R.J. Marks II and M.W. Hall, "Differintegral interpolation from a bandlimited signal's samples", IEEE Transactions on Acoustics, Speech and Signal Processing, vol. ASSP-29, pp.872-877 (1981).

V. Conclusion

Because a symmetric Toeplitz matrix is a doubly symmetric matrix, its eigenvectors are either symmetric or skew symmetric (provided the eigenvalues are distinct). The corresponding eigenfilters have their zeros either on the unit circle or they come in inverse pairs, with one zero inside the unit circle and the second outside the unit circle. We have shown that a symmetric Toeplitz matrix is a special case where the eigenfilters corresponding to the maximum and minimum eigenvalues, if distinct, have their zeros *on* the unit circle. The same property may or may not hold for the other eigenfilters. We have also shown that even if all the eigenfilters of a doubly symmetric matrix have their zeros on the unit circle, the matrix need not be Toeplitz. If some eigenvalue has multiplicity greater than one, the corresponding eigenvectors need not even be symmetric, although they could be chosen to be symmetric.

Acknowledgment

The author wishes to thank H. Gish for many helpful discussions during the course of this work.

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Differintegral Interpolation from a Bandlimited Signal's Samples

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Abstract—**The Whittaker-Shannon cardinal series dictates that any L_2 bandlimited signal is defined everywhere by its (sufficiently closely spaced) sample values. This paper derives those interpolation functions necessary for direct evaluation of such a signal's derivatives, integrals, and fractional derivatives directly from the sample values. Generation and recursion formulas for these interpolation functions are presented.**

I. Introduction

It is well known that a low-pass bandlimited signal is uniquely specified by its sufficiently closely spaced sample values [1]-[4]. Thus, the results of all operations on the signal are also specified by these values. In this paper, we present interpolation functions which directly generate derivatives and integrals of arbitrary order directly from the signal's samples.

Manuscript received July 29, 1980.
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The classical sampling theorem dictates that each point of interpolation is determined by every sample value. The contribution of a sample value roughly decreases monotonically as the interval to the point of interpolation increases. For differintegral interpolation, the amount of contribution from each sample value increases with the order of differentiation or integration. Conventional numerical analysis techniques, for example, utilize only a few adjacent sample values to interpolate a derivative at a point [5], [6]. The interpolatory relations presented in this paper relate the contributions of each sample value to each point of interpolation.

One technique to generate differintegrals stems from the derivative and integration theorems of Fourier analysis [7]. A signal's spectrum is appropriately weighted and inverse transformed. The use of the FFT for digital implementations makes such a technique attractive. Recent fast convolution algorithms, however, have been shown in many cases to require fewer operations [8]. The interpolatory relations presented in this paper—each in convolution form—thus afford a potentially more efficient computational technique for performing differintegrals.

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The discrete convolution operation can be expressed as a matrix-vector product. There has been much recent interest and success in the utilization of both coherent and noncoherent optical processors to perform such operations.[1] The capacity for information throughput is incredibly high. The interpolatory relations in this paper can be directly applied to fabricate optical transmittances for use in many of these processors to perform differintegral operations on unaliased data.

II. PRELIMINARIES

The classical Whittaker-Shannon [1], [2] cardinal series is [3], [4]

$$x(t) = \sum_{n=-\infty}^{\infty} x(t_n) \operatorname{sinc} 2W(t - t_n) \tag{1}$$

where

$$\operatorname{sinc} t = \frac{\sin \pi t}{\pi t}. \tag{2}$$

This uniformly convergent series [10] is applicable to all $x(t)$ in L_2 when

$$x(t) = \int_{-W}^{W} X(f)\, e^{j2\pi ft}\, df \tag{3}$$

where

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt. \tag{4}$$

The sample locations in (1) are at $t_n = n/2W$ where $2W$, the signal's bandwidth,[2] is defined in (3). This class of bandlimited L_2 signals will be denoted by B_W. The signal samples uniquely specify $x(t)$ via (1). In this paper, we derive the interpolatory relations to generate differintegrals of $x(t)$ directly from its sample values. The case of general linear operations on $x(t)$ has been treated elsewhere [11].

III. INTEGER DERIVATIVE INTERPOLATION

The interpolation formula for the pth derivative of a bandlimited signal originates directly from (1):

$$x^{(p)}(t) = (2W)^p \sum_{n=-\infty}^{\infty} x(t_n)\, d_p[2W(t - t_n)] \tag{5}$$

where

$$x^{(p)}(t) = \left(\frac{d}{dt}\right)^p x(t) \tag{6}$$

and

$$d_p(t) = \left(\frac{d}{dt}\right)^p \operatorname{sinc} t \tag{7}$$

[1] A list of references of work in this area can be found in [9].

[2] The spectrum of $x(t)$ can be identically zero over any subinterval of $|f| \leq W$. If nonzero only over the interval $|f| \leq B < W$, we are merely oversampling.

is the derivative kernel. From the derivative theorem of Fourier analysis, we can equivalently write

$$\begin{aligned} d_p(t) &= \int_{-1/2}^{1/2} (j2\pi f)^p\, e^{j2\pi ft}\, df \\ &= \frac{(-1)^p p!}{\pi t^{p+1}} [\sin(\pi t) \cos_{p/2}(\pi t) \\ &\quad - \cos \pi t \sin_{(p-1)/2}(\pi t)] \end{aligned} \tag{8}$$

where, in the second step, we have used [12]. The incomplete sine and cosine are defined, respectively, as

$$\cos_a(t) = \sum_{n=0}^{[a]} \frac{(-1)^n t^{2n}}{(2n)!} \tag{9a}$$

$$\sin_a(t) = \sum_{n=0}^{[a]} \frac{(-1)^n t^{2n+1}}{(2n+1)!}. \tag{9b}$$

The notation $[a]$ denotes the greatest integer less than or equal to a. To allow for $p = 0$ in (8), we set $\sin_{-1/2}(t) = 0$. Then $d_0(t) = \operatorname{sinc}(t)$.

For large t and even p, the $\cos_{p/2}(\pi t)$ term in (8) dominates. For odd p, $\sin_{(p-1)/2}(\pi t)$ dominates. This observation leads to the following asymptotic relation for $d_p(t)$ for large t:

$$d_p(t) \to \begin{cases} (-1)^{p/2} \pi^p \operatorname{sinc} t; & p \text{ even} \\ (-1)^{(p-1)/2} \pi^p \dfrac{\cos \pi t}{\pi t}; & p \text{ odd.} \end{cases} \tag{10}$$

Convolution of $(2W)^p d_p(2Wt)$ with any $x(t) \in B_W$ yields $x^{(p)}(t)$. To show this, we write

$$\begin{aligned} &(2W)^p \int_{-\infty}^{\infty} x(\tau)\, d_p[2W(t - \tau)]\, d\tau \\ &= (-1)^p \int_{-\infty}^{\infty} x(\tau) \left(\frac{d}{d\tau}\right)^p \operatorname{sinc} 2W(t - \tau)\, d\tau \\ &= \int_{-W}^{W} X(f)\, (j2\pi f)^p\, e^{j2\pi ft}\, df \\ &= x^{(p)}(t) \end{aligned} \tag{11}$$

where, in the second step, we have used the power theorem of Fourier analysis [13]. This result is a generalization of that of Gallagher and Wise [10] who noted that the first derivative of a bandlimited signal can be achieved by a convolution with an appropriately scaled first-order spherical Bessel function $j_1(t) = -(d/dt) \operatorname{sinc}(t/\pi)$.

Using $d_q(t)$ as the signal in (11) gives the recurrence relation

$$d_{p+q}(t) = \int_{-\infty}^{\infty} d_q(\tau)\, d_p(t - \tau)\, d\tau. \tag{12}$$

Thus, higher order kernels can be generated by convolution of lower ordered kernels.

A second obvious recurrence relation is

$$d_{p+q}(t) = \left(\frac{d}{dt}\right)^q d_p(t). \tag{13}$$

Using this expression with $q = 1$ and the relations

$$\frac{d}{dt}\cos_n(t) = -\sin_{n-1}(t) \tag{14a}$$

$$\frac{d}{dt}\sin_n(t) = \cos_n(t) \tag{14b}$$

gives, via (8), a third recurrence formula:

$$\frac{d}{dt}d_p(t) = d_{p+1}(t) = \begin{cases} \dfrac{-(p+1)}{t}d_p(t) + \dfrac{(-1)^{p/2}\pi^p}{t}\cos \pi t; & p \text{ even} \\ \dfrac{-(p+1)}{t}d_p(t) - \dfrac{(-1)^{(p-1)/2}\pi^p}{t}\sin \pi t; & p \text{ odd.} \end{cases} \tag{15}$$

Alternate derivative interpolation can be achieved by recognizing that $x(t) \in B_W$ implies $x^{(p)}(t) \in B_W$. Using (1), we then have

$$x^{(p)}(t) = \sum_{m=-\infty}^{\infty} x^{(p)}(t_m) \operatorname{sinc} 2W(t - t_m). \tag{16}$$

Thus, the signal derivative is uniquely specified by its sample values which, from (5), can be computed by the discrete convolution

$$x^{(p)}(t_m) = (2W)^p \sum_{n=-\infty}^{\infty} x(t_n)\, d_p(m-n) \tag{17}$$

where, from (8),[3]

$$d_p(m) = \begin{cases} \dfrac{-(-1)^{m+p}p!}{\pi m^{p+1}} \sin_{(p-1)/2}(\pi m); & m \neq 0 \\ (-1)^{p/2}\dfrac{\pi^p}{p+1}\delta_{p,\text{even}}; & m = 0. \end{cases} \tag{18}$$

Here, $\delta_{n,m}$ denotes the Kronecker delta. Note that the discrete derivative kernel is independent of the signal bandwidth. Plots of $|d_p(m)|$ are shown in Figs. 1 and 2.

Using (9b), the asymptotic behavior for $d_p(m)$ for large m is found to be

$$d_p(m) \to \begin{cases} \dfrac{(-1)^{m+(p/2)}p\pi^{p-2}}{m^2}; & p \text{ even} \\ \dfrac{(-1)^{m+[(p-1)/2]}\pi^{p-1}}{m}; & p \text{ odd.} \end{cases} \tag{19}$$

A recurrence relation for the discrete derivative kernel follows from the use of $d_q(n)$ as the signal in (17):

[3]To derive the $m = 0$ case in (18), it is easiest to use the integral in (8) with $t = m = 0$.

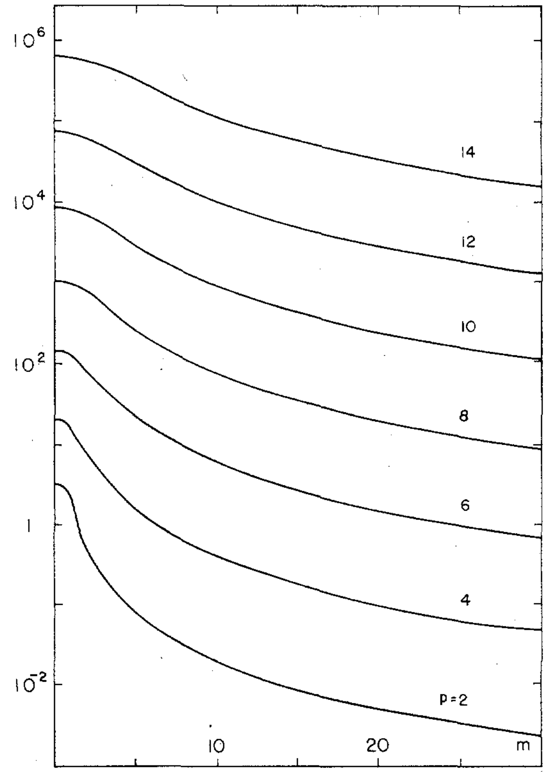

Fig. 1. Plots of $|d_p(m)|$ for even p. Points are connected for clarity.

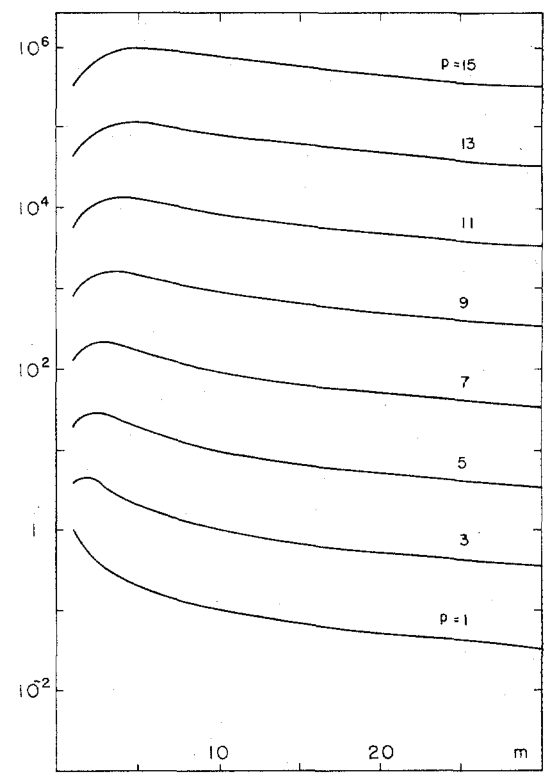

Fig. 2. Plots of $|d_p(m)|$ for odd p.

$$d_{p+q}(m) = \sum_{n=-\infty}^{\infty} d_q(n)\, d_p(m-n). \tag{20}$$

This is the discrete equivalent of (12).

A second recurrence relation immediately follows from (15) for $m \neq 0$:

$$d_{p+1}(m) = \begin{cases} \dfrac{-(p+1)}{m} d_p(m) + \dfrac{(-1)^{m+(p/2)}\pi^p}{m}; & p \text{ even} \\[2ex] \dfrac{-(p+1)}{m} d_p(m); & p \text{ odd.} \end{cases} \tag{21}$$

The discrete derivative kernel is square summable. Since $d_p(m)$ is simply the mth Fourier coefficient of $(j2\pi f)^m$ for $|f| \leq \frac{1}{2}$, we have

$$\sum_{m=-\infty}^{\infty} |d_p(m)|^2 = \int_{-\infty}^{\infty} |d_p(t)|^2 \, dt = \int_{-1/2}^{1/2} |(j2\pi f)^p|^2 \, df = \frac{\pi^{2p}}{2p+1}. \tag{22}$$

The discrete derivative kernel can be utilized to couple a bandlimited signal's Taylor series and sampling theorem expansion. If $x(t) \in B_W$, it is analytic everywhere [14]. Thus, its Taylor series about t_m is

$$x(t) = \sum_{p=0}^{\infty} \frac{(t - t_m)^p}{p!} x^{(p)}(t_m) \tag{23}$$

converges for all t. Substituting (17) gives

$$x(t) = \sum_{p=0}^{\infty} \frac{(2Wt - m)^p}{p!} \sum_{n=-\infty}^{\infty} x(t_n)\, d_p(m - n). \tag{24}$$

Since the series is absolutely convergent (see the Appendix), we can interchange the summation order:

$$x(t) = \sum_{n=-\infty}^{\infty} x(t_n) \sum_{p=0}^{\infty} \frac{(2Wt - m)^p}{p!} d_p(m - n). \tag{25}$$

The sum over p is recognized as the Taylor series expansion of sinc $2W(t - t_n)$ about $t = t_m$. Thus, (25) reduces to the cardinal series in (1).

IV. Integer Integral Interpolation

We define the pth integral of $x(t)$ by

$$x^{(-p)}(t) = \int_0^t \cdots \int_0^t x(t)\,(dt)^p \tag{26}$$

where we have chosen the lower limit of integration to be zero. (In general, $x^{(-p)}(t) \notin B_W$.) The corresponding interpolation relation from the sample values of $x(t)$ is

$$x^{(-p)}(t) = \frac{1}{(2W)^p} \sum_{m=-\infty}^{\infty} x(t_m)\, \text{sinc}^{(-p)}(2Wt - m) \tag{27}$$

where

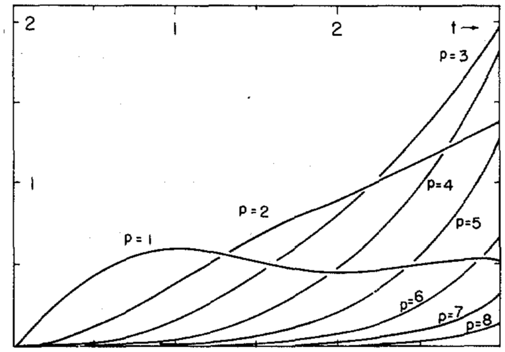

Fig. 3. Plots of $d_{-p}(t)$.

$$\text{sinc}^{(-p)}(t - m) = d_{-p}(t - m) - \sum_{k=0}^{p-1} \frac{t^k}{k!} d_{k-p}(-m) \tag{28}$$

and

$$d_{-p}(t) = \text{sinc}^{(-p)} t = t^p \sum_{q=0}^{\infty} \frac{(-1)^q (\pi t)^{2q}}{(2q+1)(2q+p)!}. \tag{29}$$

Equation (28) can easily be verified by induction using the recursion relation

$$\int_0^t d_{-p}(t - m)\, dt = d_{-p-1}(t - m) - d_{-p-1}(-m). \tag{30}$$

Plots of $d_{-p}(t)$ are shown in Fig. 3.

We can simplify the series expression in (29) by recognizing that

$$(\pi t)^p \frac{d}{dt} t^{1-p} d_{-p}(t) = \sum_{q=0}^{\infty} \frac{(-1)^q (\pi t)^{2q+p}}{(2q+p)!} \equiv \theta_p(t). \tag{31}$$

One can easily show that

$$\theta_p(t) = \begin{cases} (-1)^{p/2} [\cos \pi t - \cos_{(p-2)/2}(\pi t)]; & p \text{ even} \\ (-1)^{(p-1)/2} [\sin \pi t - \sin_{(p-3)/2}(\pi t)]; & p \text{ odd.} \end{cases} \tag{32}$$

It follows that, for $p > 1$,

$$d_{-p}(t) = t^{p-1} \int_0^t \frac{\theta_p(\tau)}{(\pi\tau)^p} d\tau. \tag{33}$$

Recognizing further that

$$\left(\frac{d}{dt}\right) d_{-p}(t) = d_{-p+1}(t) \tag{34}$$

then gives the recurrence relation (for $p > 1$)

$$(p - 1)\, d_{-p}(t) = t d_{-p+1}(t) - \pi^{-p} \theta_p(t). \tag{35}$$

By induction, we can show that application of (35) n times yields

$$d_{-p}(t) = \frac{(p-n-1)!}{(p-1)!} t^n d_{-p+n}(t) - \sum_{k=0}^{n-1} \frac{t^k (p-k-2)! \theta_{p-k}(t)}{(p-1)! \pi^{p-k}}. \quad (36)$$

The closed-form expression for the integral kernel then follows for $n = p - 1$:

$$d_{-p}(t) = \frac{1}{(p-1)!} \left[t^{p-1} d_{-1}(t) - \sum_{k=0}^{p-2} (p-k-2)! t^k \pi^{k-p} \theta_{p-k}(t) \right]. \quad (37)$$

Note that

$$d_{-1}(t) = \frac{1}{\pi} S_i\left(\frac{t}{\pi}\right) \quad (38)$$

where

$$S_i(t) = \int_0^t \frac{\sin t}{t} dt \quad (39)$$

is the well-tabulated sine integral [15].

V. Fractional Derivative Interpolation

For p a rational number, we have from (1)

$$x^{(p)}(t) = \sum_{n=-\infty}^{\infty} x(t_n) \operatorname{sinc}^{(p)}(2Wt - n). \quad (40)$$

The fractional derivative of any analytic function $\phi(t)$ can be written as [16]

$$\phi^{(p)}(t) = t^{-p} \sum_{k=0}^{\infty} \frac{t^k \phi^{(k)}(0)}{\Gamma(k-p+1)} \quad (41)$$

where we have chosen the "lower integration limit" as zero for $p < 0$. Thus,

$$\operatorname{sinc}^{(p)}(2Wt - n) = \sum_{k=0}^{\infty} \frac{(2Wt)^{k-p} d_k(-n)}{\Gamma(k-p+1)}.$$

Therefore, fractional derivatives can be generated from a series of weighted discrete derivative kernels. For $p \geqslant 0$, $\operatorname{sinc}^{(P)}(t)$ is bandlimited.

Appendix

Here, we show that the Taylor series for all $x(t) \in B_W$ is absolutely convergent. Expanding $x(t)$ about $t = \tau$ gives

$$x(t) = \sum_{m=0}^{\infty} \frac{(t-\tau)^m}{m!} x^{(m)}(\tau).$$

The series converges absolutely if

$$S \equiv \sum_{m=0}^{\infty} \frac{|t-\tau|^m}{m!} |x^{(m)}(\tau)| < \infty.$$

From the derivative theorem for Fourier transforms,

$$S = \sum_{m=0}^{\infty} \frac{|t-\tau|^m}{m!} \left| \int_{-W}^{W} (j2\pi f)^m x(f) e^{j2\pi f\tau} df \right|$$

$$\leqslant \sum_{m=0}^{\infty} \frac{|t-\tau|^m}{m!} \left[\int_{-W}^{W} (2\pi f)^{2m} df \right]^{1/2} \cdot \left[\int_{-W}^{W} |X(f)|^2 df \right]^{1/2}$$

where, in the second step, we have used Schwarz's inequality. Since $x(t) \in L_2$,

$$E^2 = \int_{-W}^{W} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

is finite. Thus,

$$S \leqslant \sqrt{2W} E \sum_{m=0}^{\infty} \frac{(2\pi W|t-\tau|)^m}{m! \sqrt{2m+1}} < \sqrt{2W} E \sum_{m=0}^{\infty} \frac{(2\pi W|t-\tau|)^m}{m!} = \sqrt{2W} E e^{2\pi W|t-\tau|}.$$

This bound is finite for all finite t and τ. Thus, any Taylor series for any $x(t) \in B_W$ is absolutely convergent.

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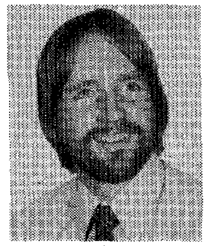

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Floating Point Roundoff Error in the Prime Factor FFT

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***Abstract*—The prime factor fast Fourier transform (PF FFT), developed by Kolba and Parks, makes use of recent computational complexity results by Winograd to compute the DFT with a fewer number of multiplications than that required by the FFT. Patterson and McClellan have derived an expression for the mean squared error (MSE) in the PF FFT, assuming finite precision fixed point arithmetic. In this paper, we derive an expression for the MSE in the PF FFT, assuming floating point arithmetic. This expression is quite complicated, so an upper bound on the MSE is also derived which is easier to compute. Simulation results are presented comparing the error in the PF FFT with both the derived bound and the error observed in a radix-2 FFT.**

I. INTRODUCTION

FAIRLY recently, a new class of algorithms has emerged for computing the discrete Fourier transform (DFT) with a fewer number of multiplications than that required by the fast Fourier transform (FFT). The first of these algorithms was developed by Winograd [1], [2] and makes use of his formulation for performing convolution with the minimum number of multiplications [3]. This algorithm has been termed the Winograd Fourier transform algorithm (WFTA) [4]. An unnested version of the WFTA has been proposed by Kolba and Parks and termed the prime factor FFT (PF FFT) [5].

It is of interest to investigate the effects of finite register length in these new algorithms. Patterson and McClellan have derived expressions for the average mean squared error (MSE) in both the WFTA and PF FFT, assuming a statistical error model and fixed point arithmetic [6]. In this paper, we restrict attention to the PF FFT and consider the case with floating point arithmetic. Section II briefly introduces the PF FFT, reviews the standard floating point error model, and develops an expression for the error vector at the output of a one-dimensional Winograd DFT. In Section III, we derive a rather cumbersome expression for the floating point MSE in the PF FFT. This expression is of little practical use, so in Section IV, we proceed to bound it by a quantity which is easier to compute. Simulation results are presented in Section V, comparing the observed error in the PF FFT with both the derived bound and the error in a radix-2 FFT.

Manuscript received June 23, 1980. This work was sponsored by the Air Force Office of Scientific Research, USAF under Grant AF-AFOSR-76-3083 through Princeton University and by the Joint Services Electronics Program through the University of Illinois. Portions of this paper were presented at the IEEE International Conference on Acoustics, Speech, and Signal Processing, Denver, CO, April 9-11, 1980.

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II. PRELIMINARIES

A. PF FFT Algorithm

A one-dimensional Winograd-type DFT algorithm can be represented in matrix notation as

$$Y = CDAy \tag{1}$$