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On the Stability of Mu–Varying Dynamic Equations on Stochastically Generated Time Scales

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***Abstract*—In their 2001 paper, Pötzsche, Siegmund and Wirth gave necessary and sufficient conditions for an LTI system on a time scale to have exponentially stable solutions based on pole placement. We find simple conditions for the stability of mu–varying scalar dynamic equations on time scales which are stochastically generated. As a special case, we examine the region in the complex plane which will guarantee the exponential stability of solutions of LTI systems. Via a decay analysis, we show how the tendency of the solution to grow or decay at each time step is determined by the pole placement within the region of exponential stability[1].**

I. Foundations

A time scale, $\mathbb{T}$, is any closed subset of the real line. We restrict attention to causal, unbounded time scales [5]. The *forward jump operator* [3], [7], $\sigma(t)$, is defined as the point immediately to the right of t, in the sense that $\sigma(t) = \inf\{s \in \mathbb{T} \; \forall \; s > t\}$. The *graininess* is the distance between points defined as $\mu(t) := \sigma(t) - t$. For $\mathbb{R}$, $\sigma(t) = t$ and $\mu(t) = 0$.

The time scale or *Hilger derivative* of a function $x(t)$ on $\mathbb{T}$ is defined as

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}. \tag{I.1}$$

On $\mathbb{R}$, this is interpreted in the limiting case and $x^{\Delta}(t) = \frac{d}{dt}x(t)$. The *Hilger integral* can be viewed as the antiderivative in the sense that, if $y(t) = x^{\Delta}(t)$, then for $s, t \in \mathbb{T}$,

$$\int_{\tau=s}^{t} y(\tau)\Delta\tau = x(t) - x(s).$$

The solution to the differential equation

$$x^{\Delta}(t) = zx(t); \; x(0) = 1,$$

is $x(t) = e_z(t, 0)$ where [3], [7]

$$e_z(t, s) := \exp\left(\int_{\tau=s}^{t} \frac{\mathrm{Log}(1 + \mu(\tau)z)}{\mu(\tau)}\Delta\tau\right).$$

For each $t \in \mathbb{T}$, define the *Hilger Circle* by

$$\mathcal{H}_{\mu(t)} = \{z \in \mathbb{C} | |1 + z\mu(t)| < 1, z \neq -1/\mu(t)\}$$

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Note that $\mathcal{H}_{\mu(t)}$ is a disc of radius $1/\mu(t)$ contained in the left half–plane tangent to the imaginary axis.

For an introduction to time scales, there is an online tutorial [7] or, for a more thorough treatment, see the text by Bohner and Peterson [3].

II. Definitions and Preliminary Work

Definition II.1. Let $t_0 \in \mathbb{R}$ and $\{M_i\}_{i=0}^{\infty}$ be a sequence of random variables with range $(0, \infty)$. A *stochastic time scale* with initial time t_0 generated by $\{M_i\}_{i=0}^{\infty}$ is the random set

$$\tilde{\mathbb{T}} = \{t_0\} \cup \{t_0 + \sum_{i=0}^{n} M_i | n \in \mathbb{N}^0\}$$

Note that a realization of the random set $\tilde{\mathbb{T}}$ will yield a time scale as it will be a set of isolated points, and hence closed in the subspace topology on $\mathbb{R}$.

Definition II.2. Let $\tilde{\mathbb{T}}$ be a stochastically generated time scale with initial time t_0 generated by $\{M_i\}_{i=0}^{\infty}$ and let $\lambda : [0, \infty) \to \mathbb{C}$. Let $T_i = t_0 + \sum_{i=0}^{i-1} M_i$. We say the random sequence $\{x(T_i)\}_{i=1}^{\infty}$, where $x(T_i)$ satisfies

$$x(T_i) = (1 + \lambda(M_{i-1})M_{i-1})x(T_{i-1}) \text{ for some } \lambda \in \mathbb{C},$$

is *exponentially stable almost surely* if and only if with probability one there exists a constant $\alpha < 0$ such that for every $T_i \in \tilde{\mathbb{T}}$ there exists a $K = K(T_i) \geq 1$ with

$$|x(T_k)| \leq Ke^{\alpha(T_k - T_i)}$$

for $k \geq i$.

Remark II.1. Note that on any realization $\mathbb{T}$ of a stochastic time scale $\tilde{\mathbb{T}}$, the realization of the sequence $\{x(T_i)\}_{i=1}^{\infty}$ as in the above definition is the solution of the dynamic equation $x^{\Delta} = \lambda(\mu(t))x$ on $\mathbb{T}$. Using this fact, we will say the dynamic equation

$$x^{\Delta} = \lambda(\mu(t))x$$

is exponentially stable almost surely on $\tilde{\mathbb{T}}$ if and only if the random sequence $\{x(T_i)\}_{i=1}^{\infty}$ is exponentially stable almost surely.

III. Main Result

Our main result requires a lemma, which is a modest generalization of proposition 6 from Pötzsche et al. [6].

Lemma III.1. *Let $\mathbb{T}$ be a time scale which is bounded above and let $\lambda : [0,\infty) \to \mathbb{C}$. The scalar system*

$$x^{\Delta} = \lambda(\mu(t))x \tag{III.1}$$

is exponentially stable if and only if one of the following conditions is satisfied:
(i) $\limsup_{T\to\infty} \frac{1}{T-t_0}\int_{t_0}^{T} \lim_{s\to\mu(t)} \frac{\ln|1+\lambda(s)s|}{s}\Delta t < 0$.
(ii) $\forall\, T \in \mathbb{T}\ \exists\, t \in \mathbb{T}$ *with* $t > T$ *such that* $1+\mu(t)\lambda(\mu(t)) = 0$

Proof: Follow the proof of proposition 6 in Pötzsche et al. [6]. No step in the proof relies explicitly on $\lambda(\mu(t))$ being a constant. ■

Theorem III.2. *Let $\lambda : [0,1] \to \mathbb{C}$ and $\{M_i\}_{i=0}^{\infty}$ be a sequence of independent identically distributed nonnegative random variables. Assume further that for all $M \in \{M_i\}_{i=0}^{\infty}$ $P[M = 0] = 0$ and $P[1 + \lambda(M)M = 0] = 0$. Let $\mathbb{T}$ be a stochastically generated time scale with initial time t_0 generated by $\{M_i\}_{i=0}^{\infty}$. Let $T_i = t_0 + \sum_{i=0}^{i-1} M_i$. Then the scalar dynamic equation* (III.1) *is exponentially stable almost surely on $\mathbb{T}$ if and only if for all $M \in \{M_i\}_{i=0}^{\infty}$, $E[\ln|1+\lambda(M)M|] < 0$.*

Proof: " $\Leftarrow$" If $E[\ln|1+\lambda(M)M|] < 0$, then by the strong law of large numbers,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i|}{n} \\ =& \lim_{n\to\infty} \frac{\sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i|}{n} \\ =& E[\ln|1+\lambda(M)M|] < 0 \end{aligned}$$

almost surely. The above implies

$$\limsup_{n\to\infty} \sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i| < 0$$

almost surely, or equivalently,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{T_n - t_0} \sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i| \\ =& \limsup_{n\to\infty} \frac{1}{T_n - t_0} \int_{t_0}^{T_n} \frac{\ln|1+\lambda(M_i)M_i|}{M_i} \Delta t \\ <& \ 0 \end{aligned}$$

almost surely. Thus by Lemma III.1, the dynamic equation (III.1) is exponentially stable almost surely.

" $\Rightarrow$" We show the contrapositive. First note for all $t > t_0$, $1 + \mu(t)\lambda(\mu(t)) \neq 0$ almost surely since $P[1+\lambda(M_i)M_i = 0] = 0$, $i \in \mathbb{N}^0$, thus the second condition of Lemma III.1 does not hold.

If $E[\ln|1+\lambda(M)M|] \geq 0$, then by the strong law of large numbers,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i|}{n} \\ =& \lim_{n\to\infty} \frac{\sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i|}{n} \\ =& E[\ln|1+\lambda(M)M|] \geq 0 \end{aligned}$$

almost surely. The above implies

$$\limsup_{n\to\infty} \sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i| \geq 0$$

almost surely, or equivalently,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{T_n - t_0} \sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i| \\ =& \limsup_{n\to\infty} \frac{1}{T_n - t_0} \int_{t_0}^{T_n} \frac{\ln|1+\lambda(M_i)M_i|}{M_i} \Delta t \\ \geq& \ 0 \end{aligned} \tag{III.2}$$

almost surely. Thus by Lemma III.1, the dynamic equation (III.1) is not exponentially stable almost surely. ■

Remark III.3. If M is an a continuous random variable which admits a probability density function $f : D \to [0,\infty)$ with support D, the condition $E[\ln|1+\lambda(M)M|] < 0$ becomes

$$\int_D f(\mu)\ln|1+\lambda(\mu)\mu| d\mu < 0.$$

We note that the above gives an up or down test for whether a given function λ makes (III.1) exponentially stable. The function space of all such functions λ is quite complicated. In fact, the space is not even linear. We can, however, study certain classes of functions. Letting $\lambda(\mu(t)) = \frac{e^{z\mu(t)}-1}{\mu(t)}$, the cylinder transformation, a particularly important function in the study of time scales, we find

$$\begin{aligned} E[\ln|1+\lambda(M)M|] &= E\left[\ln\left|1+\frac{e^{zM}-1}{M}M\right|\right] \\ &= \text{Re}(z)E[M] < 0 \end{aligned}$$

if and only if $\text{Re}(z) < 0$ and M has finite mean. This should agree with our intuition, as the region of exponential stability for the equation $\dot{x} = zx$ on $\mathbb{R}$ is $\{z \in \mathbb{C} | \text{Re}(z) < 0\}$.

Remark III.4. If M is a discrete random variable with finitely many possible values $\mu_1, \mu_2, \ldots \mu_n$ with a probability mass function $g : D \to [0,\infty)$, the condition $E[\ln|1+\lambda M|] < 0$ becomes

$$\sum_{i=1}^{n} f(\mu_i)\ln|1+\lambda(\mu_i)\mu_i| < 0,$$

or

$$\prod_{i=1}^{n} |1+\lambda(\mu_i)\mu_i|^{f(\mu_i)} < 1.$$

Consider the special case where $\lambda(\mu)$ is constant, that is, we have the equation

$$x^{\Delta} = \lambda x. \tag{III.3}$$

Then the above agrees with the result by Davis et al. in [4] where the asymptotic weights $d_k = f(\mu_k)$. We can view their concept of asymptotic equivalence class as the set of all time scales which are distributed the same in the tail.

Remark III.5. In the proof of proposition 6 of Pötzsche et al. [6], a formula for a suitable α in the bounding exponential function $Ke^{-\alpha t}$ is given by, in our case,

$$\begin{aligned} \alpha &= -\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i|}{T_n - t_0} \\ &= -\limsup_{n\to\infty} \frac{n}{\sum_{i=0}^{n-1} M_i} \frac{\sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i|}{n} \\ &= -\frac{E[\ln|1+\lambda(M)M|]}{E[M]} \\ &> 0. \end{aligned}$$

Remark III.6. We can view the solution of the deterministic equation (III.1) on a stochastic time scale as the solution of the stochastic equation $x_{n+1} = (1 + \lambda(M_n)M_n)x_n$ on the deterministic time scale $\mathbb{Z}$. The problem of stability of stochastic systems has been studied in [1]. It is known that the stochastic difference equation $x_{n+1} = a_n x_n$, where $\{a_n\}$ is a sequence of ergodic scalar random variables is exponentially stable almost surely if and only if $E[a_n] < 0$. This result matches our result, as the sequence of random variables $\{1 + \lambda(M_n)M_n\}$ is a sequence of independent random variables, and hence is a sequence of ergodic random variables. The definition of exponential stability almost surely for stochastic difference equations differs, however, from the definition presented in this paper, which is the definition commonly used in the time scales literature. The two definitions may or may not be equivalent.

Corollary III.7. *Let $\{M_i\}_{i=0}^{\infty}$ be states at step i of an ergodic Markov chain with finitely many states $\mu_1, \mu_2, \ldots \mu_n$, all of which are nonzero. Let $\lambda \in \mathbb{C}$ such that $|1 + \lambda\mu_k| \neq 0$ for $1 \leq k \leq n$. Let $\tilde{\mathbb{T}}$ be a stochastically generated time scale with initial time t_0 generated by $\{M_i\}_{i=0}^{\infty}$. Define π to be the unique stationary discrete distribution associated with the Markov chain. Then the scalar dynamic equation* (III.3) *is exponentially stable almost surely on $\tilde{\mathbb{T}}$ if and only if $\sum_{i=1}^{n} \pi(\mu_i) \ln|1 + \lambda\mu_i| < 0$.*

Corollary III.8. *Let $\{M_i\}_{i=0}^{\infty}$ be a sequence of nonnegative independent random variables and let $\tilde{\mathbb{T}}$ be a stochastically generated time scale with initial time t_0 generated by $\{M_i\}_{i=0}^{\infty}$. Assume, for $\lambda : [0, \infty) \to \mathbb{C}$,*

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} Var[\ln|1 + \lambda(M_i)M_i|] < \infty.$$

Assume further that $P[M_i = 0] = 0$ and $P[1 + \lambda(M_i)M_i = 0] = 0$ for $i \in \mathbb{N}^0$. Then the scalar dynamic equation (III.3) *is exponentially stable almost surely on $\tilde{\mathbb{T}}$ if and only if $\lim_{n\to\infty} E\left[\frac{\sum_{i=0}^{n} \ln|1+\lambda(M_i)M_i|}{n+1}\right] < 0$.*

Proof: Use Kolmogorov's Strong Law of Law Numbers and follow the proof of Theorem 1. ■

The next proof requires a corollary to the Borel-Cantelli lemma, which we state here

Lemma III.9. *Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of random variables and let $a \in \mathbb{R}$ be such that*

$$\sum_{n=0}^{\infty} P[X_n \geq a] < \infty.$$

Then

$$\limsup_{n\to\infty} X_n < a.$$

We can now prove the following corollary.

Corollary III.10. *Let $\{M_i\}_{i=0}^{\infty}$ be a sequence of nonnegative random variables and let $\lambda : [0, \infty) \to \mathbb{C}$ such that*

$$\begin{aligned} &\sum_{n=0}^{\infty} P\left[\sum_{k=0}^{n} \ln|1 + \lambda(M_i)M_i| \geq 0\right] \\ = &\sum_{n=0}^{\infty} P\left[\frac{\sum_{k=0}^{n} \ln|1 + \lambda(M_i)M_i|}{n+1} \geq 0\right] < \infty. \end{aligned}$$

Assume further that $P[M_i = 0] = 0$ and $P[1 + \lambda(M_i)M_i = 0] = 0$ for $i \in \mathbb{N}^0$. let $\tilde{\mathbb{T}}$ be a stochastically generated time scale with initial time t_0 generated by $\{M_i\}_{i=0}^{\infty}$. Then the scalar dynamic equation (III.3) *is exponentially stable almost surely on $\tilde{\mathbb{T}}$.*

Proof: Condition (III.4) yields, by (III.9), that

$$\limsup_{n\to\infty} \ln|1 + \lambda(M_i)M_i| < 0$$

almost surely. With this, we proceed as in the "$\Leftarrow$" part of the proof of Theorem 1. ■

IV. Examples

We now examine the behaviour of (III.3) on a stochastically generated time scale $\mathbb{T}_\Gamma$ generated by independently identically distributed random variables taken from a Gamma Distribution with shape parameter 2 and rate parameter 2, whose probability density function we call f. Notice that such a stochastically generated time scales falls under the scope of (III.2), and by Remark III.4, given $\lambda \in \mathbb{C}$, (III.3) is exponentially stable on $\mathbb{T}_\Gamma$ if and only if

$$\int_0^{\infty} f(\mu) \ln|1 + \lambda\mu| d\mu < 0.$$

The region in the complex plane where this inequality holds, which we will call the region of stability, is shown in Figure 1.

We choose two values of λ, $\lambda_1 = 1 + .25i$ and $\lambda_2 = -2 + .67i$ and generate six realizations of the time scale using each $\lambda_i, i = 1, 2$. The results are shown in Figure 2

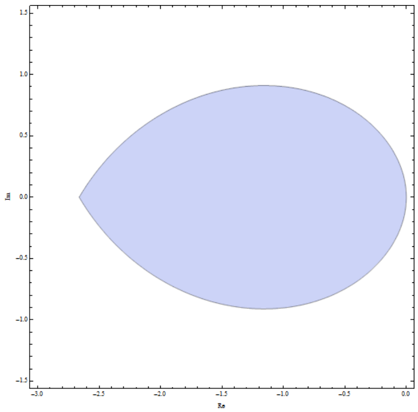

Fig. 1. The region of stability for the stochastically generated time scale $\mathbb{T}_\Gamma$.

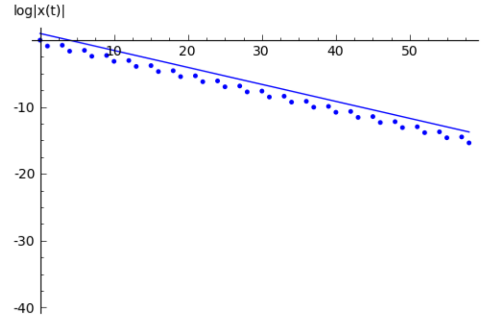

Fig. 4. The solution of (III.3) on $\mathbb{T}_{1,2}$ for $\lambda = -.9 + .4i$ along with the predicted decay rate.

and Figure 3 along with the theoretical decay rate as in a Remark III.5.

Note that the solution of (III.3) with $\lambda = \lambda_1$ decays fairly regularly and does not require an extremely large multiplier on the bounding exponential.

The solution of (III.3) with $\lambda = \lambda_2$, on the other hand, is very irregular in its behavior , having swings in the order of magnitude of x as large as 10^{100}. Amazingly (III.3) is exponentially stable by (III.2), but it has a very slow decay rate and does not decay reliably locally.

We note that this analysis is not limited to time scales picked randomly from a distribution. If we know the frequency with which different grainінesses appear in the tail of the time scale, the same results hold. To see this, we consider $\mathbb{T}_{1,2} = \{0, 1, 3, 4, 6, \dots, k, k+1, k+3, k+4, \dots\}$ which is a time scale where the graininess alternates between 1 and 2. Thus we can think of this as a particular instance of a time scale generated by a random variable with probability mass function

$$f(t) = \begin{cases} 1/2 \text{ if } t = 1 \\ 1/2 \text{ if } t = 2 \end{cases} .$$

The condition on λ for stability of (III.3) on $\mathbb{T}_{1,2}$ is

$$\sum_{i=1}^{2} f(i) \ln |1 + \lambda i| < 0$$

Solutions of (III.3) for λ satisfying the above condition is shown in Figure 4 with along with the theoretical decay rate which we mentioned in Remark III.5.

V. Decay Analysis

The example that showed the exponential stability of (III.3) with $\lambda = \lambda_2$ on $\mathbb{T}_\Gamma$ should give us some concern with this framework. After all, in applications we would not call a system with a state variable whose magnitude reached 10^{100} "stable"! We now consider how to analyze the probability that the state variable will have a magnitude below a certain tolerance $\tau > 0$. Throughout this section we will denote the conditional probability of an event A given another event B by $P[A; B]$.

Let $\{x(T_k)\}_{k=0}^{\infty}$ be the solution of (III.3) with initial condition $x(t_0) = k$ where $|k| = 1$ on a time scale with generated by the independent identically distributed random variables $\{M_i\}_{i=0}^{\infty}$. Note

$$P[|x(t_0)| < \tau] = \begin{cases} 0 \text{ if } \tau \leq 1 \\ 1 \text{ if } \tau > 1. \end{cases}$$

For the sake of the simplicity, assume further that the M_i's are continuous random variables which admit a probability distribution function f with support $[0, \infty)$. To find the probability that the magnitude of the state variable is beneath the tolerance after one step, write $\lambda = x + iy$ and calculate

$$\begin{aligned} & P[|x(T_1)| < \tau] \\ = \; & P[|x(t_0)(\lambda M_0 + 1)| < \tau] \\ = \; & P[|x(t_0)||(x + iy)M_0 + 1| < \tau] \\ = \; & P[(M_0 x + 1)^2 + M_0^2 y^2 < \tau^2] \\ = \; & P[M_0^2(x^2 + y^2) + 2M_0 x + (1 - \tau^2) < 0] \\ = \; & P[M_0^2 |\lambda|^2 + 2M_0 \operatorname{Re}(\lambda) + (1 - \tau^2) < 0] \\ = \; & P[c_1(\tau) < M_0 < c_2(\tau)] \\ = \; & \int_{c_1(\tau)}^{c_2(\tau)} f(\mu) d\mu \end{aligned}$$

where $c_1(\tau) = \frac{-\operatorname{Re}(\lambda) - \sqrt{(\operatorname{Re}(\lambda))^2 - |\lambda|^2(1-\tau^2)}}{|\lambda|^2}$ and $c_2(\tau) = \frac{-\operatorname{Re}(\lambda) + \sqrt{(\operatorname{Re}(\lambda))^2 - |\lambda|^2(1-\tau^2)}}{|\lambda|^2}$ are obtained via the quadratic formula with the assumption $c_i(\tau) = 0$ if the equations above yield imaginary or negative numbers, $i = 1, 2$. We note that if $\tau \geq 1$ then $c_1(\tau)$ and $c_2(\tau)$ are real–valued. This is not necessarily the case if $\tau < 1$, since the solution cannot decay arbitrarily fast. The smallest factor the solution can decay by is $\hat{\tau}$ such that $\operatorname{Re}(\lambda)^2 - |\lambda|^2(1 - \hat{\tau}^2) = 0$.

Note that since the $M_i's$ are IID random variables, the method above shows the probability that the solution grows by a factor bounded by τ on the any single time step. By letting $\tau = 1$, we obtain the probability that the solution will not grow the next time step,

$$p := P[|x(T_1)| < 1] = P[|x(T_k)| < c; |x(T_{k-1})| = c]$$

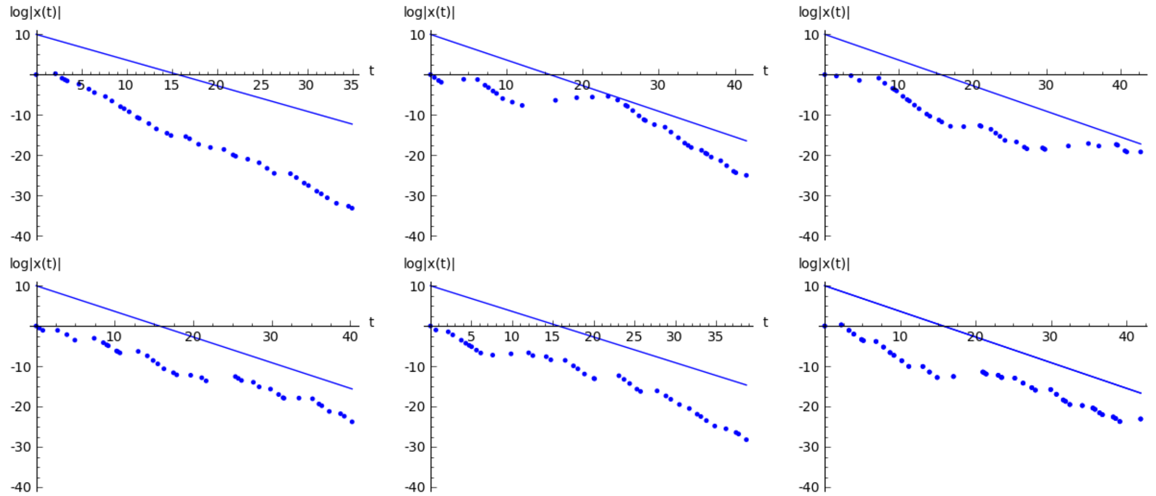

Fig. 2. $\log(|x(t)|)$ with $\lambda_1 = -1 + .25i$ on six different time scales generated from the gamma distribution with shape parameter 2 and rate parameter 2.

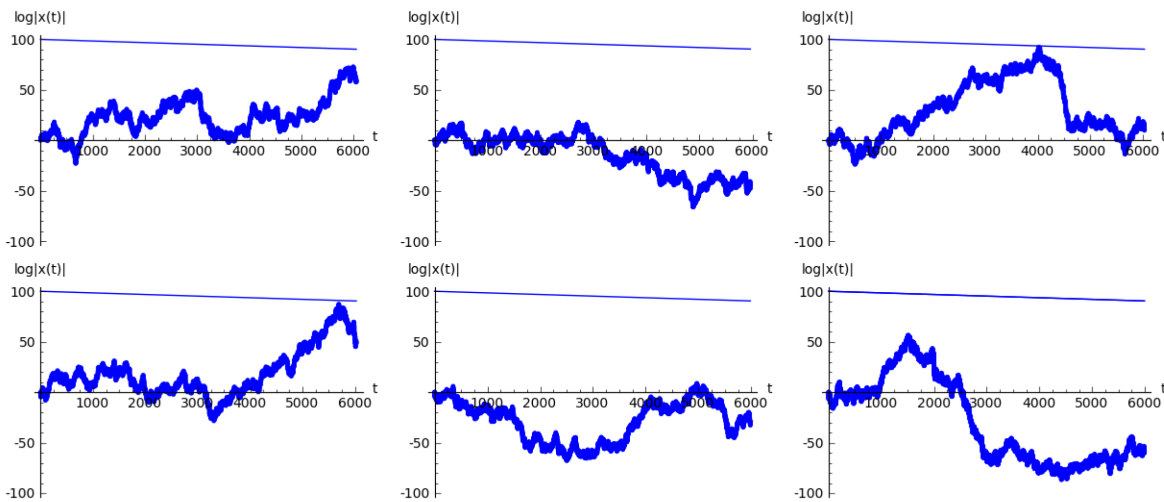

Fig. 3. $\log(|x(t)|)$ with $\lambda_2 = -2 + .67i$ on six different time scales generated from the gamma distribution with shape parameter 2 and rate parameter 2.

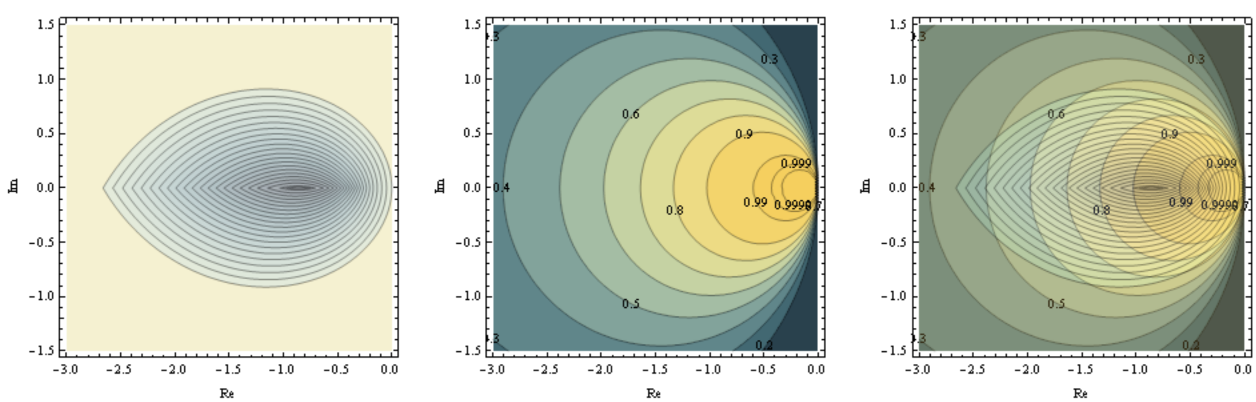

Fig. 5. All figure are for the stochastically generated time scale $\mathbb{T}_\Gamma$. Left: Contour plot of the decay rate α in the region of stability. Center: Contour plot of p in the left–half complex plane. Right: Contour plot of p in the left–half complex plane with the decay rate α in the region of stability overlaid.

for any $c > 0$. This can be a very useful design parameter, as we would like to choose λ so that the probability of decay in the state variable is sufficiently large or one, ensuring "local stability." This design parameter may be more useful than calculating the probability that the magnitude of the state variable is beneath a tolerance at T_k, as it involves k integrations, as we now show. Note that

$$P[|x(T_1)| < \tau] = \int_{c_1(\tau)}^{c_2(\tau)} f(\mu) d\mu := F(\tau)$$

where $F(\tau)$ is increasing as $c_2(\tau)$ is increasing and $c_1(\tau)$ is decreasing and $f(\mu) \geq 0$ for $\mu \geq 0$. Thus $F(\tau)$ is a CDF for the random variable $|x(T_1)|$. Note

$$\begin{aligned} F'(\tau) &= f(c_2(\tau))c_2'(\tau) - f(c_1(\tau))c_1'(\tau) \\ &= \frac{\tau(f(c_2(\tau)) + f(c_1(\tau)))}{\sqrt{(\mathrm{Re}(\lambda))^2 - |\lambda|^2(1-\tau^2)}} \\ &:= h(\tau) \end{aligned}$$

is therefore a PDF for $|x(T_1)|$. Now, by the law of total probability,

$$\begin{aligned} &P[|x(T_2)| < \tau] \\ =& \int_0^\infty h(l) P[|x(T_2)| < \tau; |x(T_1)| = l] dl \\ =& \int_0^\infty h(l) P[|1 + \lambda M_1| < \tau/l] dl \\ =& \int_0^\infty h(l) P[|1 + \lambda M_0| < \tau/l] dl \\ =& \int_0^\infty h(l) \int_0^{\tau/l} h(\mu) d\mu dl \\ =& \int_0^\infty \int_0^\tau h(l) h\left(\frac{\mu}{l}\right) \frac{1}{l} d\mu dl \\ =& \int_0^\tau \int_0^\infty h(l) h\left(\frac{\mu}{l}\right) \frac{1}{l} dl d\mu \end{aligned}$$

so $k(\tau) := \int_0^\infty h(l) h\left(\frac{\tau}{l}\right) \frac{1}{l} dl$ is a probability distribution function for the random variable $|x(T_2)|$.

By induction it is easy to show

$$\begin{aligned} &P[|x(T_k)| < \tau] \\ =& \int_0^\tau \underbrace{\int_0^\infty \cdots \int_0^\infty}_{k-1 \text{ times}} h(s_1) h\left(\frac{s_2}{s_1}\right) \cdots h\left(\frac{\mu}{s_{k-1}}\right) \\ & \frac{1}{s_1 s_2 \cdots s_{k-1}} ds_1 \ldots ds_{k-1} d\mu. \end{aligned}$$

Rather than calculate the above integral, we may find the parameter p, an easy–to–calculate number that yields important information about tendency of the system to decay. On one hand, choosing λ such that p is near one helps ensure the magnitude of the state variable will not become extremely large. On the other hand, this choice of λ may yield a slow decay rate. The best performance will be had by balancing the p and the decay rate α according to some metric. For example, we may wish to maximize the decay rate subject to $p > c$ with $0 \leq c < 1$. Instead of building an optimization algorithm for this problem, we simply plot both p and α as a function of λ. Such a plot is shown in Figure 5 for $\mathbb{T}_\Gamma$.

We note that the value of p is constant along any circle tangent to the imaginary axis since

$$p = P[|1 + \lambda M_0| < 1] = P[0 < M_0 < -2\,\mathrm{Re}(\lambda)/|\lambda|^2]$$

and $-2\,\mathrm{Re}(\lambda)/|\lambda|^2 = 1/c$ for each λ on a circle of radius c tangent to the imaginary axis. Thus the contours of constant probability of decay are Hilger circles. If the support of the distribution of graininess is bounded by $\mu_{\max} \in (0, \infty)$ and λ is on a circle of radius smaller than $1/\mu_{\max}$, then $-2\,\mathrm{Re}(\lambda)/|\lambda|^2 > \mu_{\max}$. Thus

$$p = P\left[0 < M_0 < -2\frac{\mathrm{Re}(\lambda)}{|\lambda|^2}\right] \geq P[0 < M_0 < \mu_{\max}] = 1,$$

so $p = 1$. This shows if λ is in the smallest possible Hilger circle, then the solution will decay at each step with probability one.

Remark V.1. It is well known that the smallest Hilger circle is contained in the region of exponential stability. In general, the Hilger circle corresponding to a probability of decay $\beta < 1$ is not contained in the region of stability. To see this, Consider a time scale generated by the probability mass function

$$f(\mu) = \begin{cases} \beta \text{ if } \mu = 1 \\ 1 - \beta \text{ if } \mu = 2 \end{cases}$$

Then the probability β contour is a Hilger circle of radius one, but the region of exponential stability is strictly contained in the Hilger circle of radius one because the region of exponential stability is the weighted geometric mean of a Hilger circle of radius one and a Hilger circle of radius $1/2$ [4].

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