Myths Concerning Woodward's Ambiguity Function: Analysis and Resolution

CHARLES BAYLIS
LAWRENCE COHEN
DYLAN EUSTICE
ROBERT MARKS, II
Baylor University
Waco, TX, USA

Woodward's ambiguity function measures the ability of a radar signal to simultaneously measure the range of an object via time delay and its velocity using Doppler shift. The ambiguity function is a foundational staple in radar signal processing. Six myths concerning Woodward's ambiguity function (AF) are addressed in this paper: 1) the AF is uniquely defined, 2) the magnitudes of the various definitions of the AF are the same, 3) the AFs of the baseband and corresponding radio frequency signal are the same, 4) the maximum of a correlation's magnitude determines optimality, and 5) multiplying a signal by a complex linear chirp rotates the AF and 6) the AF is not invertible. Each myth is explained, analyzed, and resolved. In discussing myth 6, the formula for inversion of the ambiguity function to its spawning signal is derived.

Manuscript received October 11, 2015; revised February 15, 2016; released for publication May 20, 2016.

DOI. No. 10.1109/TAES.2016.150735.

Refereeing of this contribution was handled by R. Narayanan.

This work has been funded under a grant from the National Science Foundation (Award Number ECCS-1343316).

Authors' address: Baylor University, ECE, One Bear Place #97356, Waco, TX 76798-7356. Corresponding author is R. Marks, E-mail: (r.marks@ieee.org).

I. INTRODUCTION

Woodward's ambiguity function, a two-dimensional function remnant of the correlation of a transmitted and received signal pair in both Doppler and time delay, is a staple of radar signal processing. Despite its fundamental role, however, the ambiguity function is inconsistently treated in the literature. The ambiguity function is defined differently by different sources. The reason for different definitions is not always evident. Other stated properties of the ambiguity function in the literature are misleading. For example, despite statements to the contrary, the ambiguity function is directly invertible to its spawning signal to within a multiplicative constant phase. Likewise, the ambiguity function of a baseband signal is not the same as when the signal is heterodyned to a radio frequency (RF) signal. We collect these discrepancies concerning the ambiguity function and dub them myths. Each myth is either resolved or shown to be inaccurate. Doing so unifies and clarifies discrepancies in the ambiguity function seen in the literature.

The narrow band ambiguity function for monostatic radar describes the ability of a narrow band RF waveform $g(t)$ to determine the range and Doppler of a target without indeterminateness. We will use the definition [7, 11, 12, 16, 19–21, 32]:

$$\chi(\tau, u) := \int_{-\infty}^{\infty} g(t) g^*(t-\tau) e^{-j2\pi ut} dt \tag{1}$$

where τ and u are the range and Doppler errors relative to the actual range and Doppler of a measured target. A detailed tutorial on the ambiguity function is given by Eustice et al. [10].

We have collected myths concerning the ambiguity function from both implicit and explicit treatment in texts, presentations, and the literature. Each of these myths is analyzed and resolved.

II. FOUNDATIONS

A baseband signal, $x(t)$, is bandlimited with frequency components no greater than B Hertz. Thus,

$$\int_t x(t) e^{-j2\pi ut} dt = 0 \text{ for } |u| > B. \tag{2}$$

The baseband signal is expressed in terms of its in-phase and quadrature components $i(t)$ and $q(t)$

$$x(t) = |x(t)| e^{j\angle x(t)} = i(t) + jq(t).$$

The baseband signal is heterodyned into an RF signal $s_\nu(t)$ with carrier frequency ν as illustrated in Fig. 1. Given the real baseband signals $i(t)$ and $q(t)$ and carrier frequency ν, the RF signal generated is

$$\begin{aligned} s_\nu(t) &= i(t)\cos(2\pi\nu t) + q(t)\sin(2\pi\nu t) \\ &= \Re x(t) e^{-j2\pi\nu t} \\ &= |x(t)|\cos(2\pi\nu t - \angle x(t)). \end{aligned} \tag{3}$$

Note that, because they are complex, the temporal signals $x(t)$ and $x(t)e^{-j2\pi\nu t}$ do not physically exist. Nevertheless,

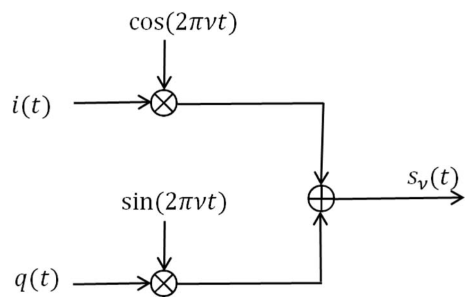

Fig. 1 Generation of RF signal in (3). Sin and cos can be generated by single oscillator using 45° phase shift.

we can calculate ambiguity functions for the complex signal $g(t) = x(t)$ and for the real signal $g(t) = s_W(t)$ using (1).

Properties of the ambiguity function proven in Appendix (part A) include 1) the origin of the ambiguity function is

$$\chi^2(0,0) = \int_t |g(t)|^2 dt \tag{4}$$

and 2) the ambiguity function magnitude at the origin is a maximum

$$|\chi(\tau,u)| \le \chi(0,0). \tag{5}$$

Now we address some myths about the ambiguity function and offer resolutions.

III. MYTHS

A. Myth 1: The Ambiguity Function Is Uniquely Defined

1) *The Truth*: We see phrases like "The ambiguity function . . . is defined as . . ." [8, 19, 23, 30].[1] But there are variations in the definition. Even Woodward, who first proposed the ambiguity function, is not consistent in his definition. In his 1953 book, using the same notation as in (1), Woodward [33] defines his first ambiguity function as [28, 29, 31]

$$\chi_1(\tau,u) = \int_{-\infty}^{\infty} g(t) g^*(t+\tau) e^{-j2\pi ut} dt. \tag{6}$$

Later, in 1967, Woodward opts for the noncausal and more symmetric definition [35][2]

$$\chi_2(\tau,u) = \int_{-\infty}^{\infty} g^*\left(t-\frac{\tau}{2}\right) g\left(t+\frac{\tau}{2}\right) e^{-j2\pi ut} dt, \tag{7}$$

which he notes can also be written as

$$\chi_2(\tau,u) = e^{-j\pi u\tau} \int_{-\infty}^{\infty} g^*(t) g(t+\tau) e^{-j2\pi ut} dt. \tag{8}$$

Similar to (7) is Abramovich and Frazer's definition [1]

$$\chi_3(\tau,u) = \int_{-\infty}^{\infty} g^*\left(t+\frac{\tau}{2}\right) g\left(t-\frac{\tau}{2}\right) e^{-j2\pi ut} dt \tag{9}$$

and the variation [6, 14, 17, 23]

$$\chi_4(\tau,u) = \int_{-\infty}^{\infty} g\left(t+\frac{\tau}{2}\right) g^*\left(t-\frac{\tau}{2}\right) e^{j2\pi ut} dt. \tag{10}$$

Lastly, we have Skolnik's definition [27]

$$\chi_5(\tau,u) = \int_{-\infty}^{\infty} g(t) g^*(t+\tau) e^{j2\pi ut} dt. \tag{11}$$

There are therefore many similarly appearing albeit different definitions of the ambiguity function.

B. Myth 2: The Magnitudes of the Various Definitions of the Ambiguity Function Are the Same

Myth 2 can be stated as follows: *Many definitions exist for the ambiguity function. Some of these definitions are limited to narrowband signals* [2, 29], *and some of them are the magnitude squared of other definitions* [9, 13, 25].

1) *The Truth*: Using the hedge term "some [of these definitions]" in this quote from Weiss [32] is technically correct, but there are also some ambiguity function definitions that are not equal to "the magnitude squared of other definitions." Including (1), we have six definitions of the ambiguity function. From (6),

$$\chi_1(\tau,u) = \chi(-\tau,u). \tag{12}$$

Likewise, from (8) and (12),

$$\begin{aligned}\chi_2(\tau,u) &= e^{-j\pi u\tau}\left[\int_{-\infty}^{\infty} g(t) g^*(t+\tau) e^{j2\pi ut} dt\right]^* \\ &= e^{-j\pi u\tau}\, \chi_1^*(\tau,-u) \\ &= e^{-j\pi u\tau} \chi^*(-\tau,-u).\end{aligned} \tag{13}$$

Similar analysis gives

$$\chi_3(\tau,u) = e^{-j\pi u\tau} \chi(\tau,u).$$

Comparing (9) and (10) reveals that

$$\chi_4(\tau,u) = \chi_3^*(\tau,u) \quad = e^{j\pi u\tau} \chi^*(\tau,u).$$

For Skolnik's definition in (11)

$$\chi_5(\tau,u) = \chi(-\tau,-u)$$

In summary, the six ambiguity functions are related by

$$\chi(\tau,u) = \begin{cases} \chi_1(-\tau,u) \\ \chi_2^*(-\tau,-u) e^{-j2\pi u\tau} \\ \chi_3(\tau,u) e^{j\pi u\tau} \\ \chi_4^*(\tau,u) e^{-j\pi u\tau} \\ \chi_5(-\tau,-u). \end{cases}.$$

Thus,

$$|\chi(\tau,u)| = \begin{cases} |\chi_1(-\tau,u)| \\ |\chi_2(-\tau,-u)| \\ |\chi_3(\tau,u)| \\ |\chi_4(\tau,u)| \\ |\chi_5(-\tau,-u)| \end{cases}. \tag{14}$$

[1]Some define a version of $|\chi|$ [18, 26] or $|\chi|^2$ [3, 8, 16, 27] as the ambiguity function.
[2]Also used by Barbarossa et al. [4, 5] and Papoulis [24].

These transpositional relationships are not as bad as they look, however, because of symmetry properties of the ambiguity function.

a) *General*: For an arbitrary complex signal, the ambiguity function has the following symmetry:

$$\chi^*(-\tau, u) = \chi(\tau, -u)e^{-j2\pi u\tau} \tag{15}$$

from which it follows that

$$|\chi(-\tau, u)| = |\chi(\tau, -u)| \tag{16}$$

and, trivially, $|\chi(\tau, u)| = |\chi(-\tau, -u)|$. Applying the general symmetry property to (14) reveals that

$$|\chi(\tau, u)| = \begin{cases} |\chi_1(-\tau, u)| \\ |\chi_2(\tau, u)| \\ |\chi_3(\tau, u)| \\ |\chi_4(\tau, u)| \\ |\chi_5(\tau, u)| \end{cases} \tag{17}$$

and we have equality except for Woodward's first ambiguity function χ_1 in (6).

PROOF From the definition of the ambiguity function in (1)

$$\chi^*(-\tau, u) = \int_{-\infty}^{\infty} g^*(\xi)g(\xi + \tau)e^{j2\pi u\xi}d\xi.$$

Setting $t = \xi + \tau$ gives

$$\chi^*(-\tau, u) = \int_{-\infty}^{\infty} g^*(t - \tau)g(t)e^{j2\pi u(t-\tau)}dt$$
$$= e^{-j2\pi u\tau}\int_{-\infty}^{\infty} g(t)g^*(t - \tau)e^{j2\pi ut}dt,$$

which gives the desired answer in (15). ∎

b) *For real signals*: If $g(t)$ is real, then

$$\chi(\tau, u) = \chi^*(\tau, -u). \tag{18}$$

Thus,

$$|\chi(\tau, u)| = |\chi(\tau, -u)|. \tag{19}$$

Therefore, when $g(t)$ is real, the magnitude of all the ambiguity functions thus far defined are equal

$$|\chi(\tau, u)| = |\chi_m(\tau, u)|; m = 1, 2, 3, 4, 5. \tag{20}$$

Note, however, that the baseband signal $x(t)$ used to generate the RF signal $s_\nu(t)$, is generally complex and, therefore, the universal equality in (20) does not apply to $g(t) = x(t)$.

PROOF When $g(t)$ is real, $g = g^*$ and the ambiguity function in (1) becomes

$$\chi(\tau, u) = \int_{-\infty}^{\infty} g(t)g(t - \tau)e^{-j2\pi ut}dt.$$

Thus,

$$\chi^*(\tau, u) = \int_{-\infty}^{\infty} g(t)g(t - \tau)e^{j2\pi ut}dt$$

from which (18) follows. ∎

Equation (17) therefore applies, and Woodward's first ambiguity function is not equal to the magnitude of other definitions.

c) *A resolution*: An alternative to Fig. 1 is to use –sin instead of sin. Using the notation already established, the baseband signal is now

$$\hat{x}(t) = i(t) - jq(t)$$
$$= x^*(t)$$

and, in lieu of (3), we have the RF signal

$$\hat{s}_\nu(t) = i(t)\cos(2\pi\nu t) - q(t)\sin(2\pi\nu t)$$
$$= \Re x^*(t)e^{-j2\pi\nu t}$$
$$= |x(t)|\cos(2\pi\nu t + \angle x(t)).$$

From (1), the ambiguity function of $x(t)$ is[3]

$$\chi_x(\tau, u) = \int_{-\infty}^{\infty} x(t)x^*(t - \tau)e^{-j2\pi ut}dt. \tag{21}$$

Thus, for $\hat{x}(t)$,

$$\chi_{\hat{x}}(\tau, u) = \chi_{x^*}(\tau, u)$$
$$= \int_{-\infty}^{\infty} x^*(t)x(t - \tau)e^{-j2\pi ut}dt$$
$$= e^{-j2\pi u\tau}\int_{-\infty}^{\infty} x(\xi)x^*(\xi + \tau)e^{-j2\pi u\xi}d\xi$$
$$= e^{-j2\pi u\tau}\chi_x(-\tau, u) \tag{22}$$

where we have used $\xi = t - \tau$. Hence, using (17)

$$|\chi_{\hat{x}}(\tau, u)| = |\chi_x(-\tau, u)|$$
$$= |\chi_1(\tau, u)|.$$

Thus, the freedom to choose in each case whether the baseband signal is modulated by sin or –sin in the generation of the RF signal allows all of the definitions of ambiguity function considered to have equal magnitude. In practice, the choice of sin and –sin is inconsequential as long as consistency in the analysis is maintained.[4]

C. Myth 3: The Ambiguity Function of Baseband Signals and the Corresponding RF Signals Are the Same

Here is a mathematically correct but unrealizable derivation that seems to support this myth. If $x(t)$ is a complex baseband signal and

$$\tilde{s}_\nu(t) = x(t)e^{-j2\pi\nu t} \tag{23}$$

is the RF signal centered at frequency ν, then the ambiguity for $\tilde{s}_\nu(t)$ is

[3]When useful, we will subscript the ambiguity function with its corresponding function as is done here.

[4]For example, one can choose either a $\mp$ in the definition of the Fourier transform as $X(u) = \int_t x(t)e^{\mp j2\pi ut}dt$. Either works as long as the inverse transform is [22] $x(t) = \int_u X(u)e^{\pm j2\pi ut}du$.

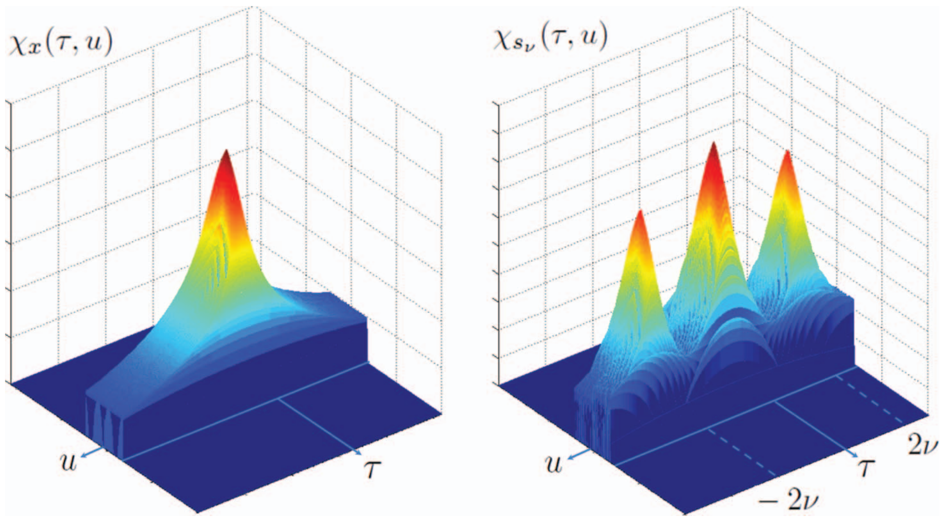

Fig. 2 Log plots of magnitude of ambiguity function for (left) $\chi_x(\tau, u)$ in (21) for baseband signal $x(t)$ equal to simple pulse, and (right) ambiguity function for RF signal in (27) using same pulse. (Negative values of log are set to zero.)

$$\begin{aligned}\chi_{\tilde{s}}(\tau, u) &= \int_t \tilde{s}_\nu(t)\tilde{s}_\nu^*(t-\tau)e^{-j2\pi ut}dt \\ &= \int_t (x(t)e^{-j2\pi\nu t}) \\ &\quad \times (x^*(t-\tau)e^{j2\pi\nu(t-\tau)})e^{-j2\pi ut}dt \\ &= e^{-j2\pi\nu\tau}\int_t x(t)x^*(t-\tau)e^{-j2\pi ut}dt \\ &= e^{-j2\pi\nu\tau}\chi_x(\tau, u) \end{aligned} \tag{24}$$

where the ambiguity of the baseband signal is given in (21). Therefore, the magnitudes of the baseband and RF signals are identical:

$$|\chi_{\tilde{s}}(\tau, u)| = |\chi_x(\tau, u)|.$$

The problem, however, is that the temporal transmitted signal $\tilde{s}_\nu(t)$ in (23) is complex and therefore does not exist. The signal $s_\nu(t)$ in (3), on the other hand, is real.

1) *The Truth*: The ambiguity functions of the baseband signal $x(t)$ and real heterodyned RF signal $s_\nu(t)$ differ in two ways: 1) the ambiguity function of the RF signal contains high frequency beat terms on the u axis in the ambiguity plane centered at twice the RF frequency and 2) the RF ambiguity function's baseband term is equal to the ambiguity function of $x(t)$ modulated in τ by a sinusoid at the RF frequency ν. To show this, start with the ambiguity function of the RF signal

$$\chi_{s_\nu}(\tau, u) = \int_t s_\nu(t)s_\nu^*(t-\tau)e^{-j2\pi ut}dt.$$

Using the definition in (3) and

$$\Re z = \frac{1}{2}(z + z^*) \tag{25}$$

gives

$$\begin{aligned}\psi_{s_\nu}(\tau, u) &= \int_t \Re(x(t)e^{-j2\pi\nu t}) \\ &\quad \times \Re(x(t-\tau)e^{-j2\pi\nu(t-\tau)})e^{-j2\pi ut}dt \\ &= \frac{1}{4}\int_t (x(t)e^{-j2\pi\nu t} + x^*(t)e^{j2\pi\nu t}) \\ &\quad \times (x(t-\tau)e^{-j2\pi\nu(t-\tau)} \\ &\quad + x^*(t-\tau)e^{j2\pi\nu(t-\tau)})e^{-j2\pi ut}dt. \end{aligned} \tag{26}$$

The two auto terms are complex conjugate pairs. The cross terms are also conjugate pairs. Using (25), simplification of the FOIL terms of (26) gives

$$\begin{aligned}\chi_{s_\nu}(\tau, u) &= \frac{1}{2}\Re\left(e^{-j2\pi\nu\tau}\int_t x(t)x^*(t-\tau)e^{-j2\pi ut}dt\right) \\ &\quad + \frac{1}{2}\Re\left(e^{j2\pi\nu\tau}\int_t x(t)x(t-\tau)e^{-j2\pi(u-2\nu)t}dt\right)\end{aligned}$$

or

$$\chi_{s_\nu}(\tau, u) = \frac{1}{2}\Re(e^{-j2\pi\nu\tau}\chi_x(\tau, u) + e^{j2\pi\nu\tau}\psi_x(\tau, u-2\nu)) \tag{27}$$

where we define the pseudo-ambiguity function as

$$\psi_x(\tau, u) = \int_t x(t)x(t-\tau)e^{-j2\pi ut}dt. \tag{28}$$

The pseudo-ambiguity function is similar to the ambiguity function in (1) except there are no conjugations.

The ψ_x terms in (27) corresponds to a beat frequency. On the ambiguity (τ, u) plane, the pseudo-ambiguity function is shifted so far up and down the frequency axes (twice the RF carrier frequency!) that there is no overlap with the ambiguity function, which is centered at the origin. This is illustrated in Fig. 2 for a small value of ν where the magnitude of the ambiguity functions of $x(t)$ and $s_\nu(t)$ are shown. The higher ordered pseudo-ambiguity functions on both sides of the ambiguity function are apparent in the right-hand plot. For these plots, the carrier ν

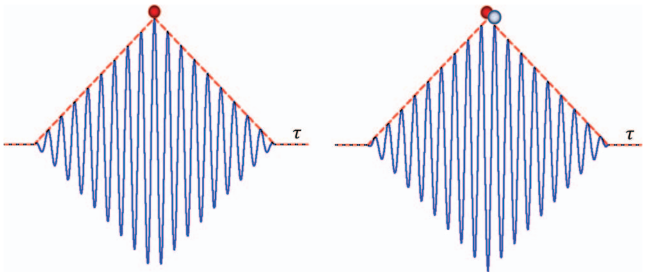

Fig. 3 Illustration of difference between finding maximum of amplitude-modulated waveform and maximum of its envelope.

was on the same order as the bandwidth B. For RF signals, however, the carrier ν is so large that the pseudo-ambiguity functions are greatly separated from the baseband signal. One never considers an unreasonably large Doppler shift corresponding to twice the carrier frequency. The pseudo-ambiguity function term is thus of no practical concern.

The baseband ambiguity function in (27) of interest is the baseband (BB) component

$$\begin{aligned}\chi_{s_\nu}^{BB}(\tau, u) &= \frac{1}{2}\Re e^{-j2\pi\nu\tau}\chi_x(\tau, u)\\ &= \frac{1}{2}|\chi_x(\tau, u)|\\ &\quad\times \cos(2\pi\nu\tau - \angle\chi_x(\tau, u)).\end{aligned} \tag{29}$$

Even so, $\chi_{s_\nu}^{BB}$ is modulated by a sinusoid with frequency centered about the carrier. This is illustrated in the left-hand plot in Fig. 3 where a representation of the $u = 0$ slices of the ambiguity functions in Fig. 2 are plotted. For a rectangular pulse, the cross section of the ambiguity function of the baseband signal $\chi_x(\tau, 0)$ is the triangle shown by the dashed line. The cross section of the ambiguity function of the RF signal $\chi_{s_\nu}^{BB}(\tau, 0)$ is represented as the corresponding sinusoidal modulation in the left-hand plot.

The ambiguity function for the RF baseband signal in (29) oscillates. The function $\frac{1}{2}|\chi_x(\tau, u)|$, though, is seen to be the envelope of $\chi_{s_\nu}^{BB}(\tau, u)$.

D. Myth 4: The Maximum of a Correlation's Magnitude Determines the Optimal Solution

1) *The Truth*: The ambiguity function is a correlation. The location of the maximum of the real part of ambiguity function $\Re\,\chi(\tau, u)$ (not $|\chi(\tau, u)|$) determines the optimal Doppler and range. For the specific case of the ambiguity function, these values are the same. More deeply, however, there is logistically a difference between the more difficult task of finding the maximum of a highly oscillating function and identifying the maximum of the envelope of the function. Here are the details [29, 34].

We transmit the RF signal $s_\nu(t)$ in (3) and receive the signal $s_{\nu-\Delta\nu}(t - \Delta t)$ where $\Delta\nu$ and Δt are, respectively, the unknown Doppler shift and time delay (range) of the target [15]. To find the unknown Doppler and range, we compare the received signal with the family of signals $s_{\nu-u}(t - \tau)$ for all τ and u to find the best estimates of $\Delta\nu$ and Δt. This is easily done by finding the nearest neighbor in the mean square error sense. The best estimate of the Doppler and time delay $\Delta\nu^\dagger$ and $\Delta t^\dagger$ are thus given by

$$(\Delta t^\dagger, \Delta\nu^\dagger) = \arg\min_{\tau,u} \|s_{\nu-\Delta\nu}(t - \Delta t) - s_{\nu-u}(t - \tau)\|^2. \tag{30}$$

Expanding (30) gives

$$\begin{aligned}&(\Delta t^\dagger, \Delta\nu^\dagger)\\ &= \arg\min_{\tau,u}\Big(\|s_{\nu-\Delta\nu}(t - \Delta t)\|^2\\ &\quad + \|s_{\nu-u}(t-\tau)\|^2 - 2\int_t s_{\nu-\Delta\nu}(t - \Delta t)s_{\nu-u}(t-\tau)dt\Big).\end{aligned} \tag{31}$$

The two terms containing $\|s\|^2$ can be simplified. Specifically, as proven in the Appendix (part B):

a) If the carrier frequency plus or minus the Doppler shift is large in comparison with the baseband bandwidth B, then

$$|\nu - \Delta\nu| > B. \tag{32}$$

When this is true,

$$\|s_{\nu-\Delta\nu}(t - \Delta t)\|^2 = \frac{1}{2}\|x(t)\|^2. \tag{33}$$

b) Likewise, if the search for the Doppler shift excludes looking at frequencies near the carrier so that

$$|\nu - u| > B,$$

then

$$\|s_{\nu-u}(t-\tau)\|^2 = \frac{1}{2}\|x(t)\|^2. \tag{34}$$

Using the results in (33) and (34) then changes (31) into

$$\begin{aligned}(\Delta t^\dagger, \Delta\nu^\dagger) = \arg\min_{\tau,u}\Big(&\|x(t)\|^2\\ &-2\int_t s_{\nu-\Delta\nu}(t-\Delta t)s_{\nu-u}(t-\tau)dt\Big).\end{aligned} \tag{35}$$

The term $\|x(t)\|^2$ is not a functions of either τ or u and thus plays no role in finding the minimum. Therefore (35) can be simplified from a minimization to a maximization problem

$$(\Delta t^\dagger, \Delta\nu^\dagger) = \arg\max_{\tau,u}\left(\int_t s_{\nu-\Delta\nu}(t-\Delta t)s_{\nu-u}(t-\tau)\right)dt. \tag{36}$$

We can simplify bookkeeping by defining

$$\zeta_{s_\nu}(\tau, u) := \int_t s_{\nu-\Delta\nu}(t-\Delta t)s_{\nu-u}(t-\tau)dt \tag{37}$$

so that (36) becomes

$$(\Delta t^\dagger, \Delta\nu^\dagger) = \arg\max_{\tau,u}\zeta_{s_\nu}(\tau, u).$$

Details can be fleshed out of the inner product in (36) by using the expressions in (3) and

$$\zeta_{s_\nu}(\tau,u) = \int_t \Re\left(x(t-\Delta t)e^{-j2\pi(\nu-\Delta\nu)(t-\Delta t)}\right) \times \Re\left(x(t-\tau)e^{-j2\pi(\nu-u)(t-\tau)}\right)dt.$$

Applying (25) gives

$$\begin{aligned}\zeta_{s_\nu}(\tau,u) = \frac{1}{4}\int_t &(x(t-\Delta t)e^{-j2\pi(\nu-\Delta\nu)(t-\Delta t)} \\ &+ x^*(t-\Delta t)e^{j2\pi(\nu-\Delta\nu)(t-\Delta t)}) \\ &\times (x(t-\tau)e^{-j2\pi(\nu-u)(t-\tau)} \\ &+ x^*(t-\tau)e^{j2\pi(\nu-u)(t-\tau)})dt. \qquad (38)\end{aligned}$$

Although the expression looks formidable, it generates a useful and interesting result. The auto products in (38) are complex conjugate pairs as are the cross products. Using (25) again, we get

$$\begin{aligned}\zeta_{s_\nu}(\tau,u) = &\frac{1}{2}\Re\left[e^{-j2\pi(\nu-u)(\tau-\Delta t)}\right. \\ &\left.\times\int_\xi x(\xi)x^*(\xi-(\tau-\Delta t))e^{-j2\pi\xi(u-\Delta\nu)}d\xi\right] \\ &+\frac{1}{2}\Re\left[e^{j2\pi(\nu-u)(\tau-\Delta t)}\right. \\ &\times\int_\xi x(\xi)x(\xi-(\tau-\Delta t)) \\ &\left.\times e^{-j2\pi\xi(2\nu-(u-\Delta\nu))}d\xi\right]\end{aligned}$$

where we have set $\xi = t - \Delta t$. This expression can be written in terms of the ambiguity function χ_x and the pseudo-ambiguity function ψ_x of the signal x as

$$\begin{aligned}\zeta_{s_\nu}(\tau,u) = &\frac{1}{2}\Re\left[e^{-j2\pi(\nu-u)(\tau-\Delta t)}\chi_x(\tau-\Delta t, u-\Delta\nu)\right] \\ &+\frac{1}{2}\Re\left[e^{j2\pi(\nu-u)(\tau-\Delta t)}\right. \\ &\left.\times\psi_x(\tau-\Delta t, 2\nu-(u-\Delta\nu))\right]. \qquad (39)\end{aligned}$$

This relation is akin to the ambiguity function for the RF signal in (27). The pseudo-ambiguity function ψ_x is shifted on the ambiguity plane on the u axis to be roughly centered at twice the carrier frequency ν. It can therefore be ignored because the Doppler shift being sought will not be in that region. The remaining baseband expression is

$$\begin{aligned}&\zeta_{s_\nu}^{BB}(\tau,u) \\ &\coloneqq \left[\int_t s_{\nu-\Delta\nu}(t-\Delta t)s_{\nu-u}(t-\tau)dt\right]_{BB} \\ &= \frac{1}{2}\Re\left[e^{-j2\pi(\nu-u)(\tau-\Delta t)}\chi_x(\tau-\Delta t, u-\Delta\nu)\right]. \qquad (40)\end{aligned}$$

$$\begin{aligned}= &\frac{1}{2}|\chi_x(\tau-\Delta t, u-\Delta\nu)| \\ &\times\cos(2\pi(\nu-u)(\tau-\Delta t) - \angle\chi_x(\tau-\Delta t, u-\Delta\nu)). \qquad (41)\end{aligned}$$

Keeping the search on the ambiguity plane only within the baseband region, (37) becomes

$$(\Delta t^\dagger, \Delta\nu^\dagger) = \arg\max_{\tau,u}\zeta_{s_\nu}^{BB}(\tau,u) \qquad (42)$$

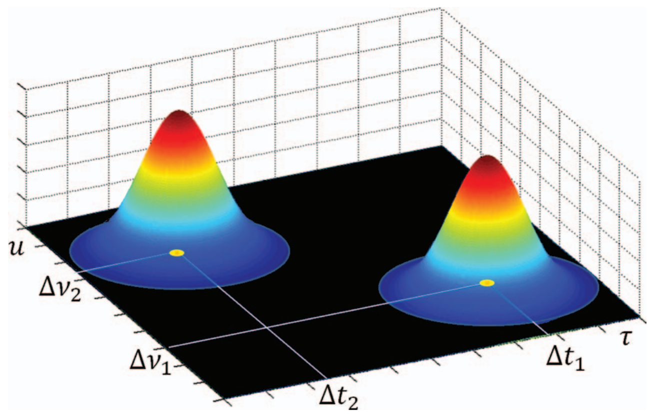

Fig. 4 Correlation for all range values τ and all Doppler shifts u gives matched filter output that peaks at target's true range Δt and Doppler $\Delta\nu$. Shown here are results of two cases, $(\tau,u) = (\Delta t_1, \Delta\nu_1)$ and $(\tau,u) = (\Delta t_2, \Delta\nu_2)$. Independent of measured range and Doppler, function centered around measured values always has same shape. This invariant shape, when shifted to origin, is ambiguity function. Shape of ambiguity function is determined solely by transmitted signal.

Because $|\cos| \le 1$,

$$|\zeta_{s_\nu}^{BB}(\tau,u)| \le \frac{1}{2}|\chi_x(\tau-\Delta t, u-\Delta\nu)|.$$

Indeed, $|\zeta_{s_\nu}^{BB}(\tau,u)|$ is the envelope of $\frac{1}{2}|\chi_x(\tau-\Delta t, u-\Delta\nu)|$. From (40), we have equality at

$$\zeta_{s_\nu}^{BB}(\Delta t, \Delta\nu) = \frac{1}{2}\Re\chi_x(0,0) = \frac{1}{2}\|g(t)\|$$

which we know from (5) is the maximum value of $|\chi_x|$. Because $|\cos| \le 1$, the solution to (42) is the obvious one

$$(\Delta t^\dagger, \Delta\nu^\dagger) = (\Delta t, \Delta\nu).$$

We have two takeaways:

1) Maximizing the oscillatory function in (41) turns out to be the same as maximizing its envelope. This is illustrated in the left-hand plot in Fig. 3 where a representation of the $u = 0$ slices of the ambiguity functions in Fig. 2 are plotted. For a rectangular pulse, the cross section of the ambiguity function of the baseband signal is the triangle shown by the dashed line. The maximum of the triangular signal, as marked by the dot, is the same as the maximum of the oscillating ambiguity function of the RF signal. Although exact for the ambiguity function as a consequence of (5), estimating the maximum of the envelope of an oscillatory signal is a good approximation to maximizing the oscillation as is illustrated in by the right-hand plot in Fig. 3. The maximum of the oscillation is marked by a dot just to the right and just below the dot marking the maximum of the envelope. For high frequency oscillations, the location of both points are nearly identical.

2) As seen in Fig. 4, the (modulated) ambiguity function in (41) is always centered at the optimal solution $(\Delta t, \Delta\nu)$. The shape around the centered function is independent of the measured range and Doppler.

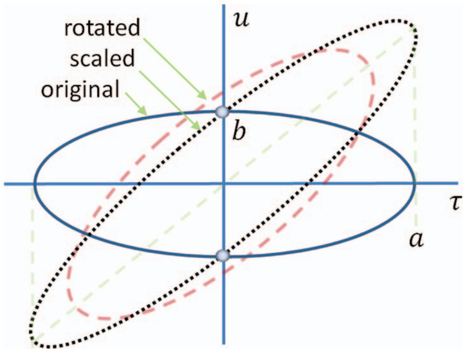

Fig. 5 Illustration of effect on ambiguity function by chirping. Ellipse labeled "original" corresponds to contour of ambiguity function of Gaussian pulse. Common equation for ellipse is $(\tau/a)^2 + (u/b)^2 = 1$. Rotated ellipse is congruent to original. Scaled version of ellipse in (43) becomes $(\tau/a)^2 + ((u - \alpha\tau)/b)^2 = 1$. Although still ellipse, there is marked difference with rotated version.

E. Myth 5: Multiplying a Signal by a Complex Linear Chirp Rotates the Ambiguity Function

1) *The Truth*: Multiplying a signal by a complex chirp imposes a frequency dependent shift of the ambiguity function that can resemble a rotation. This can be easily shown. If $g(t)$ has an ambiguity function of $\chi_x(\tau, u)$, then $g(t)e^{-j\pi\alpha t^2}$ has a corresponding ambiguity function of

$$\begin{aligned}\chi_y(\tau, u) &= \int_t \left(g(t)e^{-j2\pi\alpha t^2}\right) \\ &\quad\times \left(g^*(t-\tau)e^{j\pi\alpha(t-\tau)^2}\right) e^{j2\pi ut}dt \\ &= e^{-j\pi\alpha\tau^2}\int_t g(t)g^*(t-\tau)e^{j2\pi(u-\alpha\tau)t}dt \\ &= e^{-j\pi\alpha\tau^2}\chi_x(\tau, u-\alpha\tau) \qquad (43)\end{aligned}$$

Thus, $|\chi_y(\tau, u)| = |\chi_x(\tau, u - \alpha\tau)|$ and, as illustrated in Fig. 5, the ambiguity function is shifted rather than rotated. See the caption for details.

F. Myth 6: The Ambiguity Function Is Not Invertible

"Unfortunately, no analytic method exists for calculating a signal given its ambiguity function (inverse ambiguity transform); thus the design of a radar signal with desirable characteristics of the ambiguity function is based primarily on the radar designer's prior knowledge of radar waveforms and his or her expertise in such designs." [18]

1) *The Truth*: This myth is strictly true in the sense that

- $g(t)$ will have the same ambiguity function as

$$f(t) = e^{j\phi}g(t) \qquad (44)$$

when ϕ is a real constant because, with reference to (1), $g(t)g^*(t-\tau) = f(t)f^*(t-\tau)$ and

- an inversion is not immediate evident when given only the magnitude $|\chi(\tau, u)|$.

Levanon and Mozeson [18], whom we quoted above, define the ambiguity function as the magnitude of Skolnik's definition in (11). Given the ambiguity in (1), $g(t)$ can be found to within a constant phase term as in (44) where

$$g(t) = \sqrt{\frac{\chi(0,0)}{\int_\tau |\int_\nu \chi^*(-\tau,\nu)d\nu|^2 d\tau}}\int_u \chi^*(-t,u)du. \qquad (45)$$

The proof is provided in Appendix (part C).

IV. CONCLUSIONS

Using analysis built on fundamental foundations, six myths about Woodward's ambiguity function, of varying significance, have been successfully analyzed and resolved.

APPENDIX:

A. Some Elementary Properties of the Ambiguity Function

1) *Proof of* (4): This follows immediately from the definition of ambiguity function in (1).

2) *Proof of* (5): Using the Schwartz inequality and (4) gives

$$\begin{aligned}|\chi(\tau,u)|^2 &= \left|\int_t g(t)g^*(T-\tau)e^{-j2\pi ut_{dt}}\right|^2 \\ &\leq \int_t |g(t)|^2 dt \int_t \left|g^*(T-\tau)e^{-j2\pi ut}\right|^2 dt \\ &= \|g(t)\|^2 \\ &= |\chi(0,0)|^2.\end{aligned}$$

B. Proof of (33) and (34)

Expanding (33) gives

$$\|s_{\nu-\Delta\nu}(t-\Delta t)\|^2 = \int_t |s_{\nu-\Delta\nu}(t-\Delta t)|^2\, dt.$$

Because the centering of the integrand does not affect the integration result

$$\|s_{\nu-\Delta\nu}(t-\Delta t)\|^2 = \int_t |s_{\nu-\Delta\nu}(t)|^2 dt.$$

Using (3) and a trigonometry identity gives

$$\begin{aligned}&\|s_{\nu-\Delta\nu}(t-\Delta t)\|^2 \\ &= \int_t (|x(t)|\cos(2\pi(\nu-\Delta\nu)t - \angle x(t)))^2.dt \\ &= \frac{1}{2}\int_t |x(t)|^2 \frac{1}{2}(1+\cos((4\pi(\nu-\Delta\nu)t - 2\angle x(t))).\,dt \\ &= \frac{1}{2}\|x(t)\|^2 + \frac{1}{2}\int_t |x(t)|^2\cos((4\pi(\nu-\Delta\nu)t - 2\angle x(t)))dt \\ &= \frac{1}{2}\|x(t)\|^2 + \frac{1}{2}\Re\int_t \left(|x(t)|^{e^{j\angle x(t)}}\right)^2 e^{-j4\pi(\nu-\Delta\nu)^t\, dt} \\ &= \frac{1}{2}\|x(t)\|^2 \frac{1}{2}\Re\int_t x^2(t)e^{-j4\pi(\nu-\Delta\nu)^t dt}. \qquad (46)\end{aligned}$$

Because $g(t)$ is bandlimitied with bandwidth B as defined in (2), we are assured that

$$\int_t x^2(t)e^{-j2\pi ut}dt = 0 \text{ for } |u| > 2B. \tag{47}$$

It follows that the integral in (46) has the property that

$$\int_t x^2(t)e^{-j4\pi(\nu-\Delta\nu)^t}dt = 0 \text{ for } |\nu - \Delta\nu| > B.$$

Note that the condition $|\nu - \Delta\nu| > B$ is identical to the criterion in (32) and that, because the second term in (46) is zero, we have proven (33).

The proof of (34) follows the same steps.

C. Proof of the Ambiguity Function Inverse in (45)

For a fixed τ, the ambiguity function in (1) is a simple one-dimensional Fourier transform that can be inverted as

$$g(t)g^*(t-\tau) = \int_{-\infty}^{\infty} \chi(\tau, u)e^{j2\pi ut}du. \tag{48}$$

Setting $t = 0$, and conjugating gives

$$g^*(0)g(-\tau) = \int_{-\infty}^{\infty} \chi^*(\tau, u)du$$

or

$$g(\tau) = \frac{\int_{-\infty}^{\infty} \chi^*(-\tau, u)du}{g^*(0)}. \tag{49}$$

Given $\chi(\tau, u)$, it is therefore possible to regain $g(t)$ to within a multiplicative $g^*(0)$[5]. We now show we can evaluate $g^*(0)$ to within a constant phase term $e^{j\alpha}$.

Equation (49) can be interpreted as

$$h(\tau) := Cg(\tau) = \int_{-\infty}^{\infty} \chi^*(-\tau, u)du \tag{50}$$

where $h(\cdot)$ and $\chi(\tau, u)$ are known and $C \neq 0$ is an unknown constant. (For our derivation, $C = g^*(0)$ but this is not important for the analysis to follow.) From the definition of the ambiguity function in (1), it follows that

$$\chi(0, 0) = E_g$$

where a signal's energy is

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt.$$

Because χ is known, so is $\chi(0, 0) = E_g$. From (50),

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = |C|^2 \int_{-\infty}^{\infty} |g(t)|^2 dt$$

or equivalently,

$$E_h = |C|^2 E_g = |C|^2 \chi(0, 0).$$

[5] 5Equation (49) can be interpreted as a scaled projection of the ambiguity function onto τ. The procedure does not work if $g(0) = 0$. The expression in (48), though, can be evaluated at other values of t. For example, if $t = 1$, (48) becomes $g(1)g^*(1-\tau)$ from which we can find $g(t)$ to within a proportionality constant. Alternately, the projection of the ambiguity onto u gives the Fourier transform of g scaled by $G(0) = \int_t g(t)dt$ [10].

Because $h(t)$ is known, so is E_h, and we conclude the unknown scaling constant is

$$|C|^2 = \frac{E_h}{\chi(0, 0)}.$$

Thus, for ϕ an arbitrary real constant

$$C = e^{j\phi}\sqrt{\frac{E_h}{\chi(0, 0)}}. \tag{51}$$

Then

$$g(t) = \sqrt{\frac{\chi(0, 0)}{E_h}} \int_{u=-\infty}^{\infty} \chi^*(-t, u)du.$$

The inversion in (45) then follows from (49) and (51) for $C = g^*(0)$. A similar analysis can be applied to the inversion of the other of ambiguity function definitions $\chi_m(\tau, u)$, $m = 1, 2, 3, 4, 5$.

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Charles Baylis (S'03—M'08) is an associate professor in the Department of Electrical and Computer Engineering at Baylor University, where he directs the Wireless and Microwave Circuits and Systems (WMCS) Program. Dr. Baylis received the B.S., M.S., and Ph.D. degrees in electrical engineering from the University of South Florida in 2002, 2004, and 2007, respectively. His research focuses on spectrum issues in radar and communication systems, and has been sponsored by the National Science Foundation, the Naval Research Laboratory. He has focused his work on the application of microwave circuit technology and measurements, combined with intelligent optimization algorithms, to creating reconfigurable transmitters. He serves as the general chair of the annual Texas Symposium on Wireless and Microwave Circuits and Systems.

Lawrence Cohen (M'87—SM'12) has been involved in electromagnetic compatibility (EMC) engineering and management, shipboard antenna integration, and radar system design for 33 years. In this capacity, he has worked in the areas of shipboard electromagnetic interference problem identification, quantification and resolution, modestirred chamber research, and radar absorption material design, test, and integration. In March of 2007, Mr. Cohen acted as the Navys principal investigator in the assessment of radar emissions on a WiMAX network. Additionally, he has acted as the principal investigator for various radar programs, including the radar transmitter upgrades. Currently, he is involved with identifying and solving spectrum conflicts between radar and wireless systems, as well as research into spectrally cleaner power amplifier designs, tube, and solid state. Mr. Cohen received a B.S. degree in electrical engineering from The George Washington University in 1975 and a M.S. degree in Electrical Engineering from Virginia Tech in 1994. He is certified as an EMC engineer by the National Association of Radio and Telecommunications Engineers. Larry served as the technical program chairman for the IEEE 2000 International Symposium on EMC and was elected for a threeyear term to the IEEE EMC Society Board of Directors in 1999 and 2009. He is also a member of the IEEE EMC Society Technical Committee 6 for Spectrum Management. For the past 26 years, he has been employed by the Naval Research Laboratory in Washington, DC.

Dylan Eustice earned his Masters degree in Electrical and Computer Engineering from Baylor University in 2015. He grew up in Houston, Texas and currently works as a Research Scientist at Numerica Corporation in Fort Collins, Colorado.

Robert Marks, II (S'71—M'72—SM'83—F'94) is a distinguished professor of electrical and computer engineering at Baylor University, Waco, TX. He is a Fellow of IEEE and the Optical Society of America. His most recent books are Handbook of Fourier Analysis and Its Applications (Oxford University Press, 2009), and Introduction of Evolutionary Informatics (Singapore: World Scientific, 2017) with William A. Dembski and Winston Ewert. His Erdős—Bacon number is five.