R.J. Marks II, "Coherent optical extrapolation of two-dimensional signals: processor theory", Applied Optics, vol. 19, pp.1670-1672 (1980).

Coherent optical extrapolation of 2-D band-limited signals: processor theory

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Given a truncated portion of a band-limited image with known x and y bandwidths, the extrapolation problem is to determine the signal over all x and y. An iterative extrapolation algorithm, recently proposed by Gerchberg, requires only the repeated operations of Fourier transformation and truncation. A coherent optical processor is presented that implements Gerchberg's iterative extrapolation algorithm in two dimensions. Iteration is performed by simple passive feedback.

I. Introduction

The problem of signal extrapolation has long intrigued both mathematicians and engineers. Simply stated, the problem is this: Given a signal in a given signal class over a finite interval, how does one determine the signal over a larger interval? Specific examples of the signal extrapolation problem include prediction, spectrum estimation, and superresolution (resolution beyond the Rayleigh limit). In this paper, a coherent processor is developed that is, in principle, capable of performing extrapolation of 2-D band-limited signals at the speed of light.

The most familiar form of signal extrapolation is the conventional Taylor series. Here, the signal class consists of all functions that are analytic over a given interval. Given such a function over a finite interval, it is possible, in principle, to compute all derivatives of the signal at some interior point. The resulting Taylor series specifies the signal over the entire interval of analyticity. In practice, of course, this method of signal extrapolation is impractical due to the ever-increasing uncertainty in measuring higher-order derivatives.

A second method of extrapolation applicable to the class of band-limited signals has been presented by Slepian and Pollak.[1-2] Here, the known portion of the signal is expanded in an orthonormal series of prolate spheroidal wave functions. The resulting expansion coefficients can then be used in a series expansion for the entire signal. In numerical application, these wave functions are unfortunately most difficult to deal with.[3]

A third, more recent, method of band-limited signal extrapolation has been developed by Gerchberg.[4] Sabri and Steenaart[5] made a digital adaptation of the scheme and obtained some quite remarkable results.

The purpose of this paper is to outline a coherent optical processor implementation of Gerchberg's algorithm for extrapolation of 2-D band-limited signals. Analysis and experimental results will subsequently be reported. In Sec. II, Gerchberg's algorithm is discussed in detail. A coherent processor for performing the algorithm is developed in Sec. III. Section IV contains some concluding remarks.

II. Gerchberg's Algorithm

We here consider the class of low-pass (L_2) band-limited signals. Let $u(x)$ be a signal with Fourier spectrum

$$U(f) = \mathcal{F}[u(x)] = \int_{-\infty}^{\infty} u(x)\exp(-j2\pi fx)dx. \quad (1)$$

Then $u(x)$ is in our signal class if it is square integrable and

$$u(x) = \int_{-W}^{W} U(f)\exp(j2\pi fx)df. \quad (2)$$

The parameter $2W$ is the signal's bandwidth.

Suppose $u(x)$ is known only over a finite interval, say, $-a \le x \le a$. Denote this truncated signal by

$$u_T(x) = u(x)G\left(\frac{x}{2a}\right), \quad (3)$$

where the gate function is defined by

$$G(x) = \begin{cases} 1; & |x| \le ½, \\ 0; & |x| > ½. \end{cases}$$

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Received 19 November 1979.
0003-6935/80/101670-03$00.50/0.

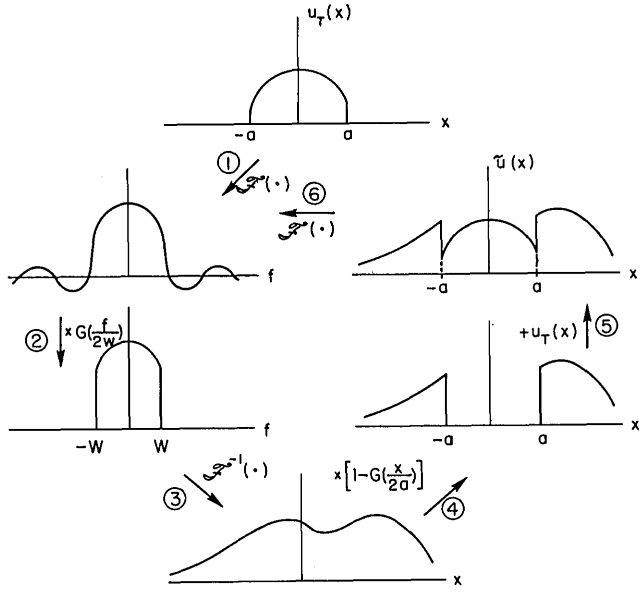

Fig. 1. Illustration of Gerchberg's algorithm for extrapolation of band-limited signals.

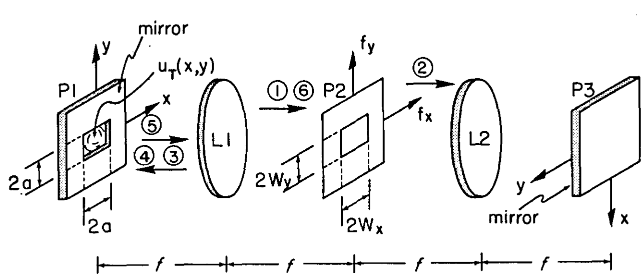

Fig. 2. Coherent optical processor for implementing Gerchberg's algorithm in two dimensions.

Gerchberg's algorithm, pictured in Fig. 1, allows for the extrapolation of $u(x)$ given $u_T(x)$. In step 1, we perform a Fourier transform of $u_T(x)$. Since the signal's bandwidth is $2W$, this spectrum is appropriately chopped in the second step. In step 3, this truncated spectrum is inverse-Fourier transformed. The inverse transform operator is defined as

$$\mathcal{F}^{-1}[S(f)] = \int_{-\infty}^{\infty} S(f)\exp(j2\pi fx)df. \tag{4}$$

The resulting signal is set to zero for $|x| \leq a$ in step 4 and is replaced in this interval by the original truncated signal in step 5. The result, $\tilde{u}(x)$, is Fourier transformed in step 6, and the cycle is repeated. In the limit, $\tilde{u}(x)$ will approach the desired result: $u(x)$. Three distinctly different proofs of the algorithm are given by Gerchberg,[4] Papoulis,[6] and Youla.[7]

Mathematically, Gerchberg's algorithm can be written

$$u(t) = \sum_{n=0}^{\infty} \mathcal{H}^n u_T(t). \tag{5}$$

The linear operator $\mathcal{H}$ is defined by

$$\mathcal{H}s(t) = 2W[s(t) * \text{sinc}2Wt]\left[1 - G\left(\frac{t}{2a}\right)\right], \tag{6}$$

where $\text{sinc}x = \sin\pi x/\pi x$, $*$ denotes convolution, and

$$\mathcal{H}^o s(t) = s(t). \tag{7}$$

We note as a matter of interest that Fienup[8] has used a (nonlinear) iterative algorithm with similar flavor to digitally reconstruct an object from the modulus of its Fourier transform.

III. Coherent Optical Extrapolator

A coherent processor capable of executing Gerchberg's algorithm in two dimensions is pictured in Fig. 2. In plane *P1*, the 2-D truncated signal $u_T(x,y)$ is input into the system. The input is assumed zero outside the square $|x| \leq a, |y| \leq a$. Outside this aperture is a mirror whose purpose will be explained shortly.

Lens *L1* performs a Fourier transform on the input corresponding to step 1 in Fig. 1. Thus, $\mathcal{F}[u_T(x,y)]$ is incident on plane *P2* where a rectangular aperture is placed. The dimensions of the aperture are determined by the known (x,y) bandwidths of $u(x,y)$. That is, if

$$u(x,y) = \int_{-W_x}^{W_x}\int_{-W_y}^{W_y} U(f_x,f_y)\exp[j2\pi(f_x x + f_y y)]df_x df_y, \tag{8}$$

where the spectrum of $u(x,y)$ is

$$U(f_x,f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y)\exp[-j2\pi(f_x x + f_y y)]dxdy, \tag{9}$$

the dimension of the aperture in spatial frequency is $2W_x \times 2W_y$.[9] The field amplitude immediately to the right of plane *P2* is thus the truncated version of the input's Fourier transform. This corresponds to step 2 in Fig. 1.

Incident on the mirror in plane *P3* is the Fourier transform of the field amplitude exiting plane *P2*. This is reflected by the mirror and is Fourier transformed again by the lens *L2* (right to left). The result, except for coordinate reversal, is that the same field amplitude previously exiting plane *P2* (left to right) is now incident on plane *P2* (right to left). This field amplitude thus passes through the aperture unaltered. Since we have a coordinate reversal, the inverse-Fourier transform of this field amplitude is incident on plane *P1* (right to left). This corresponds to step 3 in Fig. 1.

Steps 4 and 5 take place simultaneously. The unwanted center portion of the signal exits through the mirror's aperture and is lost to the system. The remainder of the signal, corresponding to the tails in the function shown between steps 4 and 5 in Fig. 1, is reflected back into the processor. The truncated signal $u_T(x,y)$ is, of course, still being input into the system. The net result is that the field amplitude exiting plane *P1* (left to right) is the $\tilde{u}(x,y)$ corresponding to Fig. 1. This is put into the system, and step 6 is performed. After a number of cycles the extrapolated signal will appear on plane *P1* (and also on plane *P3*).

Mathematically, we are implementing the 2-D generalization of Eqs. (5) and (6). That is,

$$u(x,y) = \sum_{n=0}^{\infty} \mathcal{H}^n u_T(x,y), \quad (10)$$

where, now,

$$\mathcal{H}s(x,y) = 4W_x W_y[s(x,y) * (\mathrm{sinc}2W_x x\ \mathrm{sinc}2W_y y)] \times \left[1 - G\left(\frac{x}{2a}\right) G\left(\frac{y}{2a}\right)\right]. \quad (11)$$

The $*$ here denotes 2-D convolution.

In practice, the coherent optical extrapolator must be augmented to include a method to bleed off the extrapolated signal for detection purposes. This can be done either (1) by utilization of a highly reflective pellicle (and appropriate imaging optics) in place of the right-hand mirror in Fig. 2 or (2) by placing a highly transmitting pellicle at an angle within the beam path in the processor. In either case, the extrapolation operation on a first-order analysis becomes

$$\sum_{n=0}^{\infty} (p\mathcal{H})^n u_T(x,y), \quad (12)$$

where $p < 1$ denotes the pellicle loss. The effect of this loss term on the extrapolation process has not yet been determined.

IV. Discussion

There are many aspects of the extrapolation processor that have not been addressed. These include (1) the relative inexactitude of analog processors vs sensitivity of extrapolation schemes and (2) the extreme sensitivity of extrapolation methods to input noise. These aspects will severely degrade the analytically predicted processor performance. Preliminary results, however, seem most promising.[10] Analysis and further experimental results will be reported subsequently.

This work was generously sponsored by the National Science Foundation under grant ENG 79 08009.

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Books *continued from page 1669*

sired modern physics and electronics and is used in disciplines as diverse as astrophysics and fusion research. Other kinds of devices considered include electroluminescent devices (other than the diodes mentioned), which emit light when voltages are applied; liquid crystals, which become optically anisotropic in electric fields (due to molecular orientation effects); ferroelectric ceramics (field-induced optical anisotropy in the solid state); devices based on electrochromic and electrophoretic effects (still developmental and based on color changes induced by applied fields, the first effect being primarily molecular or solid state, the latter due to particle orientation); and solid-state image pickup devices. The last are not actually display devices themselves but couple visual images to them. The fundamental concepts and equations are introduced, the physical characteristics affecting performance are well presented, many useful tabulated data from a variety of sources are given, and up-to-date references to relevant literature are supplied. While previous chapters cover fields in which many good texts have existed for a long time, this one covers areas of application where the information is mostly scattered in many journals. The author has done good service in pulling so much information together so well.

Chapter 4 (146 pages) covers the systems and equipment in which CRT display devices are used. The discussion is more from a systems engineering viewpoint than from that of specific applications; the latter view is adopted in the relatively short sixth chapter (39 pages). A major chunk of TV receiver design comes under the rubric of this chapter, and relevant aspects of it are treated; but the bulk of the text is concerned with displays in computer and instrumentation systems. The material is well organized and authoritative. Again, there is coverage of many fields, including display aspects of the burgeoning computer graphics. Character generators, refresh systems, alphanumeric and graphic terminals, analog and digital TV, problems of large screen systems, dedicated microprocessors at terminals, storage systems, input devices by which humans interact with the system directly (e.g., light pens) as well as those that take display information from elsewhere in the system are all treated. Again there is much useful tabulated information culled from a large variety of sources.

The fifth chapter (81pages) considers equipment and systems in which alphanumeric and matrix systems rather than CRTs are used. These systems are newer, are in full possession of fields like watches and calculators, as far as displays are concerned, and are likely to become increasingly competitive with, or to displace, CRTs in areas like instrumentation indicators and some large screen displays. The complimentary remarks of the previous paragraph apply here too, but the newness of, and activity in, the field may require extended discussion in a future edition; the previous chapter's material is more stable. The sixth chapter, already noted, gives more information on systems using these devices.

The short seventh, and last, chapter (39 pages) is devoted to evaluating how well the overall performance of a display system actually lives up to specifications. It considers various measurements, e.g., photometric, for parameters affecting how well humans will be able to use the display and then discusses parameters specific to the two main classes of display device. For CRTs these are luminance, contrast ratio, resolution, modulation transfer function, deflection settling time, dynamic focus, positional accuracy, pattern distortion and linearity, and their application of TV systems. The treatment of matrix devices is less standardized, and the discussion is briefer.

It is now appropriate to weigh the merits and shortcomings of the book. In this reviewer's opinion the former are major, the latter minor in comparison. The organization is logical, the treatment is comprehensive, and the references are surprisingly up-to-date and adequate as guides to the literature. The numerous tables and graphs are a valuable compendium of practical information, now made conveniently accessible. The author has delivered what he promised and has compressed a great amount of applied science and engineering into one volume.

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