

Holographic representation of space-variant systems: system theory

Robert J. Marks II and Thomas F. Krile

System theory for holographic representation of linear space-variant systems is derived. The utility of the resulting piecewise isoplanatic approximation (PIA) is illustrated by example application to the invariant system, ideal magnifier, and Fourier transformer. A method previously employed to holographically represent a space-variant system, the discrete approximation, is shown to be a special case of the PIA.

I. Introduction

By holographically recording an arbitrarily complex linear space-invariant system's transfer function, one may duplicate input-output relationships of the system with access solely to the hologram. An arbitrary input is Fourier transformed by a thin lens, multiplied by the holographically recorded transfer function placed in the lens's back focal plane, and reimaged. The output is equivalent to that of the recorded system with the same input. This type of system representation may be used to condense a multielement space-invariant linear system into a single hologram.

Unfortunately, there are many common optical systems of interest that are space variant to which this scheme cannot be directly applied. Recent efforts to holographically record space-variant systems employing a variation of the transfer function method, however, have proven successful. Either by input isoplanatic patch division or sampling, successful volume hologram' representations of the space-variant nonunity magnification imaging system have been obtained.¹⁻⁵

It is thus of practical importance to investigate that system theory necessary for holographically representing space-variant systems. The approach employed herein divides the space-variant system input plane into isoplanatic patches to which corresponding transfer functions are assigned. The resulting piecewise isoplanatic approximation (PIA) output is

representative of the true system output. Although the isoplanatic patch concept is not new,⁶ its use has been primarily confined to system analysis within a single patch.

The PIA is herein derived, and examples of its application to various systems are offered. Equivalence of true and PIA outputs of space-invariant systems is established. The discrete approximation (DA) previously employed for space-variant system representation¹ is shown to be a special case of the PIA.

II. Foundations

For notational and continuity purposes, a brief review of classical linear system theory is now offered after Goodman.⁷ Analysis is restricted to one dimension with no loss of generality.

A general system consisting of an input, a black box, and an output may be modeled by the mathematical operator $\mathbf{S}[\]$ such that

$$g_0(x) = \mathbf{S}[g_i(x)], \quad (1)$$

where $g_0(x)$ is an output corresponding to an input $g_i(x)$. A system is said to be linear if and only if

$$\mathbf{S}[ag_i(x) + bf_i(x)] = a\mathbf{S}[g_i(x)] + b\mathbf{S}[f_i(x)], \quad (2)$$

where a and b are constant. Such systems may be described by the superposition integral

$$g_0(x) = \int_{-\infty}^{\infty} g_i(\xi)h(x - \xi; \xi)d\xi, \quad (3)$$

where, after the notation of Lohmann and Paris,⁸

$$h(x - \xi; \xi) = \mathbf{S}[\delta(x - \xi)] \quad (4)$$

is the system's line spread function, and $\delta(x)$ is the Dirac delta.

When this work was done, both authors were with the Rose-Hulman Institute of Technology, Department of Electrical Engineering, Terre Haute, Indiana 47803. R. J. Marks II is presently with Texas Tech University, Department of Electrical Engineering, Lubbock, Texas 79409.

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A linear system is said to be space invariant, or isoplanatic, if

$$h(x - \xi; \xi) = h(x - \xi), \quad (5)$$

that is, the line spread function is a function only of the difference $(x - \xi)$. For such systems, Eq. (3) becomes the convolution integral

$$g_0(x) = \int_{-\infty}^{\infty} g_i(\xi)h(x - \xi)d\xi, \quad (6)$$

which is expressed in shorthand notation as

$$g_0(x) = g_i(x) * h(x). \quad (7)$$

For the space-variant linear system, one needs to know the line spread function from every point on the input plane to define completely the system, while the invariant case demands only one defining relationship.

The input-output expression of Eq. (7) may also be expressed in the frequency domain as

$$G_0(f_x) = H(f_x)G_i(f_x), \quad (8)$$

where $G_0(f_x)$, $H(f_x)$, and $G_i(f_x)$ are the respective Fourier transforms of $g_0(x)$, $h(x)$, and $g_i(x)$. For example,

$$H(f_x) = \int_{-\infty}^{\infty} h(x) \exp(-j2\pi f_x x) dx \quad (9)$$

is termed the system transfer function.

The straightforward relationships in Eqs. (7) and (8) give rise to the feasibility of holographic recording of space-invariant systems described in Sec. I. A further generalization is needed for recording techniques for the space-variant case.

III. Piecewise Isoplanatic Approximation

In the above analysis, a system was given a specific classification of variant or invariant by the criterion of Eq. (5). Lohmann and Paris⁸ have proposed that variant systems might be assigned degrees of invariance. It follows that a space-variant linear system having a high degree of invariance might successfully be analyzed in a manner similar to that used for invariant systems. To do this, the input plane is divided into a number of regions, or isoplanatic patches, in which the line spread function essentially meets the invariance criterion. Each input function is divided into similar regions that are convolved with corresponding line spread functions and superimposed to yield an approximated output.

A. Derivation

A linear system composed of invariant input regions is termed piecewise isoplanatic. The distribution of these regions on the input plane may be expressed as

$$\sum_n \mu(x - l_n)\mu(-x + u_n), \quad (10)$$

where l_n and u_n are, respectively, the lower and upper endpoints of the n th patch, and $\mu(x)$ is the unit step function defined as

$$\mu(x) = \begin{cases} 1; & x \geq 0, \\ 0; & x < 0. \end{cases} \quad (11)$$

The summation is assumed to cover the region of interest on the input plane. We also assume adjacent patches are joined so that

$$u_n = l_{n+1} \quad (12)$$

The n th isoplanatic patch thus has a width of $u_n - l_n$ and is centered at $x = (l_n + u_n)/2$.

In order to define completely the piecewise isoplanatic system, knowledge of the system's line-spread function for each patch is needed. We thus assume line sources are conveniently placed at some input plane point x_n where

$$l_n \leq x_n \leq u_n, \quad (13)$$

and that we have knowledge of

$$\begin{aligned} h_n(x - x_n) &= h(x - x_n; x_n) \\ &= \mathcal{S}[\delta(x - x_n)] \end{aligned} \quad (14)$$

for all n .

Consider now the input-output relationship of the piecewise isoplanatic system. An input, $g_i(x)$, must first be divided into isoplanatic regions. Specifically,

$$g_i(x) = \sum_n g_n(x - x_n), \quad (15)$$

where

$$g_n(x - x_n) = g_i(x)\mu(x - l_n)\mu(-x + u_n). \quad (16)$$

An illustration of extraction of $g_n(x)$ from $g_i(x)$ is offered in Fig. 1.

The output of the piecewise invariant system is found through substitution of Eq. (15) into Eq. (1);

$$g_0(x) = \sum_n \mathcal{S}[g_n(x - x_n)]. \quad (17)$$

The summation sign is extracted from the system operator due to superposition [(Eq. (2))]. Each argument of the system operator is invariant and can be expressed via the convolution integral [Eq. (6)]:

$$\mathcal{S}[g_n(x - x_n)] = \int_{-\infty}^{\infty} g_n(\xi - x_n)h_n(x - \xi)d\xi. \quad (18)$$

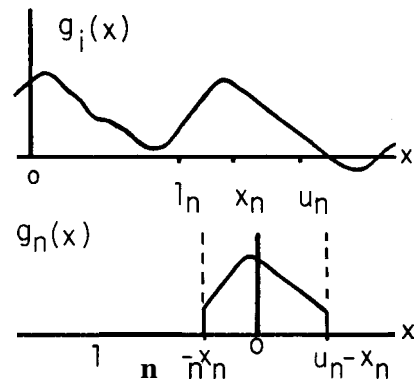


Fig. 1. Extraction of the n th isoplanatic region $g_n(x)$ from an input $g_i(x)$.

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