2p-2q)\gamma s]}{\left[\frac{1}{2}+j\omega\right]^k}$$

$$+\ 2^{-N}\sum_{m=0}^{N}\binom{N}{m}\exp(-m\gamma s)\exp[j\omega(N-2m)\gamma s]$$

Let $\hat{A}(\omega) = \hat{F}_N(\omega) - 2^{-N}\sum_{m=0}^{N}\binom{N}{m}\exp(-m\gamma s)$

$$\cdot\ \exp[j\omega(N-2m)\gamma s] .$$

Notice that $\hat{A}(\omega)$ belongs to L_2 and thus possesses an inverse Fourier transform $A(x)$, defined as a limit in the mean.

$$A(x) = \sum_{k=1}^{N}\binom{N}{k}2^{-(N+k)}\sum_{p=0}^{k}\binom{k}{p}(-1)^p$$

$$\cdot\sum_{q=0}^{N-k}\binom{N-k}{q}\exp[-(p+q)\gamma s]$$

$$\cdot\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\{j\omega[(N-2p-2q)\gamma s+x]\}}{\left[\frac{1}{2}+j\omega\right]^k}\,d\omega .$$

Using contour integration, the residue theorem, and Jordan's lemma [11], we obtain, for $k\geq 1$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\{j\omega[(N-2p-2q)\gamma s+x]\}}{\left[\frac{1}{2}+j\omega\right]^k}\,d\omega =$$

$$\frac{[x+(N-2p-2q)\gamma s]^{k-1}}{(k-1)!}\exp\{-\tfrac{1}{2}[x+(N-2p-2q)\gamma s]\}$$

$$\cdot\ u[x+(N-2p-2q)\gamma s] .$$

After simplification, we obtain

$$A(x) = N!\ \exp[-\tfrac{1}{2}(x+N\gamma s)]\sum_{k=1}^{N}\frac{2^{-(N+k)}}{(k-1)!}$$

$$\cdot\sum_{p=0}^{k}\frac{(-1)^p}{p!(k-p)!}\ \cdot\sum_{q=0}^{N-k}\frac{[x+(N-2p-2q)\gamma s]^{k-1}}{q!(N-k-q)!}$$

$$\cdot\ u[x+(N-2p-2q)\gamma s] .$$

Let

$$B(x) = \int_{-\infty}^{x}A(v)\,dv .$$

Upon performing the integration (see [12]), we obtain

$$B(x) = N!\ \exp[-\tfrac{1}{2}N\gamma s]\sum_{k=1}^{N}2^{-(N+k)}\sum_{p=0}^{k}\frac{(-1)^p}{p!(k-p)!}$$

$$\cdot\sum_{q=0}^{N-k}\frac{u[x+(N-2p-2q)\gamma s]}{q!(N-k-q)!}$$

$$\cdot\sum_{r=0}^{k-1}\frac{[(N-2p-2q)\gamma s]^{k-r-1}}{(k-r-1)!}$$

$$\cdot\sum_{w=0}^{r}\frac{2^{w+1}}{(r-w)!}\Big\{(-1)^{r-w}[(N-2p-2q)\gamma s]^{r-w}$$

$$\cdot\ \exp[\tfrac{1}{2}(N-2p-2q)\gamma s] - e^{-x/2}\,x^{r-w}\Big\} .$$

Thus we get

$$\hat{A}(\omega) = \int_{-\infty}^{\infty}A(x)e^{-j\omega x}\,dx = \int_{-\infty}^{\infty}e^{-j\omega x}\,dB(x) .$$

Letting

$$C(x) = 2^{-N}\sum_{m=0}^{N}\binom{N}{m}\exp(-m\gamma s)\ u[x-(N-2m)\gamma s] ,$$

we get

$$\int_{-\infty}^{\infty}e^{-j\omega x}\,dC(x) = 2^{-N}\sum_{m=0}^{N}\binom{N}{m}\exp(-m\gamma s)$$

$$\cdot\ \exp[j\omega(N-2m)\gamma s] .$$

Thus

$$\hat{F}_N(\omega) = \int_{-\infty}^{\infty}e^{-j\omega x}\,dB(x) + \int_{-\infty}^{\infty}e^{-j\omega x}\,dC(x)$$

$$= \int_{-\infty}^{\infty}e^{-j\omega x}\,dF_N(x) .$$

Therefore, we see that $F_N(x) = B(x) + C(x)$.

Letting $F_N^{(o)}(x)$ denote the distribution function of the test statistic under the hypothesis H_o, we have, after a straightforward manipulation,

$$F_N^{(o)}(x) = 2^{-N} \sum_{k=1}^{N} \binom{N}{k} \sum_{p=0}^{k} (-1)^p \binom{k}{p} \sum_{q=0}^{N-k} \binom{N-k}{q} \cdot \Big[\exp[-(p+q)\gamma s] - \exp[-\tfrac{1}{2}(x+N\gamma s)] \cdot e_{k-1}\Big\{\tfrac{1}{2}x+(N-2p-2q)\gamma s\Big\}\Big] \cdot u[x+(N-2p-2q)\gamma s] + 2^{-N} \sum_{m=0}^{N} \binom{N}{m} \exp(-m\gamma s) \cdot u[x+(N-2m)\gamma s] \,, \quad (6)$$

where $e_k(\cdot)$, the incomplete exponential, is defined as

$$e_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!} \ .$$

We now consider the signal present case, i.e. H_1. We let $F_N^{(1)}(x)$ denote the distribution function of the test statistic under H_1. Since the Laplace pdf is symmetric, it can be shown [1] that

$$F_N^{(1)}(x) = 1 - F_N^{(o)}(-x) \quad . \quad (7)$$

Eqs. (6) and (7) thus completely determine the performance of the Neyman-Pearson optimal detector.

The Linear Detector

By a linear detector, we mean a scheme such as that illustrated in Figure 1, but where the function $g(\cdot)$ is $g(x) = x$. That is, the test statistic is simply the sum of the observations. The linear detector is Neyman-Pearson optimal for Gaussian noise and is a commonly used detector.

Consider the signal absent case, i.e. H_o. In this situation, the test statistic is given by

$$t = \sum_{i=1}^{N} X_i \ ,$$

where the X_i are independent identically distributed random variables with the pdf of Eq.(1). Let $p_N(x)$ denote the pdf of t. Then we have [3, p.24]

$$p_N(x) = \frac{\gamma e^{-\gamma|x|}}{(N-1)!} \sum_{k=0}^{N-1} 2^{-(N+k)} \frac{(N+k-1)!}{k!(N-k-1)!} (\gamma|x|)^{N-k-1} .$$

After a straightforward integration [12], we obtain $G_N^{(o)}(x)$, the distribution function of the test statistic of the linear detector under H_o,

$$G_N^{(o)}(x) = \begin{cases} \frac{1}{2} + \sum_{k=0}^{N-1} 2^{-(N+k)} \binom{N+k-1}{k} \Big[1 - e^{-\gamma x} \cdot e_{N-k-1}(\gamma x)\Big] \,, & x \geq 0 \\ 1 - G_N^{(o)}(-x) \,, & x < 0 \ . \end{cases} \quad (8)$$

In the signal present case, the test statistic is given by

$$t = \sum_{i=1}^{N} X_i + Ns \ ,$$

where, once again, the X_i are independent identically distributed random variables with the density function of Eq. (1). Let $G_N^{(1)}(x)$ denote the distribution function of the test statistic of the linear detector under H_1. Then we have

$$G_N^{(1)}(x) = G_N^{(o)}(x-Ns) \quad . \quad (9)$$

Eqs. (8) and (9) completely determine the performance of the linear detector.

Examples

As an example, assume that $\gamma=1$, $s=1$, $N=6$, and that we set $\alpha=0.1$. Then for the optimal detector, we find that the threshold is given by $T \approx 0.4044$ and the detection probability is given by $\beta \approx 0.809$. For the linear detector, we get that the threshold is $T \approx 4.3265$ and the detection probability is given by $\beta \approx 0.696$.

Now we set $\alpha=0.01$, with $\gamma=1$, $s=1$, $N=6$, as above. For the optimal detector, we find that $T \approx 2.7321$ and that $\beta \approx 0.418$. For the linear detector, we get that $T \approx 8.43$ and $\beta \approx 0.23$.

As another example, assume that $\gamma=1$, $s=1$, $N=36$, and that we set $\alpha=0.1$. For the optimal detector, we find that the threshold is given by $T \approx -6.9234$ and the detection probability is given by $\beta \approx 0.999957$. For the linear detector, we get that $T \approx 10.824$ and $\beta \approx 0.9982$.

Now we set $\alpha=0.01$, with $\gamma=1$, $s=1$, and $N=36$. For the optimal detector, we get $T \approx -1.5093$ and $\beta \approx 0.998$. For the linear detector, we find that $T \approx 19.9$ and $\beta \approx 0.97$.

As above, we let $\gamma=1$, $s=1$, $N=36$; and now we set $\alpha=0.001$. For the optimal detector we find that $T \approx 2.503$ and $\beta \approx 0.9837$. For the linear detector, we get that $T \approx 26.7935$ and $\beta \approx 0.8625$.

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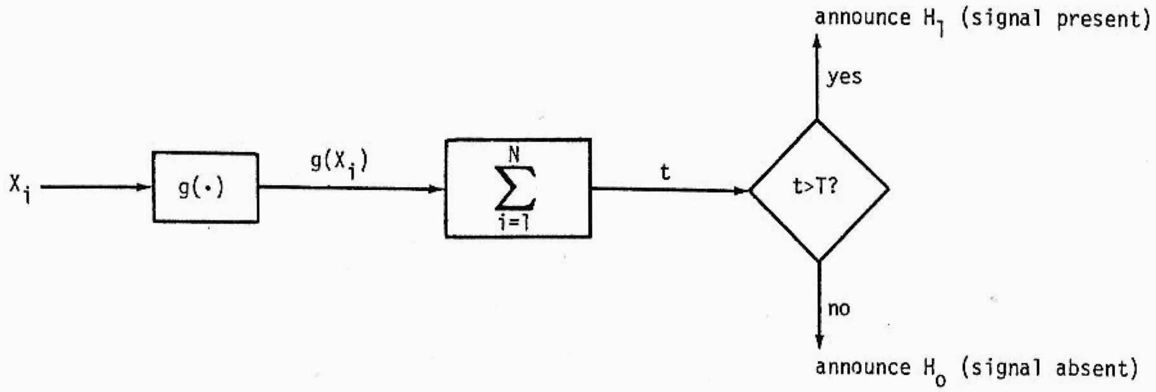

Figure 1

The Structure of the Optimal Detector

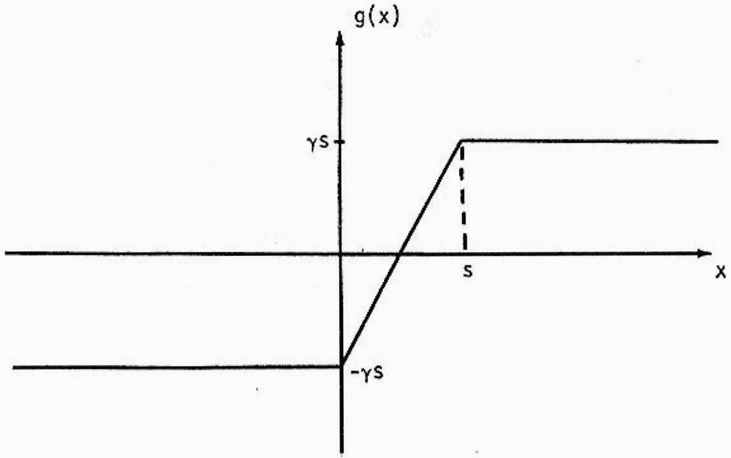

Figure 2

The Nonlinearity in the Optimal Detector