R.J. Marks II and S.M. Tseng, "Effect of sampling on closed form bandlimited signal interval interpolation", Applied Optics, vol. 24, pp.763-765 (1985); Erratum, vol. 24, p.2490 (1985).

Effects of sampling on closed form bandlimited signal interval interpolation

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A bandlimited signal is to be recovered over a finite interval. There are two closed form sampling theorem-type algorithms that can perform this restoration. This paper explores the effects of sampling rate on the restoration uncertainty. Increasing the amount of data near the interpolation interval can either increase or decrease the restoration noise level. In the former case, there is an optimum sampling rate. In the latter, the reduction can be only slight.

I. Introduction

Let $f(x)$ denote a deterministic bandlimited signal with a bandwidth $2W$, that is, $f(x) = \int_{-W}^{W} F(u) \exp(j2\pi ux)du$, where $F(u) = \int_{-\infty}^{\infty} f(x) \exp(-j2\pi ux)dx$. As shown in Fig. 1(a), we lose a portion of the signal. The corresponding interpolation problem has application in the areas of tomography[1–3] and speckle spectra restoration.[4]

A number of iterative algorithms, although initially formulated for extrapolation, are applicable to this problem.[5–9] If we exclude techniques that require inversion of large matrices,[9–11] there are, to the authors' knowledge, only two closed form algorithms for this restoration problem. Both require only a portion of the known data.

The first restoration algorithm is discrete. As shown in Fig. 1(b), the known data are sampled at a rate $2B$ that exceeds the Nyquist rate $2W$. The unknown samples in the interpolation interval (denoted by open dots) can be restored.[12] The entire signal can then be obtained from conventional sampling theorem interpolation.[13]

The second technique is analog.[14] From the known data, we obtain the continuously sampled signal shown in Fig. 1(c). This signal is multiplied by an off-line computed periodic function and is then low pass filtered. The filter output is the signal $f(x)$. A coherent optical implementation of the analog restoration algorithm has been proposed.[15]

Both of these algorithms allow freedom in choosing the sampling rate equal to $2B$ in the discrete and $1/T$ in the analog case. Our purpose is to examine the sampling rate effect on the restoration noise level. In both cases, we will demonstrate that increasing the amount of data near the interpolation interval need not yield a significantly lower restoration noise level.

II. Preliminaries

Define $p_\tau(x)$ as one for $|x| < \tau$ and zero otherwise. The known signal in Fig. 1(a) is thus $g(x) = f(x)[1 - p_\tau(x)]$. Both restoration algorithms are linear. Thus adding noise $\xi(x)$ to $g(x)$ will produce an output of $f(x) + \eta(x)$, where $\eta(x)$ is the algorithm response to $\xi(x)$ alone. When referring to a specific algorithm we will use a subscript of d (for discrete) or a (for analog) for η. In all cases, we will assume that $\xi(x)$ is real, zero mean, and wide sense stationary with autocorrelation $R_\xi(x - y) = E[\xi(x)\xi(y)]$, where E denotes expectation.

III. Discrete Algorithm

Assume that the correlation length of the noise is sufficiently short to give sample wise white noise: $R_\xi(n/2B) = \overline{\xi^2}\delta_n$, where δ_n is the Kronecker delta, and the overbar denotes the expectation operation. Define $r = B/W$. Using the results in Ref. 16 (which have been placed in closed form in Ref. 17), we generate the $\overline{\eta_d^2(0)}/\overline{\xi^2}$ vs $1/r$ plots in Fig. 2 for various numbers of missing samples. The positions for 1,2,3,4, . . . , missing samples are $(0),(0,\Delta),(-\Delta,0,\Delta),(-\Delta,0,\Delta,2\Delta), \ldots$, respectively, where $\Delta = 1/2B$.

Consider now the case where the interpolation interval's width 2τ slightly exceeds the Nyquist interval $1/2W$. Then, when sampling is performed at the Nyquist rate, there will be at least one lost sample. As we

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Received 15 March 1984.
0003-6935/85/060763-03$02.00/0.

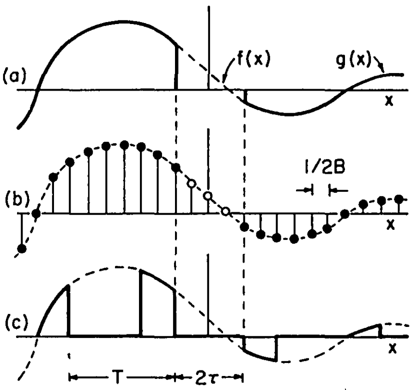

Fig. 1. (a) Interval interpolation problem. We wish to find the bandlimited signal $f(x)$ from $g(x)$. (b) Illustration of the data needed for the discrete restoration algorithm. The open dots denote samples to be restored from the known data. (c) Illustration of the data needed for the continuous restoration algorithm.

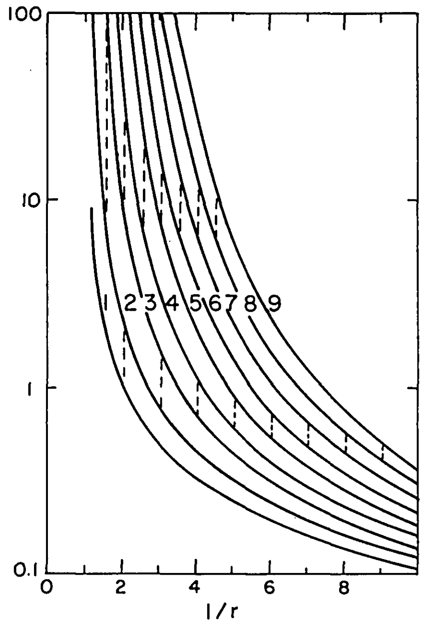

Fig. 2. Normalized restoration noise level, $\overline{\eta_d^2(0)}/\overline{\xi^2}$, vs normalized sampling rate for sample wise white data noise and various numbers of lost samples. For the interval interpolation problem, the restoration noise level follows the lower saw-toothed-type curve for one lost Nyquist interval and the upper saw-toothed-type curve for two lost Nyquist intervals. The minima on each saw-toothed curve do not decrease smoothly since for odd numbered curves the restoration is symmetric about the origin. This is not true for the even numbered curves.

increase the sampling rate, the corresponding restoration noise follows the one lost sample curve in Fig. 2 until the sampling rate becomes such that there are a minimum of two lost samples in the interpolation interval. At this point, we must jump to the two-lost-sample curve in Fig. 2. As the sampling rate increases, the restoration noise level decreases until we are required to restore three lost samples etc. The result is the lower saw-toothed shaped curve in Fig. 2. Increasing the sampling rate from 1.99 to 8.99 times the

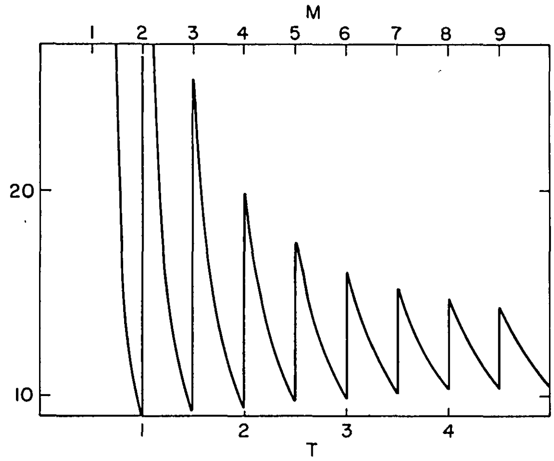

Fig. 3. Continuous sample restoration of the interval interpolation problem yields this $\overline{\eta_a^2(0)}/\overline{\xi^2}$ curve when the data are perturbed by white noise. The optimum choice of T is a bit below one ($2W = 2, 2\tau = 0.4$).

Nyquist rate decreases the noise level, $\overline{\eta_d^2(0)}/\overline{\xi^2}$, from 1 to only ~0.4 (≅4 dB).

If the interpolation interval was a bit over two Nyquist intervals, the saw-toothed curve would be initiated on the two-lost-sample curve as shown in Fig. 2.

The diminishing return observed is due to two competing phenomena. As the sampling rate increases, the amount of data near the lost sample(s) increases, thus increasing the accuracy of the lost sample estimate. The contribution of the noise, however, also increases.

In Fig. 2, the contribution of the signal is stronger than the contribution of the noise since the minima of the saw-toothed curve decrease as the sampling rate increases. The converse can also happen, for example, when using minimum mean square restoration of the lost samples.[17] In this case, increasing the sampling rate can actually increase the restoration noise level. As we shall see in the next section, either case can also happen for continuous sample restoration.

IV. Analog Algorithm

For the continuous sampled signal in Fig. 1(c), the degree of aliasing is $M = \langle 2WT \rangle$, where $\langle a \rangle$ denotes the greatest integer not exceeding a. For Mth-order aliasing, there are a total of $2M$ spectra overlapping the original zeroth-order spectrum.[14]

The restoration noise level for the continuously sampled signal restoration has been treated in Ref. 18. For continuous white noise $R_\xi(x) = \overline{\xi^2}\delta(x)$, the restoration noise level, $\overline{\eta_a^2(0)}/\overline{\xi^2}$, is shown in Fig. 3 for $2\tau = 0.4$ and $W = 1$. The jumps in the curves here are due to the increase of the aliasing order. Note that increasing T increases the minima on the saw-toothed curve. The optimum value of T in this case is 1.

The converse happens when $R_\xi(x) = \overline{\xi^2}\exp(-\alpha|x|)$. Results are shown in Fig. 4 for $2\tau = .04$, $W = 1$, and $\alpha = 2$. Increasing T decreases $\overline{\eta_a^2(0)}/\overline{\xi^2}$. If α is increased to 10, Fig. 5 results. The minima here are increasing as was the case with white noise.

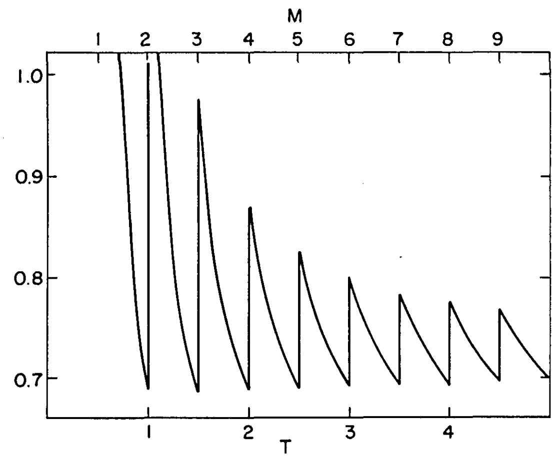

Fig. 4. Same as Fig. 3, except the noise has a Laplace autocorrelation with parameter $\alpha = 2$. The minima here increase.

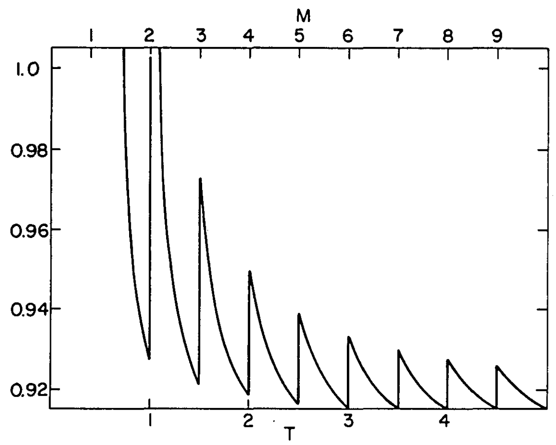

Fig. 5. Same as Fig. 4, except $\alpha = 10$. The minima here decrease. The optimal value of T is a bit below one.

V. Conclusions

When applying lost sample or continuous sample restoration to the interval interpolation problem, increasing the amount of the known signal near the interpolation interval need not decrease the restoration noise level. If concerned with the economy of data storage, a good sampling rate for the discrete case is that obtained by increasing the sampling rate above the Nyquist rate until just before one more lost sample occurs in the interpolation interval. Similarly, for continuous sampling, a good value of T is obtained by increasing T from 2τ until just before one more degree of aliasing occurs. In some cases such as in the presence of white noise, this choice of T is optimum.

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Effects of sampling on closed form bandlimited signal interval interpolation: erratum

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Received 28 May 1985.
0003-6935/85/162490-01$02.00/0.

Figures 4 and 5 in this paper[1] are described and captioned incorrectly. The relative minima increase in Fig. 4 dictating the existence of an optimal T. The minima in Fig. 5 decrease. The normalized interpolation noise level can thus be bettered by increasing T and, therefore, the order of aliasing. There appears to be no optimal value of T here.

Reference

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