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Ambiguity Function Magnitude Inversion via a Modified Gerchberg-Saxton Algorithm

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***Abstract* — Radar's narrowband ambiguity function is often described as non-invertible. The inability to uniquely invert a given ambiguity function to its constituent pulse signal is predominantly due to the common practice of expressing the function in terms of its magnitude only and removing all phase information. These phase components are critical for an inversion to the original signal. Still, a non-unique source pulse can potentially be synthesized to achieve a given ambiguity function magnitude. This paper develops a methodology based on Gerchberg-Saxton phase-retrieval algorithm for synthesizing source signals.**

***Index Terms* — Ambiguity function, Gerchberg-Saxton, inverse problems, signal synthesis.**

I. Introduction

The ambiguity function [1]-[8] provides powerful insight into how a received pulse that has been distorted by a target's Doppler effects is processed by a matched filter. As the goal of a matched filter is to detect the presence of a template signal, the magnitude of the ambiguity function is often used. However, loss of phase components in the ambiguity function results in the inability to uniquely invert to the original source pulse. This paper outlines a procedure to synthesize a source signal from a given ambiguity function based on the Gerchberg-Saxton algorithm for phase retrieval [9], [10].

The Gerchberg-Saxton algorithm originally developed for optical diffraction has since been used for phased-array beam-pattern synthesis [11]. Prior work in signal synthesis involved utilizing alternating projections [12] to form ambiguity functions with desired characteristics. Others have focused on synthesizing the ambiguity function itself [13]. The ambiguity function is a 2D function computed from a 1D source signal. As a result, the two dimensions are dependent on each other, and thus not all 2D functions are valid ambiguity functions. Ambiguity function synthesis generally involves taking a given, possibly invalid 2D function and synthesizing a valid ambiguity function with desired characteristics. This paper will explore the application of the Gerchberg-Saxton algorithm as a phase retrieval methodology to find solutions satisfying the given ambiguity function magnitudes.

II. Ambiguity Function

The ambiguity function of a potentially complex, baseband signal $x(t)$ is commonly defined as [4]

$$\chi(\tau,u) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)e^{-j2\pi ut}\,dt \qquad (1)$$

where τ is the propagation delay and u is the relative Doppler error the target and monostatic receiver. Alternative definitions of the ambiguity function exist, including those that apply to wideband signals.

The ambiguity function presented in (1) is invertible [2], [3] to an arbitrary phase. However, many sources express the ambiguity function in terms of its magnitude $|\chi(\tau,u)|$. This loss of the phase component removes the ability to invert to a uniquely shaped pulse solution.

III. Inversion Procedure

A. Inverting the Full Ambiguity Function

The ambiguity function can be expressed in terms of the pulse signal $x(t)$ via an inverse Fourier Transform.

$$x(t)x^*(t-\tau) = \int_{-\infty}^{\infty} \chi(\tau,u)e^{j2\pi ut}\,du\,. \qquad (2)$$

For any specific time ξ, setting $\tau = 0$ yields

$$|x(\xi)|^2 = x(\xi)x^*(\xi) = \int_{-\infty}^{\infty} \chi(0,u)e^{j2\pi u\xi}\,du\,. \qquad (3)$$

The phase component of $x(\xi)$ is unknown, but

$$|x(\xi)| = \left(\int_{-\infty}^{\infty} \chi(0,u)e^{j2\pi u\xi}\,du\right)^{1/2}. \qquad (4)$$

Now consider the situation where $t = \xi - \tau$, where ξ is fixed.

$$x(\xi)x^*(t) = \int_{-\infty}^{\infty} \chi(\xi - t, u)e^{j2\pi u\xi}\, du\,. \qquad (5)$$

Note that the phase of the complex number $x(\xi)$ is extraneous as the term is canceled by its conjugate. Expressing the above in terms of $x^*(t)$ yields

$$x^*(t) = \frac{1}{|x(\xi)|}\int_{-\infty}^{\infty} \chi(\xi - t, u)e^{j2\pi u\xi}\, du \qquad (6)$$

where $|x(\xi)|$ is given in (4). The function $x(t)e^{j\alpha}$ also solves the ambiguity function when α is constant. In general, $x(t)$ can be found by choosing a reference value ξ and evaluating (6). An additional constraint is that ξ must be chosen such that $|x(\xi)| \neq 0$.

The process described will invert a given ambiguity function with known phase. However, the ambiguity function is often expressed in terms of its magnitude $|\chi(\tau, u)|$. The loss of phase information will prevent a direct inversion. As a result, a phase-retrieval algorithm is appended to estimate the phase differences and determine a non-unique pulse signal.

B. Gerchberg-Saxton Algorithm

The Gerchberg-Saxton algorithm [9] is an iterative phase retrieval algorithm commonly used in optics for diffraction synthesis. The goal of the algorithm is to adjust a constrained source signal in order to acquire the specified target plane intensity. However, Gerchberg-Saxton algorithms depend on initialization and have no guarantee of convergence. The solutions found cannot be unique as any arbitrary phase applied throughout the function will still produce the same intensities [9], [10]. As a result, the application of the Gerchberg-Saxton algorithm cannot be expected to exactly reconstruct the original source angles. Any reconstruction can, for example, be shifted arbitrarily in phase.

C. Inverting the Ambiguity Function Magnitude

A modified Gerchberg-Saxton algorithm for the ambiguity function is presented in Figure 2. Only the target ambiguity function magnitude is given. Consider the ambiguity function at $\chi(0,0)$, or

$$\chi(0,0) = \int_{-\infty}^{\infty} x(t)x^*(t)\, dt\,. \qquad (7)$$

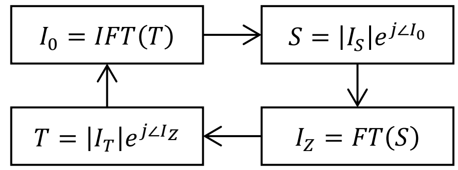

Fig. 1. The Gerchberg-Saxton algorithm is given a reference source and target intensities $|I_S|$ and $|I_T|$. It is initialized with $T = |I_T|e^{j\phi_0}$, where ϕ_0 is a uniform random phase. By iterating between target T and source S, the algorithm ideally converges to a local optimum that will generate a target with the desired intensities $|I_T|$.

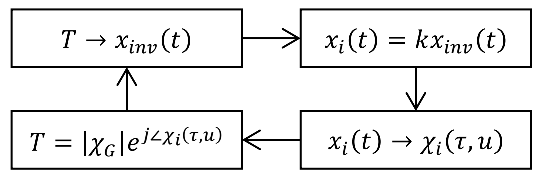

Fig. 2. Our proposed algorithm is a modified, Gerchberg-Saxton based algorithm initialized with $T = |\chi_G|e^{j\phi_0}$, where ϕ_0 is a uniform random phase. Here, $|\chi_G|$ is the given ambiguity function magnitude, $x_{inv}(t)$ is found in (6) and k is a scaling factor based on (7).

A given source signal $x(t)$ must satisfy the magnitude computed by (7). The output of an inversion found by (6) should satisfy this condition unless the input is not a valid ambiguity function. This is likely the case as the choice of phase is initially random. Thus, a scaling factor k, found by the square root of the (7) ratio is used to adjust an inverted $x(t)$ to the proper magnitude. The final modified algorithm is presented in Figure 2.

IV. RESULTS

The inversion procedure outlined in Figure 2 was applied to multiple ambiguity functions generated from known source pulses. These pulses presented here include the rectangle function, a Gaussian pulse, and a Gaussian chirp.

Figure 3 depicts the given ambiguity function magnitudes (left) and the ambiguity function of the recovered signals for each of the source pulses. While the relatively simple rectangle and Gaussian sources converge comparatively quickly, the Gaussian chirp often requires several initialization attempts to locate a desirable solution. Each figure shows the result after 200 iterations. While the original signals used to generate the target ambiguity function magnitudes were real, the recovered signals can be complex valued.

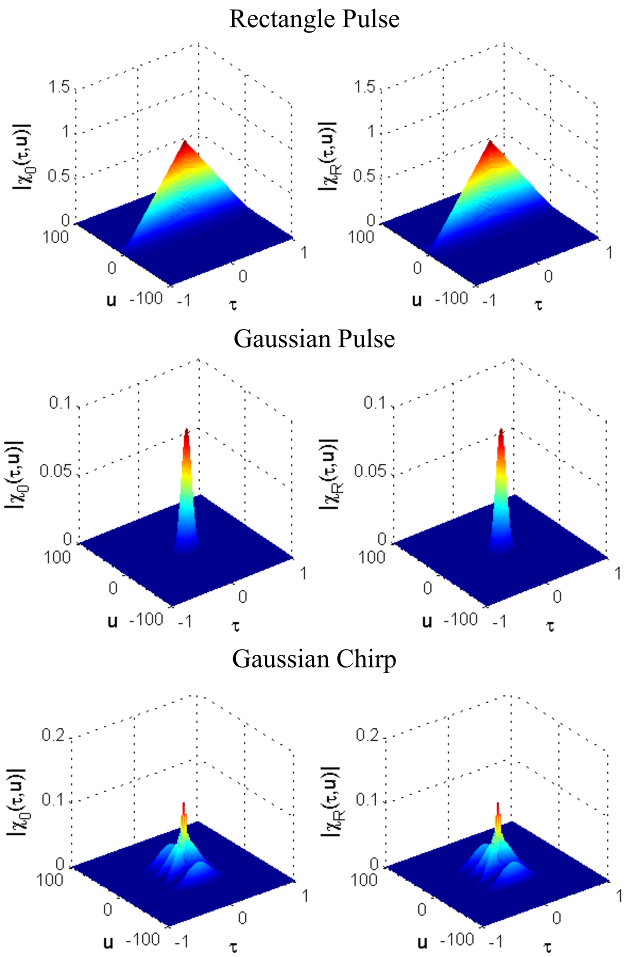

Fig. 3. The given ambiguity functions (left) and the magnitudes of the reconstructed signals (right) after 200 iterations. In general, the algorithm was successful at recovering a signal that gives the equivalent ambiguity function magnitude, yielding graphically indistinguishable results.

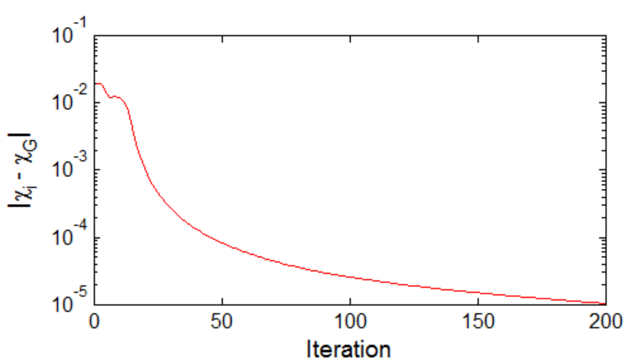

Fig. 4. An example of the RMS error of the rectangle pulse reconstruction between the given ambiguity function magnitude and the synthesized magnitude. The algorithm may not converge depending on the initialization and complexity of the source signal.

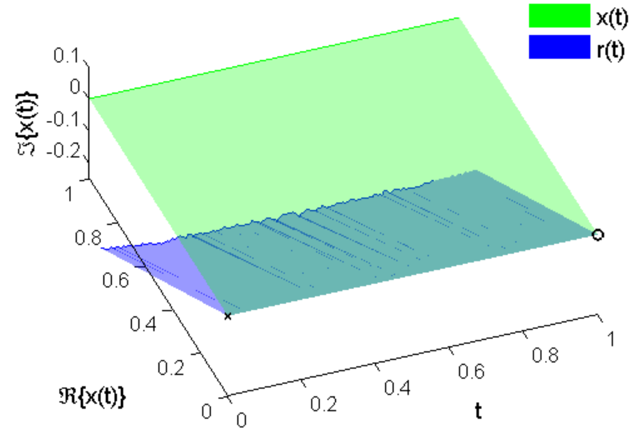

Fig. 5. The original rectangle signal $x(t)$ in green is reconstructed in $r(t)$ in blue. Note the real and imaginary axes, as denoted by $\Re$ and $\Im$. The loss of phase information in the ambiguity function coupled with the random initialization has yielded a slightly warped reconstruction.

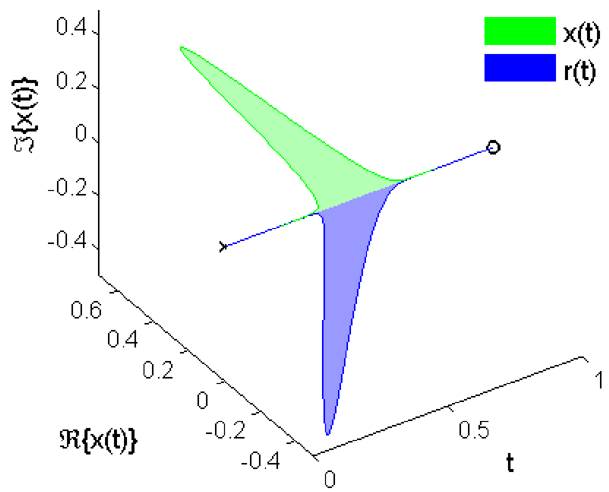

Fig. 6. The Gaussian original real signal $x(t)$ in green is reconstructed in $r(t)$ in blue. The reconstruction is slightly warped compared to the original.

The iteration error between the desired magnitude and the recovered ambiguity function magnitude is shown in Figure 4. Figure 5 shows the recovered complex signal. The arbitrary phase characteristic of the inversion algorithm results in the (blue) reconstruction to be at an arbitrary angle from the original (green) rectangle pulse. A Gaussian pulse is inverted and shown in Figure 6.

A Gaussian chirp is inverted in Figure 7. This pulse was chosen to demonstrate the ability of the procedure to invert ambiguity functions generated from non-symmetric signals with higher-frequency components.

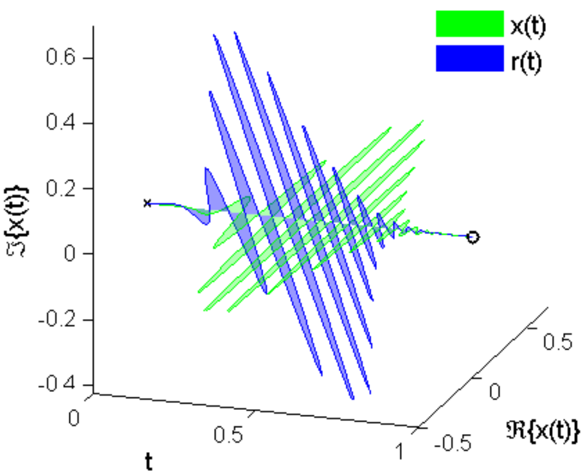

Fig. 7. A Gaussian chirp original real signal $x(t)$ in green is reconstructed as $r(t)$ in blue. This recovery is a slightly time-shifted version of the original.

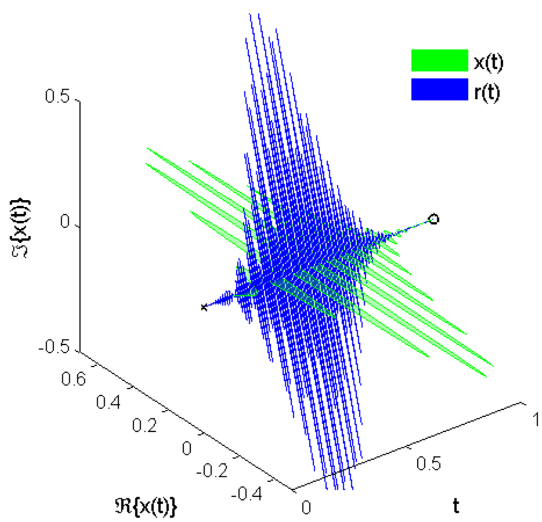

Fig. 8. An alternate recovery of the Gaussian chirp $r(t)$ has an added high-frequency component and is highly oscillatory, although the magnitudes are the same.

V. Conclusions

The modified procedure outlined in this paper can be used to generate source pulses that yield a desired ambiguity function magnitude, demonstrating that in some cases inversion from the ambiguity function magnitude is possible. Nevertheless, there are still several notable issues with the recovered signal. A disruptive high-frequency component may appear in the reconstruction due to undersampled magnitude information, however this can be resolved by ensuring that $x(t)$ is at baseband via a low pass filter. In addition, the recovered source pulse may be translated or flipped in time with no notable impact on the resulting magnitude.

Despite these issues, the resulting signals still give the desired ambiguity function magnitudes. Thus, this paper demonstrates a methodology for recovering a valid source pulse even under the non-invertible case of having the ambiguity function magnitude only.

Acknowledgement

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