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Article

Diversity Inadequacies of Parallel Universes: When the Multiverse Becomes Insufficient to Account for Conflicting Contradistinctions

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The diversity of conflicting contradistinctions available from parallel universes is commonly exaggerated. The number of required universes is shown to increase exponentially with respect to the number of desired contradistinctions. For the commonly cited upper bound of 10^{1000} *universes in the multiverse, only 3,322 binary contradistinctions are possible. What about a countably infinite number of universes? Any finite number of contradistinctions are possible in such a multiverse. If, though, there are a countably infinite number of contradistinctions, all possible cases are not realizable in a multiverse with a countably infinite number of universes.*

Multiple universes essentially existing side by side constitute the multiverse. If a multiverse exists, the number of parallel universes is a question of debate. Serious scientific conception of the multiverse dates to at least 1952 when Erwin Schrödinger suggested their existence.[1] The multiverse was subsequently predicted by string theory, a beautifully elegant model that more and more looks to be unprovable.

Claims of resources available from parallel universes are often wildly exaggerated. Such claims can be used to discount the seemingly miraculous fine tuning of our own universe in order to support life.[2] If there are numerous parallel universes each with different properties, the highly unlikely chance of the existence of our fine-tuned universe is increased.

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The enormous diversity available from parallel universes, collectively called the multiverse, is suggested by the following exchange in the sitcom *The Big Bang Theory* between consummate nerd Sheldon Cooper and Penny, the girl next door.[3]

Penny: Morning, Sheldon! Come dance with me!

Sheldon: No.

Penny: Why not?

Sheldon: While I subscribe to the many worlds theory which posits the existence of an infinite number of Sheldons in an infinite number of universes, I assure you that in none of them I am dancing.

Penny: Are you fun in any of them?

Sheldon: The math would suggest that in a few I'm a clown made of candy. But I don't dance.

Although this dialogue is written as comedy, we can and will analyze its truth using Georg Cantor's theory of infinite numbers. If there are an infinite number of universes, Sheldon could be right.

Physicist Max Tegmark makes a similar though less powerful statement with a greater degree of seriousness:

> Is there a copy of you reading this article? A person who is not you but who lives on a planet called Earth, with misty mountains, fertile fields and sprawling cities, in a solar system with eight other planets? The life of this person has been identical to yours in every respect. But perhaps he or she now decides to put down this article without finishing it, while you read on.[4]

Many physicists are champions of the multiverse. Others are skeptical of their existence.[5] "Some [scientists] even contend that studying the multiverse doesn't count as science."[6] Our purpose is not to participate in the multiverse existence debate. Nor will we discuss the various theories in physics that purport to support the existence of various models of multiverses. Interested readers can read the well-written article by Tegmark[7] or other tutorials.[8] Our purpose is to examine consequential claims often made concerning the existence of the multiverse.

Appealing to an infinite number of parallel universes lends credence to the contingency claims made by Sheldon. But assumption of anything infinite ultimately leads to logical absurdities[9] such as Hilbert's Hotel and the Tristram Shandy paradox.[10] Mathematician David Hilbert noted,

> The infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought. The role that remains for the infinite to play is solely that of an idea.[11]

There are those who disagree with Hilbert and argue the infinite does exist. Even so, the mathematical idea of the infinite due to Cantor[12] is well developed and allows us to visit Sheldon's infinite universe case later. This author is unaware of any *physical* manifestation of infinity and therefore sides with Hilbert in this philosophical debate.

The multiverse offers an explanation to the fine tuning of our universe that materialists readily embrace. Bernard Carr claims, "If you don't want God, you'd better have a multiverse."[13] Why? If there were a large number of parallel universes, the existence of an accidental fine-tuned universe becomes more probable.

The anthropic principle embraced by materialists and others explains fine tuning of our universe with the argument: "If the universe were not tuned for life, we wouldn't be here to notice it." The multiverse diminishes the need for imposition of the awkward anthropic principle.[14] We contribute to this argument in favor of creative fine tuning by noting that as many as 10^{1000} parallel universes come nowhere near to explaining meaningful diversity in the multiverse.

Let's first address the case in which the number of parallel universes is enormous but still finite. We make some simple calculations that reveal that available contradistinctions are not as great as they first might appear.

For a Finite Multiverse

There are many models of the multiverse.[15] A commonly quoted upper bound on the number of possible universes in the multiverse is the enormous number[16] $U = 10^{1000}$.[17] On first viewing, it looks as if there is nothing that we cannot do in this enormous space. A closer look shows that this is not the case.

Consider the case of a single binary contradistinction: Sheldon dances in one universe and does not dance in another. Two universes are required for this. Let's add a second contradistinction: Sheldon has read Max Tegmark's paper in one universe and has not in another. To allow both contradistinctions, four universes are required.

Let's assign a binary assignment of 0 and 1 to distinguish between two contradistinctions: 0 for NO and 1 for YES. For one contradistinction, two universes are needed: one where Sheldon dances and one where he does not.

For two contradistinctions, the four required universes for all possibilities would be tagged 00, 01, 10, and 11 for NO-NO, NO-YES, YES-NO and YES-YES. If a third contradistinction were added, eight universes are required: 000, 001, 010, 011, 100, 101, 110, and 111. Each additional contradistinction doubles the number of required universes. Therefore, if C

binary contradistinctions are desired, we need, at minimum

$$U = 2^C \text{ universes.}$$

If we perform the inverse operation, we find that the maximum allowable number of contradistinctions for universes is as follows:

$$C = \log_2 U \text{ contradistinctions.}$$

So what can be accomplished with 10^{1000} universes in the multiverse? The answer is only

$$C = \log_2 10^{1000} = 3322 \text{ contradistinctions.}$$

That is not a very big number. Making a list of contradistinctions this long is not only straightforward but reveals that we do not have the freedom of diversity initially assumed by Tegmark or Sheldon. This list might be as follows:

1. The actor who plays Sheldon dances. The actor who plays Sheldon does not dance.
2. The actor who plays Sheldon reads Tegmark's paper. The actor who plays Sheldon does not read the paper.
3. Donald Trump becomes president of the United States. Donald Trump never becomes president.
4. Elon Musk's rocket to Mars is successfully launched. The rocket is not successfully launched.
5. Mark Twain grows a moustache. Mark Twain does not grow a moustache.

⋮

3322. Blaise Pascal proposes Pascal's Wager. Pascal does not.

In this list we have used only binary contradistinctions. We could have trinary mutually exclusive contradistinctions such as the following: (1) "The Wizard of Oz" won the Academy Award for Best Picture in 1939; (2) "Gone with the Wind" won the Academy Award for Best Picture in 1939; or (3) "Mr. Smith Goes to Washington" won the Academy Award for Best Picture in 1939. Instead of multiplying the number of universes by 2, we would multiply the number of universes by 3.

Contingencies for a single event need not be binary or trinary. They can be in the millions. Some claim that there are about 10 million colors distinguishable by the discriminating human eye. If we listed all 10 million contradistinctions regarding, say, the currently most common car color on Earth, the number of universes required to cover all cases would be multiplied by 10 million.

The allowable contradistinctions grow logarithmically with respect to the number of universes. This is very slow. For the claims of Tegmark and Sheldon to be credible, the number of universes in the multiverse must be unbelievably large. The figure of 10^{1000} universes does not do it. If the number of parallel universes can be made arbitrarily large, though, all combinations from a finite contingency palate can be painted. Depending on how one defines a universe, Linde and Vanchurin have derived admitted speculative values of $10^{10^{77}}$ to $10^{10^{10,000,000}}$ parallel universes.[18] These are incredible jaw-dropping numbers considering that there are only about 10^{80} atoms in the universe.

The $U = 10^{10^{77}}$ universe count corresponds to about $C = 3 \times 10^{77}$ possible binary contradistinctions. Here is an illustration of how this can happen. Assume atoms in our universe and parallel universes are lexicographically ordered from 1 to 10^{80}. Atom #1 in our universe differs in some manner at any point of time from atom #1 in the parallel universe at some point in time. Possibly it could be displaced by a few million Planck lengths. This requires two universes. An additional parallel universe could have atom #2 likewise displaced, and so forth. Continuing with all the atoms one at a time results in the requirement of 10^{80} binary contradistinctions which cannot be contained in the $10^{10^{77}}$ parallel universes which, as we have calculated, have the capacity for only 3×10^{77} possible binary contradistinctions. And we have not considered, as yet, the contradistinction of whether you are made of candy in some parallel universe or have completed reading Tegmark's paper.

We have implicitly assumed here, as Tegmark and Sheldon did, that all of the parallel universes are similar to ours. This is not necessarily the case. The number of atoms being the same as in our illustration cannot be assumed. Indeed, we have also assumed the existence of time and atoms in the parallel universes. This may not be the case.

We have no corresponding illustration for exhausting contingencies in $U = 10^{10^{10,000,000}}$ parallel universes. If we write a one on an atom and continued to write zeros on all the remaining 10^{80} atoms in the universe, we have written the number $10^{10^{80}}$ which pales in comparison. Taking the base 2 log of $10^{10^{10,000,000}}$ we calculate the corresponding allowance of a *C* that is greater than $10^{10,000,000}$ binary contradistinctions. That is a lot! If the number of parallel universes is large enough, you might be, as claimed by Sheldon, a dancing clown made of candy in one of them.

An interesting limitation occurs, though, when the number of universes in the multiverse becomes infinite compared to which the number $10^{10^{10,000,000}}$ now pales in comparison.

An Infinite Number of Universes in the Multiverse

An infinite number of universes allows all possibilities if the list of contradistinctions is finite.

But what if the list of contradistinctions is infinite? Interestingly, allowing for all combinations from an infinite contradistinction list cannot be contained in an infinite universe multiverse.

To discuss whether there is an infinite number of universes, we must first define infinity.[19] The symbol "∞," as used in mathematics, typically is read "increasing without bound." This is not infinity. No matter how large one increases a number, even $10^{10^{10,000,000}}$, the number is infinitesimal in comparison to the truly infinite.

The mathematics of actual infinities was developed by Georg Cantor. He denoted infinities by $\aleph$, the Hebrew letter *aleph*. The infinite set of all counting numbers, $\{1, 2, 3, \cdots\}$ is said to contain $\aleph_0$ elements. A larger infinity, $\aleph_1$, is the number of points on the line segment between zero and one. Even bigger infinities can be constructed using the set of all subsets of smaller infinities. Since there are 2^N subsets of a set with *N* elements,[20] we can write $\aleph_{n+1} = 2^{\aleph_n}$. Since we can continue to make sets of all subsets forever, there is no biggest infinity.

As we struggle to intuitively grasp a physical intuitive interpretation of 4, 5, or 6 spatial dimensions, visualizing infinities above $\aleph_2$ is problematic. The set of all counting numbers has a cardinality (size) of $\aleph_0$, and $\aleph_1$ is the set of all points on a line segment between, say, zero and one. $\aleph_2$ can be visualized as the set of all possible scribbles in a square including squiggly lines, isolated points, and solid blobs. Like higher dimensions, infinities beyond $\aleph_2$ have no obvious intuitive interpretation but can be constructed mathematically by taking sets of all subsets.

Scripture refers to "eternal life,"[21] being "alive forever more,"[22] and "forever and ever."[23] We leave the question of whether these terms refer to infinite time, unbounded time, or even timelessness to Christian philosophers and theologians. We do know that Cantor had theological concerns about his results in the development of the so-called transfinite number theory of the infinite and corresponded with the Vatican about the matter. Historian Joseph W. Dauben writes,

> [Cantor] was … keenly aware of the ways in which his work might in turn aid and improve both philosophy and theology. Prompted by a strong belief in the role set theory could play in helping the Roman Catholic Church to avoid misinterpreting the nature of infinity, he undertook an extensive correspondence with Catholic theologians, and even addressed one letter and a number of his pamphlets directly to Pope Leo XIII.[24]

To understand the weirdness of the assumption of infinity, we need to explore the meaning of set cardinality (the number of elements in a set). Imagine a shepherd in the morning counting his flock. The shepherd looks at the first sheep and picks up one pebble. For the second sheep, he picks up another pebble. Repeating the process for all the sheep, the number of pebbles in the shepherd's palm is equal to the number of sheep. The cardinality of the two sets is the same because a one-to-one correspondence can be made between elements of the two sets. At the end of the day, the shepherd compares the number of pebbles collected in the morning to the number of sheep to make sure he has not lost any sheep.

Applying this to a set with an infinite number of elements yields absurdity. For example, the cardinality of the set of counting numbers $\{1,2,3,4,\cdots\}$ is the same as the cardinality of the set of even numbers, $\{2,4,6,8,\cdots\}$. For every element in the counting number set, there is a single number associated in the even numbers. Like the shepherd, we associate the 1 with 2, 2 with 4, 3 with 6, 4 with 8, and on and on forever. Even though the even number set is a subset of the counting numbers, there are the same number of elements in both sets. The same is true for multiples of 100 in the set $\{100,200,300,400,\cdots\}$. There are the same number of elements here as in the set of counting numbers. Both have a cardinality of $\aleph_0$. The math is correct, but the results seem absurd. Cantor[25] famously remarked on such strange conclusions:

Je le vois, mais je ne le crois pas!
(I see it, but I do not believe it!)[26]

The absurdity of the infinite can be used to argue that the universe must be finite in age. Tristram Shandy is composing his autobiography. One day in Tristram's life takes him one year to chronicle. Poor Tristram falls further and further behind on finishing his autobiography. If the universe is infinitely old, however, there have been, to date, as many years in the past as there have been days. Both have cardinality $\aleph_0$. Therefore, one could argue that given this infinite amount of time in the past, Tristram could have completed his autobiography today! This clear absurdity is typical of that encountered applying Cantor's transfinite numbers to reality. Tristram Shandy's paradox also points to the necessity of a finite-aged universe without an appeal to physics and the big bang.

There are other mind-bending examples of absurdity from the assumption of infinities. For example, the number of points on unit interval from zero to one, $\aleph_1$, can be placed in a one-to-one correspondence with every point in a two-dimensional unit square. For example, consider the point 0.27548294… on the unit interval. Every other digit in this number is taken to define the points $X = 0.2589\ldots$ and $Y = 0.7424\ldots$. These values define a unique coordinate on the unit square. Conversely, the value of any two coordinates on the unit square can be shuffled together to give a unique point on the unit interval. Incredibly, the number of points in a square and the line segment are the same: $\aleph_1$.

For such reasons, I believe no serious physicist should be a proponent of an infinite number of parallel universes. Assumption of infinity leads to logical absurdities. Nevertheless, because there are those who support an infinite number of parallel universes, let's perform a thought experiment and see what happens if there are. Clearly, any finite list of contradistinctions can be realized by $\aleph_0$ universes. No matter how large the number of contradistinctions, the corresponding log operation will be finite and therefore less than $\aleph_0$ and we are good to go.

For an infinite number of contradistinctions, however, this is not the case. Suppose we had a list of $\aleph_0$ binary contradistinctions. If we assume all contradistinction combinations can be assigned to a universe, we can make a table as shown (table 1). The infinite ($\aleph_0$) number of universes, U, are lexicographically ordered vertically and are numbered **1, 2, 3,** …. The contradistinctions, C, are likewise lexicographically ordered and numbered **#1, #2, #3,** ….

$U\downarrow, C\rightarrow$	**#1**	**#2**	**#3**	**#4**	**#5**	**#6**	…
1	<u>0</u>	1	1	0	1	0	…
2	1	<u>0</u>	0	1	1	1	…
3	0	1	<u>1</u>	1	1	0	…
4	0	0	1	<u>1</u>	0	0	…
5	1	1	0	0	<u>0</u>	0	…
6	1	1	1	1	1	<u>0</u>	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋯

Table 1. The infinite ($\aleph_0$) number of universes, U are lexicographically ordered vertically and are numbered **1**, **2**, **3**, …. The contradistinctions, C, are likewise lexicographically ordered and numbered **#1**, **#2**, **#3**, ….

In the table, an entry of 1 means the binary contradistinction is TRUE. An entry of 0 means FALSE. Contradistinction **#1** might correspond to finishing Tegmark's paper. In Universe **1**, the entry of 0 in the table means FALSE: you have not finished the paper. Universe **2** has a 1 for contradistinction **#1**. Therefore, the answer it TRUE. You have finished the paper.

From the table, the row for Universe **2** is 100111⋯. In Universe **2**, contradistinction **#1**, reading Tegmark's paper, is therefore TRUE, contradistinction **#2** is FALSE, **#3** is FALSE, **#4** is TRUE, **#5** is TRUE, **#6** is TRUE, and so forth. In making this list, we assume that every possible binary contradistinction combination is possible and is somewhere on the list. Using the ingenious diagonalization argument of Cantor, we can show our assumption is wrong and not all universes are listed in the table. The number of universes is therefore insufficient to include all possible binary contradistinction combinations.[27]

In the table, each element on the diagonal is underlined. Here is a list of the underlined entries: 001100⋯. If each bit is flipped, we get the complement 110011⋯. This sequence is nowhere on the list.

Let's call 110011⋯ Universe X. We know Universe X is not on the list since Universe X and Universe **1** differ on the first bit. Universe X cannot be Universe **2** because they differ on binary contradistinction **#2**. Universe X cannot be Universe **3** because they differ on the third contradistinction. Down the list we go and conclude that the universe corresponding to the bit flip of the diagonal is not on the list.[28]

The interesting conclusion is that an infinite number of contradistinctions cannot be realized by an infinite number of universes. More precisely, $\aleph_1$ universes are required to provide all of the combinations of $\aleph_0$ contradistinctions. Recall $\aleph_1$ is the number of points on a line segment. The points are so compact that given a point on the line, the nearest adjacent point cannot be identified. No matter what point is claimed to be the closest, there is a closer point midway between the points. Such is not the case with $\aleph_0$. In the set of counting numbers for example, the closest elements to 100 are 99 and 101.

Visualizing a continuum of $\aleph_1$ universes to cover all $\aleph_1$ contradistinction combinations is beyond the intuitive comprehension of your humble author.

Here is the takeaway. An infinite number of universes cannot exhaustively represent an infinite number of contradistinctions. The number of contradistinction combinations is limited to $\aleph_0$. The number of contradistinctions not covered by the infinite number of universes is greater than the $\aleph_0$ universes represented in the table and is equal to $\aleph_1 - \aleph_0 = \aleph_1$.[29]

Conclusions

Parallel universes, still a speculation, are a common source of exaggerated claims for simultaneous existence of conflicting contradistinction. A mere 10^{1000} parallel universes in the multiverse do not allow for much variation in terms of contradistinctions. The universe count, though, can always be made large enough to account for any contradistinction tally.

We have shed light on the claim Sheldon Cooper made at the beginning of this article: "While I subscribe to the many worlds theory which posits the existence of an infinite number [$\aleph_0$] of Sheldons in an infinite number [$\aleph_0$] of universes, I assure you that in none of them I am dancing." If $\aleph_0$ contradistinctions are assumed requiring $\aleph_1$ contradistinction combinations, Sheldon could, indeed, be right. There are many contradistinction combinations, including Sheldon not dancing, not realized in an infinite number of parallel universes. ☼

Notes

[1]John Gribbin, *Erwin Schrödinger and the Quantum Revolution* (New York: Random House, 2012).

[2]Paul Davies, *Cosmic Jackpot: Why Our Universe Is Just Right for Life* (New York: Houghton Mifflin Harcourt, 2007); and Martin Rees, *Just Six Numbers: The Deep Forces That Shape the Universe* (New York: Basic Books, 2008).

[3]"The Big Bang Theory—Sheldon Doesn't Dance," YouTube clip, https://youtu.be/XQ7Tj-Hxx1o.

[4]Max Tegmark, "Parallel Universes," *Scientific American* 288, no. 5 (2003): 40–51.

[5]Jim Baggott, *Farewell to Reality: How Modern Physics Has Betrayed the Search for Scientific Truth* (New York: Pegasus Books, 2014); George F. R. Ellis, "Does the Multiverse Really Exist?," *Scientific American* 305, no. 2 (2011): 38-43; George Ellis and Joe Silk, "Scientific Method: Defend the Integrity of Physics," *Nature News* 516, no. 7531 (2014): 321; Anna Ijjas, Paul J. Steinhardt, and Abraham Loeb, "Cosmic Inflation Theory Faces Challenges," *Scientific American* 316, no. 2 (2017): 32–39; and Sarah Scoles, "Can Physics Ever Prove the Multiverse Is Real?," Smithsonian.com (April 19, 2016), https://www.smithsonianmag.com/science-nature/can-physicists-ever-prove-multiverse-real-180958813/.

[6]Tom Siegfried, "Making Sense of Many Universes," *Knowable Magazine* (April 26, 2018).

[7]Max Tegmark, "Parallel Universes," in *Science and Ultimate Reality: Quantum Theory, Cosmology and Complexity*, ed. John D. Barrow, Paul C. W. Davies, and Charles L. Harper Jr. (New York: Cambridge University Press, 2004), 459–91.

[8]Bernard Carr and George Ellis, "Universe or Multiverse?," *Astronomy & Geophysics* 49, no. 2 (2008): 2–29; Steven Manly, *Visions of the Multiverse* (Pompton Plains, NJ: Career Press, 2011).

[9]Alexander R. Pruss, *Infinity, Causation, and Paradox* (New York: Oxford University Press, 2018).

[10]Michael Clark, *Paradoxes from A to Z,* 2nd edition (New York: Routledge, 2007).

[11]David Hilbert, "On the Infinite," in *Philosophy of Mathematics: Selected Readings*, ed. Paul Benacerraf and Hilary Putnam (New York: Cambridge University Press, 1983); William Lane Craig, *Time and Eternity: Exploring God's Relationship to Time* (Wheaton, IL: Crossway, 2001); and ____, *Reasonable Faith: Christian Truth and Apologetics*, 3rd edition (Wheaton, IL: Crossway, 2008), 106.

[12]Joseph Warren Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton, NJ: Princeton University Press, 1990).

[13]Quoted in Tim Folger, "Science's Alternative to an Intelligent Creator: The Multiverse Theory," *Discover* (December 2008), http://discovermagazine.com/2008/dec/10-sciences-alternative-to-an-intelligent-creator.

[14]William Lane Craig, "The Teleological Argument and the Anthropic Principle," in *The Logic of Rational Theism:*

Exploratory Essays, ed. William Lane Craig and Mark S. McLeod (Lewiston, NY: Edwin Mellen, 1990), 127–53.

[15]Tegmark, "Parallel Universes," in *Science and Ultimate Reality: Quantum Theory, Cosmology and Complexity*, ed. Barrow, Davies, and Harper. Tegmark's model does not directly espouse 10^{1000} universes and, indeed, proposes that there can be an infinite number of universes in the multiverse.

[16]More properly, 10^{1000} is the number of structures.

[17]Mary-Jane Rubenstein, *Worlds without End: The Many Lives of the Multiverse* (New York: Columbia University Press, 2014); Francisco José Soler Gil and Manuel Alfonseca, "Is the Multiverse Hypothesis Capable of Explaining the Fine Tuning of Nature Laws and Constants? The Case of Cellular Automata," (last revised March 30, 2013), https://arxiv.org/abs/1105.4278; Andrei Linde, "Towards Inflation in String Theory," *Journal of Physics: Conference Series* 24 (IOP Publishing, 2005), 151; and John Turl, "Do Many Worlds Make Light Work?," *Science & Christian Belief* 24, no. 1 (2012): 55-79.

[18]Andrei Linde and Vitaly Vanchurin, "How Many Universes Are in the Multiverse?," *Physical Review* D 81, no. 8 (2010): 083525.

[19]George Gamow, *One, Two, Three … Infinity: Facts and Speculations of Science* (Mineola, NY: Dover Publications, 1988).

[20]For example, the set {ABC} has three elements. Its subsets are Ø (the empty set), {A}, {B}, {C}, {AB}, {AC}, {BC}, {ABC}. There are therefore $2^3 = 8$ subsets of a set with three elements.

[21]Romans 6:23.

[22]Revelation 1:18.

[23]Revelation 20:10.

[24]Joseph W. Dauben, "Georg Cantor and Pope Leo XIII: Mathematics, Theology, and the Infinite," *Journal of the History of Ideas* 38, no. 1 (1977): 85–108.

[25]Fernando Q. Gouvêa, "Was Cantor Surprised?," *The American Mathematical Monthly* 118, no. 3 (2011): 198–209.

[26]Letter from Cantor to R. Dedekind regarding results of *Theory of Manifolds* (1877).

[27]Roger Penrose, *The Emperor's New Mind: Concerning Computers, Minds, and The Laws of Physics* (New York: Oxford University Press, 1989).

[28]This simple proof, due to Cantor, is referred to as diagonalization. It has many applications, including proof that the set of real numbers is greater than $\aleph_0$ and Turing's halting problem.

[29]A similar example is the set of points on the line segment [0,1] which has a cardinality of $\aleph_1$. We remove all of the points in this interval that are rational. The number of removed points is $\aleph_0$. The number of points remaining remains $\aleph_1 - \aleph_0 = \aleph_1$.

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